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PHILOSOPHICAL TRANSACTIONS.

I. *Chemical and Physiological Experiments on living Cinchonæ.*

By J. Broughton, *B.Sc., F.C.S., Chemist to the Cinchona-Plantations of the Madras Government. Communicated by Dr.* E. Frankland, *F.R.S.*

Received May 16,—Read June 16, 1870.

On the Neilgherry Mountains in South India are now growing nearly three millions of trees of cinchona of various species. The greater part of these are on plantations belonging to Government, and are the result of the introduction from South America and successful naturalization of these valuable febrifuge-yielding plants by the Government of India, under circumstances which have long since been made public.

The chemical investigations which during the last three years have been made, for the purpose of settling the various economic questions connected with the production of the febrifuge constituents of the bark, have led to some conclusions of scientific interest. I have the honour in the subsequent pages of communicating to the Royal Society the most important of these, and the experimental grounds on which they depend. These inquiries have been made under circumstances of great advantage, for the living plants have never before been under the control of the experimenter. The ability to study the changes occurring in the growing tissues cannot fail to throw light on the formation and physiological functions of the chemical constituents whose production is the object of the undertaking.

The organic principles which characterize these cinchona-barks are the alkaloids Quinine, Cinchonidine, Cinchonine, and occasionally Quinidine, the peculiar bitter principle Quinovin, the acids Quinic and Quinotannic, and in small amount another not fully investigated.

Without attempting to describe the well-known alkaloids, it may be stated that all the facts known point to a marked natural connexion between quinine and cinchonidine, notwithstanding the difference of an atom of oxygen in their composition. Thus the analysis of the individual plants will frequently give results which show the same amount of alkaloids, but which differ by the respective *quantities* of these alkaloids, while all

other constituents remain unaffected. Chemically these alkaloids greatly resemble each other, their solutions both rotate the plane of polarization to the left, and their salts and behaviour with reagents are very similar. With the exception that cinchonidine will crystallize readily from alcohol, and produces no green colour with chlorine and ammonia, their reactions differ in degree only*. The circumstances under which they mutually occur in living cinchona-barks, taken in connexion with their chemical properties, would lead me to class the alkaloids in two groups: 1st. Quinine, Cinchonidine, Quinidine; 2nd. Cinchonine. I have never yet met with the alkaloid aricine in any Indian cinchona-bark, though it has frequently been looked for.

Quinovin is a very constant constituent of all the parts of a cinchona-plant, as is also quinic acid; the latter is the acid of cinchona, as tartaric acid is that of the grape, and is found in the free state in most of the juices. The peculiar red substance known as cinchona-red, though found in all dried bark, does not exist in the living plant, and is formed by the action of the free oxygen upon the peculiar tannin: hence a slice of fresh bark always becomes brownish red when the internal tissues are exposed.

On account of their being the most vigorous, the oldest, and in suitable climates the hardiest trees, I have generally chosen the plants of *C. succirubra* for the subjects of experiment in cases where a special treatment of the plant was necessary. The bark of this kind is very rich in alkaloids and the cinchona tannin: it may contain all the alkaloids, though quinidine is rarely found; the other three invariably occur. It has a great tendency to produce cinchonidine. I have every reason for believing that the natural processes resulting in the formation of the several alkaloids are similar throughout the quinine-yielding species though differing in degree.

Occurrence of Alkaloids in various parts of the Plant.

No. 1 refers to a kind (whether species or variety is not accurately determined) which was found on the plantation among some plants raised from seed brought by Mr. Cross from the Loja district; it gives the largest yield of quinine of any kind known. The first five are quinine-yielding kinds, but in which cinchonidine may occur in greater or less degree. Nos. 6 and 7 are nearly destitute of quinine.

The thin bark on the larger roots is still richer in alkaloids, and in *C. succirubra* the yield is generally about 9·70, and may reach 12 per cent. It consists principally of cinchonidine and cinchonine. The root-bark of *C. officinalis* is also rich in alkaloids; analysis gave 5·4 per cent. The root-bark has been little investigated, as the difficulty of obtaining it would render it an unsuitable source of alkaloids. I ascribe its richness in alkaloids, for reasons given below, to the circumstance of it being shielded from the rays of the sun.

The leaves of cinchona are bitter from the presence of quinovin, and acid from free quinic acid. In many early trials I failed to get full and satisfactory evidence of the presence of alkaloids. By working with great care upon 20 lbs. of the leaves, I obtained a small amount of rough uncrystallizable alkaloid, which, by solution in weak acetic acid, exposure to a stream of air to facilitate oxidation, decolorization with animal charcoal, reprecipitation, drying, several times repeated, I procured tolerably pure and colourless. From this I obtained well-defined crystals of the quinine sulphate and oxalate, and unmistakable crystals of cinchonidine and cinchonine from their weak alcoholic solutions: the amounts are, however, very small. Other trials, conducted quantitatively, gave the following results, which are given in percentages of the dry and fresh leaves:—

	C. succirubra.		*C. officinalis.*	
	In fresh.	In dry.	In fresh.	In dry.
Total alkaloids	0·0041	0·019	0·0035	0·0111
Quinine	0·0016	0·008	0·0015	0·005
Cinchonidine and cinchonine	0·0025	0·011	0·0020	0·006
Quinine sulphate obtained in crystals	0·00078	0·0037	0·0008	0·002

It is evident from the above that the leaves are useless as a source of alkaloids. The state of their occurrence is wholly different to that in which they occur in the bark.

The wood of cinchona also contains the alkaloids in small amount; they are there associated with a red resin and the cinchona tannin. In *C. succirubra* the alkaloids may amount to 0·1 per cent., of which 0·03 is quinine and the remainder cinchonine and cin-

Wittstein's cinchonidine. In my examination of the bark of *C. peruviana*, I also found a form of cinchonidine the pearly centre of whose crystalline sulphate and other properties agreed exactly with Wittstein's description. By repeated crystallization of its sulphate from water, and then of the free base from alcohol, it was obtained apparently pure; it was then carefully compared with a standard specimen of ordinary cinchonidine in the polarimeter, and found to possess precisely the same rotatory power. I cannot therefore consider it a distinct substance; I therefore only recognize one form of cinchonidine in the text.

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Occurrence of Alkaloids in various parts of the Plant.

The bark is *par excellence* the seat of the alkaloids, that of the trunk being the richest. I here quote analyses of the trunk-bark of various species growing on the Neilgherries, showing the percentages of alkaloids in the dry bark.

	1.	2.	3.	4	5.	6.	7.
	A lanceolate-species tree, name unknown, 5 years.	*C. officinalis*, 5 years old, fine tree.	*C. officinalis*, 5 years old, mean tree.	*C. Calisaya*, 2½ years, mean tree.	*C. succirubra*, 7 years old, mean tree.	*C. peruviana*, 5 years old, mean tree.	*C. micrantha*, 4 years old, mean tree.
Total alkaloids	11·40	6·76	4·34	4·53	7·43	6·25	7·1
Quinine	9·75	3·76	2·18	3·27	1·72	0·41	0·3
Cinchonidine and cinchonine	1·65	3·00	2·16	1·26	5·71	5·84	6·8
Crystallized cinchonidine sulphate obtained	0·38	1·94	2·00	0·70	4·92	1·80	0·0
Ditto quinine sulphate	10·13	3·45	1·60	2·17	1·51	0·00	0·0
Cinchonine	......	......	......	......	......	3·84	6·8

* HOWARD (Illustrations to the 'Nuova Quinologia' of Pavon) states that the cinchonidine found in *C. peruviana* is that peculiar cinchonidine described by WITTSTEIN (J. pr. Chem. 72, 101), and generally known as

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chonidine. The wood of *C. officinalis* contains much less. All the several alkaloids have been satisfactorily identified in the bases obtained from wood, by being obtained in well-defined microscopic crystals yielding in solution their respective reactions.

The beautiful scented blossoms contain no alkaloids, but are rich in quinovin and quinic acid.

The fruit also contains no alkaloids. It is very sour from the presence of quinic acid; it is a most convenient source of this acid. The juice of the fruit mixed with milk of lime and filtered, readily yields on evaporation an abundant crop of crystals of calcic quinate.

The ripe seed contains but traces of the characteristic principles of cinchona and no alkaloids.

The milky fluid which fills the laticiferous vessels cannot be obtained in sufficient quantity for analysis. It is not bitter, and contains in all probability little or no alkaloid. These vessels are rarely contained in the *trunk*-bark of *C. succirubra*, and can therefore scarcely be supposed to be concerned (directly at least) in the production of alkaloids.

State of combination in which the Alkaloids exist in the Bark.

This, after many trials, was determined by the following method. A quantity of the best trunk-bark of *C. succirubra* was exposed to strong pressure in a very powerful screw-press, by which means the juices were pressed out so perfectly that the bark was apparently almost dry; and subsequent examination under the microscope showed that the parenchymatous cells were quite broken up, and many of the liber-cells also.

The juice at first obtained was of a greenish-yellow colour; it, however, absorbed oxygen rapidly and became red. When first obtained, it was very turbid from the presence of much alkaloid tannate of a grey colour, rapidly becoming brownish red; this soon subsided in a well-corked flask, and left the liquid clear. The supernatant liquid was bitter and strongly acid from free quinic acid; it also contained free tannin. Its specific gravity was 1·034 at 17° C.

It was now assumed that the whole of the moisture lost by the bark on drying over oil of vitriol represented the water of this juice; this assumption, if not entirely true, will not differ sufficiently from the truth to seriously affect the subsequent results. The water of hydration of the woody fibre &c. parted with under the circumstances would obviously be a very small portion of the whole; hence the specific gravity of the juice being known, the whole weight of juice present in the bark is also known.

If, now, the whole amount of alkaloid present in the bark is determined, and also that contained in solution in the juice, the difference will express the amount of alkaloids existing in an insoluble form in the living bark.

The analytical data are as follow:—

Moisture lost by bark on sulphuric acid 67·5 per cent.

	In fresh bark.	In 100 parts of juice.
Total alkaloids	6·10	1·17
Quinine	1·15	0·31
Cinchonidine	3·65	0·51
Cinchonine	1·30	0·35

whence, by an obvious calculation, in the sample of fresh bark the respective alkaloids existed as follows:—

	In natural solution.	In an insoluble state.	Total in bark.
Total alkaloids	0·82	5·28	6·10
Quinine	0·22	0·93	1·15
Cinchonidine	0·35	3·30	3·65
Cinchonine	0·24	1·06	1·30

It thus appears that the greater part of the alkaloids is contained in the bark in the solid state; none can be in the free state, on account of the excess of quinic acid and cinchona tannin. Hence it must be concluded that six-sevenths of the alkaloids exist in the cells of this bark as insoluble tannates, the remaining seventh mainly existing in solution as quinate, though the solution is of course saturated with tannate in addition.

These combinations of the alkaloids with tannin are very sparingly soluble in cold water. The whole state of things in the bark resembles the equilibrium of solution and precipitate which would result from a mixture of tannin, alkaloid, and quinic acid. The quinic acid is only able to decompose a small portion of tannate, and the tannin is not able to precipitate the whole alkaloid from solution. Actual trial with the respective substances produces these effects. It is to be remarked that there is no marked difference in the proportions of the respective alkaloids which exist, in one or the other state, in the cell-contents or in the liquid which moistens the whole. This is in harmony with the impression, to which all the facts at present known seem to conduce, that all the cinchona alkaloids are physiologically equivalent.

From the nature of the weak compound that quinovin is able to form with the cinchona alkaloids, it is evident that none could exist in the presence of quinic acid. Nearly the whole of the quinovin is to be found in the bark after the juice has been squeezed out. The clear juice only contains a very small amount, and its presence is readily explained by the faint solubility of this substance in water.

The whole of the quinovin exists therefore in the insoluble and free state.

No thoroughly accurate means of determining the amount of the cinchona tannin has yet been devised; neither gelatin, animal membrane, nor even plumbic acetate removes it completely from solution. Tartar emetic succeeds better; but the precipitate has a variable composition, and also removes the alkaloids in part. By washing these out

from the precipitated compound of oxide of antimony and tannin formed in an acetic-acid solution, and subsequently determining the amount of antimony in the precipitate, I estimated the amount of the peculiar tannin in the juice at 2·35 per cent. No verification, however, could be made, as it is scarcely possible to procure this peculiar tannin in a pure state, from its excessive proneness to oxidation.

The tannates of the alkaloids appear quite similar, when prepared artificially, to those found in the bark. When obtained as mentioned above from the barks, they are readily decomposed by mineral acids. They oxidize very rapidly, and form red substances, which, when moist, are decomposed with moderate ease by acids with separation of cinchona-red, and very readily, but not quite completely, by alkalies, which dissolve the cinchona-red and leave the alkaloids in a coloured state. If the tannates are heated to 100° and become quite dry, they occasionally form a red compound, which resists the action of dilute mineral acids even on boiling. This is a fruitful cause of inaccurate bark analysis, when the bark has been long dried in a water-bath previously. In such cases it is better to treat the bark with alcoholic potash, as recommended by De Vry (Pharm. S. Trans. vol. vi. p. 50).

Order in which the Alkaloids are formed in the living Tissues of the Bark.

Very much light has been obtained on the order in which the natural formation of the cinchona alkaloids proceeds, by making periodic examinations of rapidly forming bark-tissues.

When cinchona-bark, not exposed to the direct rays of the sun, is carefully removed in strips without injury to the delicate cambium-cells underneath, such bark is rapidly renewed by natural processes. The cambium immediately thickens, so that after a little more than a week a thin cellular layer of a light green colour has covered the wound. On the surface of this, light brown granulations are found which spread and thicken. For many months only parenchymatous cells are to be seen; but subsequently short woody cells are found, which lengthen and increase in number and thus form a liber. The so-called resin-cells do not appear, nor have I observed any laticiferous vessels in this renewed bark. A true cork is, however, formed.

Bark thus rapidly renewing has been analyzed at various intervals of the process. The following series is one of many; but I quote it here because it possesses an additional element of precision in being a set of experiments performed on the trunk-bark of the same tree of *C. succirubra*.

	After four months.	After six months.	After ten months.	After fifteen months.	After seventeen months.
Total alkaloids	4·30	5·30	5·03	6·00	5·47
Quinine	*4·20*	*4·40*	*4·02*	*3·03*	*2·65*
Cinchonidine and cinchonine	0·10	0·90	1·01	*2·97*	*1·82*
Crystallized sulphate quinine obtained	0·25	1·56	*2·20*	*2·30*	*3·12*

These analyses lead to some conclusions of importance. The alkaloid first found in the freshly forming tissues is *quinine*, although the tree is of a kind producing mainly cinchonidine. As first formed, however, the quinine refuses to form crystalline sulphates; it is "uncrystallizable quinine." After two months more growth a third part of this quinine has obtained the ability to form crystallizable salts, and a small amount of cinchonidine and cinchonine are formed, both of which are recognized by their crystalline form under the microscope. These latter constantly increase; and in the above and other similar experiments the increase seems made at the expense of the crystallizable quinine, which simultaneously diminishes. The order of formation in point of time is thus shown to be, 1st, uncrystallizable quinine; 2nd, crystallizable quinine; 3rd, cinchonidine and cinchonine.

Furthermore, the very local character of these peculiar changes seems to render it highly probable that the alkaloids are really formed *in situ* in the very tissues in which they are found,—that is, the very cells which contain them also have formed them. It has been shown above how very minute an amount of alkaloids the leaves, wood, and twigs contain. The quinine in the leaves, small though it be, forms a crystalline sulphate. The alkaloid that is found in the freshly forming bark refuses to crystallize; it is therefore scarcely credible that it owes its origin to the distant leaves (according to the prevalent notion that all the plant principles are formed therein). If this could be conceived to be the case, it would at least be expected that the crystalline principles would appear first. Also for seventeen months after its formation the newly made bark is found to differ in composition from the rest of the bark of the tree by containing much quinine and little cinchonidine, precisely the reverse to what is contained in the older portions. The conclusion thus appears to be inevitable that the alkaloids are formed *in situ*. I find that Mr. HOWARD* on other grounds also inclines to the same conclusion.

The opinions as to the actual seat of the alkaloids in cinchona-bark have been divided. WADDELL, WIGAND, and others have stated on deductive grounds that they are mainly contained in the liber; HOWARD, on the other hand, by several direct experiments made with various South-American barks, found the largest amount in the external cellular portions ('Nuova Quinologia' of Pavon, Appendix). By adopting HOWARD's method of actually separating the liber, and making an analysis both of it and the remaining cellular portion, I was enabled abundantly to corroborate his results. The following analyses illustrate mine:—

	1st Series.		2nd Series.	
	Liber.	Cellular portion.	Liber.	Cellular portion.
Total alkaloids	5·94	7·98	6·85	8·00
Quinine	0·70	2·25	0·85	3·25
Cinchonidine and cinchonine	5·24	5·74	6·00	4·75
Sulphate quinine obtained	......	......	0·90	2·80
Sulphate cinchonidine obtained	......	......	4·10	4·20

* Quinology of the East-Indian Plantations, p. 16.

The above was made with young bark whose liber contained parenchymatous as well as woody cells. In older bark the difference would be still more marked. The fact of the quinine being contained mainly in the exterior cellular tissue is in harmony with its occurrence in newly forming structures, as detailed above.

I have never yet succeeded, although many attempts have been made, in converting quinine into cinchonidine, or *vice versâ*. Nevertheless many phenomena in the living plant point to the conclusion that a real change of quinine into cinchonidine occurs therein. The order of the appearance of the alkaloids in renewing bark, the comparative absence of quinine in the highly organized liber, the changes produced by sunshine and heat, all point to such conversion as a legitimate hypothesis; it is one that is not contradicted by a single fact, and is in harmony with observations made with very diverse species of cinchona, and with the chemical similarity in the behaviour and properties of the two substances.

Influence of Sunlight and Heat on growing Cinchona-bark.

PASTEUR* first pointed out that quinine, cinchonine, &c. are converted, under the influence of a high temperature, into the isomeric uncrystallizable alkaloids, quinicine and cinchonicine. He also showed that the same change was effected by exposing the salts of the alkaloids to sunshine. On these grounds he asserted that the plan of drying cinchona-bark in the sunshine, as practised in South America, is injurious to its quality.

Many experiments made during the last two years have amply corroborated these assertions. The purest and whitest alkaloids I have been able to prepare become coloured brown when exposed to the Indian sunshine, and the change is still more rapid when exposed in the form of salts. All the four alkaloids are affected in the same manner, even when sealed *in vacuo*, but the change is more readily effected in the case of the salts of quinine. In the solid state the change is very superficial; and in cases where the uncrystallizable alkaloids are formed more abundantly, the amount produced is only a small percentage of the whole, even when the insolation has lasted weeks. Pure quinine sulphate is especially sensitive to light, but becomes far less so when a small amount of the corresponding salt of cinchonidine is present, as frequently is the case in the commercial salt.

In the course of a long series of experiments made to determine the best practical method of drying the bark for export, the effects of heat and sunshine were abundantly manifest. The change produced by sunshine on the bark requires some time to become very perceptible. It is thoroughly apparent when the drying of the fresh bark is checked by exposing it for a fortnight in a box covered with glass; a decrease in the yield of the crystalline sulphates then becomes clearly evident, and amounts to from 0·8 to 1·1 per cent. of the dried bark. If the bark be finely shred (to increase the surface of action) and exposed for three days to the cloudless Indian sun, the alkaloids also lose the power of crystallization to the amount of 0·6 per cent. In both cases I have never

* Comptes Rend. vol. xxxvii. p. 110.

succeeded in obtaining exactly the same weight of free alkaloids. After exposure the loss varies greatly, from 0·4 to 1 per cent. According to Pasteur there should be no loss of weight when the crystallizable alkaloids are converted into their isomeric crystalline forms. Direct trials showed that this loss is connected with the effect of sunshine; for when the bark is dried at 50° C., or *in vacuo* over sulphuric acid, no such loss occurs.

No alteration in the relative amounts of quinine and cinchonidine could even be detected as the effect of insolation or artificial heat.

The damaging effect of light is not practically of great importance in the commercial preparation of the bark. The dry air and fierce sun of the Neilgherries with care may be made to dry the green bark so rapidly that sufficient time does not elapse for any appreciable deterioration of quality to be effected; nevertheless the sagacious dictum of Pasteur must be held to be true.

A high temperature, or even a steam-heat, is prejudicial to the quality of cinchona-bark, for reasons before mentioned. The former converts part or the whole of the crystallizable alkaloids into their uncrystallizable isomers; the latter often causes a compound of cinchona-red to be formed with the alkaloids, which is decomposed with difficulty by acids.

The following series of analyses, though conducted with special reference to economical results and the methods of commercial manufacture, aptly illustrate the foregoing principles. In all cases the finely shred bark was dried by the several methods; the fine division exaggerates the difference of result.

Red Bark.

	Fresh bark.	Dried at 17° C. *in vacuo.*	Dried at steam-heat.	Dried in current of air at 35°–45° C.	Dried in sunshine.	Dried in sunshine slowly.
Total alkaloids	6·89	6·67	5·92	6·64	5·90	5·85
Crystalline sulphates obtained .	6·12	5·98	5·31	5·89	5·28	4·78
Comparative amounts of acid required to neutralize the alkaloids	13·2	13·1	11·8	13·1	12·0	11·6

Mixed Crown Barks.

	Fresh bark.	Dried at 17° C. *in vacuo.*	Dried at steam-heat.	Dried in current of air at 35°–45° C.	Dried in sunshine.	Dried in sunshine slowly.
Total alkaloids	3·93	3·92	3·52	3·85	3·63	3·04
Crystalline sulphates obtained ...	3·67	3·52	3·01	3·56	3·11	2·88
Comparative amounts of acid required	7·2	7·0	6·6	6·8	6·7	5·9

The effects of the sun's rays on cinchona-bark, when separated from the tree, being

thus marked, it is to be expected that their action on the growing bark produces changes of a similar character. Experiment shows that this is the case to a great extent.

A method has been introduced into cinchona-cultivation by Mr. McIvor, Superintendent of the Government Neilgherry Plantations, which, when applied to young trees, produces some remarkable results. It consists in covering the bark of the trunk with a coating of moss. After the expiration of a year or eighteen months, the bark is found to be greatly improved in quality, to have become rather thicker, and to contain a larger general yield of alkaloids as well as an augmented proportion of quinine. In illustration of this I may quote the following analyses of the barks of trees of *C. succirubra* that have been thus treated:—

	No. 1. Bark of *C. succirubra* sixteen months under moss.	No. 2. Bark of same trees renewed under moss sixteen months.	No. 3. Original bark.
Total alkaloids	10·72	8·22	6·74 per cent.
Quinine	4·31	5·14	2·40 „
Cinchonine and cinchonidine	6·41	3·08	4·34 „
Total sulphates obtained crystallized	9·27	4·67	6·06 „
Sulphate of quinine	4·02	3·87	2·21 „
„ cinchonidine	5·25	0·80	3·85 „

From certain observations I had made on the difference in the barks of trees grown in shade and sunshine, and the known changes produced by the latter both on the alkaloids and barks, I was led to believe that the singular improvement in quality produced by mossing the bark was the effect of shielding it from the prejudicial action of the sun's rays, whether acting directly or indirectly. In order to test the truth of this by experiment, I covered the trunk-bark of two trees of *C. succirubra*, one with a shield of tin plate, and the other with a double fold of black alpaca cloth. The effect of this would be to keep the bark in darkness, while the access of air was not impeded. After ten months' protection analyses were made, and the following Table expresses the results.

	Tree covered with tin plates.		Tree covered with black cloth.		
	Original bark.	Bark after ten months' protection.	Original bark.	Bark after six months' protection.	Bark after ten months' protection.
Total alkaloids	5·29	8·10	5·04	6·91	7·92
Quinine	2·16	1·65	2·26	2·03	2·34
Cinchonidine and cinchonine	3·13	6·45	2·78	4·88	5·58

The protection from light has thus, in the otherwise dissimilar instances, had the *effect of greatly increasing the amount of alkaloids.* It is remarkable, however, that the quinine has not shared in the increase, a consequence I was not prepared to find,

and for which I can only partially account. An experiment made in the warmer climate of Wynaad, in which the bark was shielded with thick coarse woollen cloth, produced similar results to the above, in which an increase of quinine was also obtained. Hence it seems demonstrated by these experiments that, even while the bark is growing, there is a constant deterioration and waste going on by the action of the sun's rays, which is prevented by removing the cause. Whether it is the light of the sunshine which produces the effect, or the comparatively high temperature produced in the outer cellular portions where the alkaloids are mainly situated, is a point still to be determined.

A study of the differences in the amounts of the respective alkaloids found in the barks grown at different elevations appears to throw some light on the effects of temperature. A series of trees was chosen of as nearly the same age as possible, and growing on soil of the same character, but growing at very various elevations, and hence under climates of very various annual mean temperatures. The trunk-bark of these trees was collected and analyzed. This was done at a time of the year when the yield of alkaloids is nearly at a minimum; there is, however, no reason for supposing this circumstance would alter the comparative value of the results. The following tabular statements express the results:—

Red Bark.

	I.	II.	III.	IV.	V.	VI.	VII.
	7770 ft.	7660 ft.	7450 ft.	6560 ft.	5450 ft.	3500 ft.	2300 ft.
Total alkaloids	2·38	2·03	4·21	5·75	3·8	4·10	4·1
Quinine	0·45	0·33	0·70	2·50	1·2	0·86	0·6
Cinchonidine and cinchonine	1·93	1·70	3·51	3·25	2·6	3·24	3·5
Crystalline sulphates obtained	1·75	1·60	3·20	4·20	3·1	3·58	3·4

Crown Bark.

	I.	II.	III.	IV.
	8000 ft.	7770 ft.	6007 ft.	5450 ft.
Total alkaloids	3·42	3·61	3·10	2·68
Quinine	2·10	1·83	0·75	0·45
Cinchonidine and cinchonine	1·32	1·78	2·35	2·23
Crystalline sulphates obtained	3·20	3·30	2·60	1·80

The analyses in the first Table were made with the product of trees which, on account of the comparatively cold climate, were stunted and unhealthy in appearance, and thus are scarcely fit subjects for the inquiry. Analyses IV., V., VI., VII. agree in showing that from a certain point an increase of mean temperature has the effect in diminishing the amount of quinine, and to a less marked extent the total alkaloids. The same effect

is also evident in the more hardy *C. officinalis.* Many other analyses of cinchona-bark grown at low elevations have abundantly shown that a high mean temperature is adverse to the production of quinine. These results, taken in conjunction with those obtained from shielded bark, seem to demonstrate that sunlight degrades the alkaloids generally, while heat mainly diminishes the amount of quinine.

It would naturally be expected that, as the barks become thickened with age, the sunlight would produce a less marked effect; this is exactly what occurs. Up to the present time the amount of alkaloids has annually increased in the bark. In the *C. succirubra* this increase has been determined as carefully as the conditions permit, and the increments during the sixth and seventh years have been 0·75 and 0·5 per cent. respectively. That some part of this increase is due to the inability of sunlight to affect the lower-seated tissues, appears certain from the fact that mossing or otherwise shielding the bark has a less effect in increasing the amount of alkaloids in the older bark than in the thin bark of younger trees. There is every reason to think that the alkaloids will increase by annually diminishing amounts until the amount formed is only equivalent to the destructive influences at work. The growth of the liber and the formation of woody cells, for the reasons given above, is adverse to the increase of yield of alkaloids.

In connexion with this subject it should be mentioned that trees of *C. officinalis*, growing in dry sunny spots, almost invariably produce bark whose main alkaloid constituent is cinchonidine. The trees grown in shade produce more quinine. The bark of trees of rapid and vigorous growth of the same species invariably contain a large amount of alkaloid, and an unusually large proportion of quinine. The latter case can be readily explained by the foregoing principles. The bark of such would be thicker, would contain much parenchymatous tissue, and would, by its more rapid formation, have been exposed for a less time to the deteriorating influences of the sun. It is very remarkable that trees of vigorous growth should yield not only more bark, but also bark in which one hundred parts should contain double the amount of alkaloids.

All cinchona-barks contain small amounts of ammonia. As it is not improbable that ammonia is a stage in the formation of alkaloid, a set of experiments has been commenced as to the action of ammoniacal manure on the yield of alkaloids in the bark; they are, however, not sufficiently advanced to be here detailed.

The Alkaloids considered as substitutes for Mineral Bases.

A speculative opinion has long been held concerning the functions of the vegeto-alkaloids in the plants producing them, whose importance is so great that its practical verification or contradiction is a primary necessity to a scientific theory of the formation of the alkaloids. The hypothesis alluded to is a corollary to the "mineral theory" of LIEBIG, and has been indirectly enunciated by that chemist himself (*Chemistry in its Applications to Agriculture*, p. 187).

According to this hypothesis, the alkaloids are substitutes for certain of the mineral bases which are the constituents of all plants, and which constitute the larger portion of

the ash obtained when the plants are burnt. The alkaloids possess most of the general properties of these mineral bases; they completely neutralize acids, and as they comport themselves in a manner chemically so similar, it is thought that their functions are identical, and that certain plants are enabled to produce in their own economy actual substitutes for these necessary mineral bases, which are frequently but meagrely furnished by the soil in which the plants grow.

The experiments of Puttfarcken (Pharm. Journ. vol. xi. p. 129) appear to corroborate this hypothesis. They were performed with the dry *Calisaya*-bark from South America, and appeared to show in the main that the bark yielded ash in inverse degree to the amount of organic bases.

The simplicity and beauty of this hypothesis has always rendered it, in the absence of more precise information, a favourite one. On my first acquaintance with living cinchona (*par excellence* the alkaloid-producing plant), there were several circumstances which seemed to show its truth; among these were the exceptionally large yield of alkaloid in the Neilgherry cinchonas, combined with the singular poverty of the soil in lime. These circumstances led me to institute a series of experiments, in which the amount of organic bases was compared with the amount of ash yielded by the same bark; this has been carried on with all the species of cinchona, and all the diversity of conditions I could devise. It is assumed as practically true that the amount of ash is proportional to the amount of mineral bases; and it has been ascertained by experiment that the amount of ammonia is nearly constant, and is too small to vitiate the results. The following Table expresses the results obtained:—

			Alkaloids, per cent.	Ash, per cent.
C. succirubra.				
C. succirubra	June	1867	5·05	2·51
,, from Coorg	July	,,	5·03	2·93
,, growing at 7770 feet	August	,,	2·38	7·80
,, ,, 7660 ,,	,,	,,	4·21	3·35
,, ,, 5450 ,,	September	,,	4·00	2·00
,, ,, 2300 ,,	November	,,	4·10	2·43
,, ,, 6900 ,,	March	1868	5·86	3·17
Mossed bark	,,	,,	10·72	2·25
Bark renewed under moss	,,	,,	8·22	2·20
,, ,, ,,	May	1869	9·27	2·52
Trunk-bark, 2 years old	March	1868	6·40	3·60
,, ,, 3 ,,	,,	,,	6·09	3·15
,, ,, 4 ,,	,,	,,	7·40	2·63
Mean ,, 7½ ,,	,,	1870	7·55	2·81
,, ,, 6½ ,,	,,	,,	6·23	2·93
,, ,, 5½ ,,	,,	,,	7·33	2·97
,, ,, 3½ ,,	,,	,,	7·61	3·09
,, ,, 2½ ,,	,,	,,	2·21	4·43
Mean branch-bark, 2½ years old	April	1868	1·91	4·71
,, ,, ,, 3½ ,,	,,	,,	4·31	3·13
,, ,, ,, 7½ ,,	,,	1870	3·20	4·92
,, ,, ,, 6½ ,,	,,	,,	2·63	5·00
,, ,, ,, 5½ ,,	,,	,,	3·07	3·64
,, ,, ,, 3½ ,,	,,	,,	4·96	3·42
,, ,, ,, 3½ ,,	,,	,,	0·81	5·03

TABLE (continued).

		Alkaloids, per cent.	Ash, per cent.
Liber of mossed red bark	March 1868	9·80	3·03
External bark of ditto	,, ,,	11·83	1·80
Liber of red bark	,, ,,	5·94	2·60
External bark of ditto	,, ,,	7·89	2·01
Liber 5½ years old	April ,,	7·10	3·10
External bark of ditto	,, ,,	8·00	2·38
Bark renewed naturally	March ,,	5·30	3·27
Bark shielded with tin plate	May ,,	7·60	2·38
Mean specimen of red bark	,, ,,	7·43	2·90
Bark from Darjeeling	,, ,,	6·85	2·70
Prize red bark	September ,,	8·10	2·10
Mossed ditto	,, ,,	5·40	1·92
Bark from east of Neilgherries	,, ,,	6·89	2·17
OTHER BARKS.			
C. officinalis	May 1867	3·72	8·77
,,	,, ,,	4·28	4·49
,,	July ,,	3·84	4·21
,,	September ,,	3·10	3·06
,, at 5500 ft.	,, ,,	2·68	2·90
,,	March 1868	3·22	2·52
,,	,, ,,	2·64	2·97
,, grown in shade	September 1869	3·40	3·89
,,	,, ,,	6·96	3·17
,,	,, ,,	3·77	4·17
,,	,, ,,	7·51	3·00
C. Pahudiana	May 1867	0·13	4·34
,, *nitida*	,, ,,	5·90	2·14
,, unknown name	,, ,,	3·96	4·58
,, *peruviana*	,, ,,	6·16	2·77
Leaves of *C. succirubra*	....	0·019	5·74
Wood ,, ,,	....	0·10	0·89
Bark of willow-trunk	....	0	4·47
,, ,, branch	....	0	5·37
,, *Cupressus lusitanica*, trunk	....	0	2·78
,, ,, ,, branch	....	0	7·72
,, *Rhodomyrtus*, trunk	....	0	3·99
,, ,, branches	....	0	5·19
var. "*lanceolata*," trunk-bark	....	5·60	4·54
,, ,, ,, ,,	....	11·40	3·34
,, ,, ,, ,,	....	11·70	2·62

The hypothesis mentioned above is not borne out by the numbers in the preceding Table.

The nature of the soil on which the trees were grown is throughout the whole very similar. As the great disturbance which might be supposed to occur from variations in the composition of the soil does not take place to any considerable extent under the conditions of the experiment, it is difficult to perceive any cause that can obscure the consequences of the hypothesis, supposing it to be true. But in order to test it still further, the bases in the ash were separately determined in a few cases in which the divergence was most marked, and the numbers obtained were compared with the amount of alka-

loid, but without at all diminishing the discrepancy. I am therefore compelled to believe either that the hypothesis is not true, or else that it is only true to a very partial extent; in fact so partially as to render it of no assistance in the scientific culture of cinchona.

In order to test the hypothesis in another manner, the soil round some young trees of *C. succirubra*, which was naturally singularly poor in lime, was made, by constant dressing with well-exposed hydrate of lime, almost into a calcareous soil. After two years and a half the bark was analyzed and its ash determined. It yielded 6·23 per cent. of total alkaloids and 2·91 per cent. of ash; in other words, the presence of abundance of a powerful base in the soil had affected neither the amounts of alkaloids nor of ash.

The crown barks offer a still further contradiction to the hypothesis. Trees growing close together may have nearly the same amount of ash and yet contain organic bases in nearly the proportion of 1 : 2.

The results of PUTTFARCKEN can be reconciled with mine. His analyses were performed with barks imported in the dry state from South America; among these would be both trunk- and branch-bark. The increased amounts of ash and small amounts of alkaloid in the latter would at first present the appearance of substitution, and would mislead the investigator. I found the same, and was for some time misled also; but a few experiments with the trunk- and branch-barks of various other trees showed me that the latter contain more mineral bases than the former, just as in cinchona, although no alkaloids are present. At the latter end of the foregoing Table some instances of this are adduced.

Furthermore, the fact that the greater part of the alkaloids exists in an insoluble state of combination in the cells of the bark, is itself somewhat adverse to the likelihood of the hypothesis being true; since it is hardly likely that in this situation they can be very active constituents of the plant.

II. *A Ninth Memoir on Quantics.* *By Professor* Cayley, *F.R.S.*

Received April 7,—Read May 19, 1870.

It was shown not long ago by Professor Gordan that the number of the irreducible covariants of a binary quantic of any order is finite (see his memoir "Beweis dass jede Covariante und Invariante einer binären Form eine ganze Function mit numerischen Coefficienten einer endlichen Anzahl solcher Formen ist," Crelle, t. 69 (1869), Memoir dated 8 June 1868), and in particular that for a binary quintic the number of irreducible covariants (including the quintic and the invariants) is $=23$, and that for a binary sextic the number is $=26$. From the theory given in my "Second Memoir on Quantics," Phil. Trans. 1856, I derived the conclusion, which, as it now appears, was erroneous, that for a binary quintic the number of irreducible covariants was infinite. The theory requires, in fact, a modification, by reason that certain linear relations, which I had assumed to be independent, are really not independent, but, on the contrary, linearly connected together: the interconnexion in question does not occur in regard to the quadric, cubic, or quartic; and for these cases respectively the theory is true as it stands; for the quintic the interconnexion first presents itself in regard to the degree 8 in the coefficients and order 14 in the variables, viz. the theory gives correctly the number of covariants of any degree not exceeding 7, and also those of the degree 8 and order less than 14; but for the order 14 the theory as it stands gives a non-existent irreducible covariant $(a, \ldots)^8(x, y)^{14}$, viz. we have, according to the theory, $5=(10-6)+1$, that is, of the form in question there are 10 composite covariants connected by 6 syzygies, and therefore equivalent to $10-6$, $=4$ asyzygetic covariants; but the number of asyzygetic covariants being $=5$, there is left, according to the theory, 1 irreducible covariant of the form in question. The fact is that the 6 syzygies being interconnected and equivalent to 5 independent syzygies only, the composite covariants are equivalent to $10-5$, $=5$, the full number of the asyzygetic covariants. And similarly the theory as it stands gives a non-existent irreducible covariant $(a, \ldots)^8(x, y)^{20}$. The theory being thus in error, by reason that it omits to take account of the interconnexion of the syzygies, there is no difficulty in conceiving that the effect is the introduction of an infinite series of non-existent irreducible covariants, which, when the error is corrected, will disappear, and there will be left only a finite series of irreducible covariants.

Although I am not able to make this correction in a general manner so as to show from the theory that the number of the irreducible covariants is finite, and so to present the theory in a complete form, it nevertheless appears that the theory can be made to accord with the facts; and I reproduce the theory, as well to show that this is so as to

exhibit certain new formulæ which appear to me to place the theory in its true light. I remark that although I have in my Second Memoir considered the question of finding the number of irreducible covariants of a given degree θ in the coefficients but of any order whatever in the variables, the better course is to separate these according to their order in the variables, and so consider the question of finding the number of the irreducible covariants of a given degree θ in the coefficients, and of a given order μ in the variables. (This is, of course, what has to be done for the enumeration of the irreducible covariants of a given quantic; and what is done completely for the quadric, the cubic, and the quartic, and for the quintic up to the degree 6 in my Eighth Memoir, Phil. Trans. 1867.) The new formulæ exhibit this separation; thus (Second Memoir, No. 49), writing a instead of x, we have for the quadric the expression $\frac{1}{(1-a)(1-a^2)}$, showing that we have irreducible covariants of the degrees 1 and 2 respectively, viz. the quadric itself and the discriminant: the new expression is $\frac{1}{(1-ax^2)(1-a^2)}$, showing that the covariants in question are of the actual forms $(a, \ldots \between x, y)^2$ and $(a, \ldots)^2$ respectively. Similarly for the cubic, instead of the expression No. 55, $\frac{1-a^6}{(1-a)(1-a^2)(1-a^3)(1-a^4)}$, we have $\frac{1-a^6x^6}{(1-ax^3)(1-a^2x^2)(1-a^3x^3)(1-a^4)}$, exhibiting the irreducible covariants of the forms $(a, \ldots \between x, y)^3$, $(a, \ldots)^2(x, y)^2$, $(a \ldots)^3(x, y)^3$, and $(a, \ldots)^4$, connected by a syzygy of the form $(a, \ldots)^6(x, y)^6$; and the like for quantics of a higher order.

In the present Ninth Memoir I give the last-mentioned formulæ; I carry on the theory of the quintic, extending the Table No. 82 of the Eighth Memoir up to the degree 8, calculating all the syzygies, and thus establishing the interconnexions in virtue of which it appears that there are really no irreducible covariants of the forms $(a, \ldots)^8(x, y)^{14}$, and $(a, \ldots{}^8 \between x, y)^{20}$. I reproduce in part GORDAN's theory so far as it applies to the quintic, and I give the expressions of such of the 23 covariants as are not given in my former memoirs; these last were calculated for me by Mr. W. BARRETT DAVIS, by the aid of a grant from the Donation Fund at the disposal of the Royal Society. The paragraphs of the present memoir are numbered consecutively with those of the former memoirs on Quantics.

Article Nos. 328 to 332.—*Reproduction of my original Theory as to the Number of the Irreducible Covariants.*

328. I reproduce to some extent the considerations by which, in my Second Memoir on Quantics, I endeavoured to obtain the number of the irreducible covariants of a given binary quantic $(a, b, \ldots \between x, y)^n$.

Considering in the first instance the covariants as functions of the coefficients $(a, b, c \ldots)$, without regarding the variables (x, y), and attending only to the following properties—1°, a covariant is a rational and integral homogeneous function of the coefficients;

2°, if P, Q, R, ... are covariants, any rational and integral function F(P, Q, R ...), homogeneous in regard to the coefficients, is also a covariant,—we say that the covariants X, Y, ... of the same degree in regard to the coefficients, and not connected by any identical equation $\alpha X+\beta Y\ldots=0$ (where α, β, ... are quantities independent of coefficients $(a, b, c, \ldots)$), are *asyzygetic* covariants, and that a covariant not expressible as a rational and integral function of covariants of lower degrees is an *irreducible* covariant; and it is assumed that we know the number of the asyzygetic covariants of the degrees 1, 2, 3....; say, these are A_1, A_2, A_3, ..., or, what is the same thing, that the number of the asyzygetic covariants of the degree θ, or form $(a, b, \ldots)^\theta$, is equal to the coefficient of a^θ in a given function

$$\phi(a)=1+A_1a+A_2a^2\ldots+A_\theta a^\theta+\ldots,$$

where I have purposely written a, as a representative of the coefficients $(a, b, c, \ldots)$, in place of the x of my Second Memoir.

329. The theory was, that determining α_1, α_2, ... by the conditions

$$\begin{aligned}&A_1=\alpha_1,\\&A_2=\tfrac{1}{2}\alpha_1(\alpha_1+1)+\alpha_2,\\&A_3=\tfrac{1}{6}\alpha_1(\alpha_1+1)(\alpha_1+2)+\alpha_1\alpha_2+\alpha_3,\\&\quad\vdots\end{aligned}$$

that is, throwing

$$1+A_1a+A_2a^2+A_3a^3+\ldots$$

into the form

$$(1-a)^{-\alpha_1}(1-a^2)^{-\alpha_2}(1-a^3)^{-\alpha_3}\ldots,$$

the index α_r would express the number of irreducible covariants of the degree r *less* the number of the (irreducible) linear relations, or syzygies, between the composite or non-irreducible covariants of the same degree. Thus $A_1=\alpha_1$, there would be α_1 covariants of the degree 1*; these give rise to $\frac{1}{2}\alpha_1(\alpha_1+1)$ composite covariants of the degree 2; or, assuming that these are connected by k_2 syzygies, the number of asyzygetic composite covariants of the degree 2 would be $\frac{1}{2}\alpha_1(\alpha_1+1)-k_2$; and thence there would be $A_2-\frac{1}{2}\alpha_1(\alpha_1+1)+k_2$, that is, α_2+k_2 irreducible covariants of the same degree; so that (irreducible invariants *less* syzygies) $(\alpha_2+k_2)-k_2$ is $=\alpha_2$.

330. The k_2 syzygies are here irreducible syzygies; for, calling P, Q, R, ... the covariants of the degree 1, there is no identical relations between the terms P^2, Q^2, PQ, ...: imagine for a moment that we could have l_2 such identical relations (viz. this might very well be the case if instead of the $\frac{1}{2}\alpha_1(\alpha_1+1)$ functions P^2, Q^2, PQ, ..., we were dealing with the same number of other quadric functions of these quantities), that is, relations not establishing any relation between P^2, Q^2, PQ, ..., and besides these k_2 non-identical relations as above; then the number of irreducible invariants would be $\alpha_2+k_2+l_2$, and the number of irreducible syzygies being as before k_2, the difference would be not α_2

* For the case of covariants, α_1 is of course $=1$; but in the investigation the term covariant properly stands for any function satisfying the conditions 1° and 2°.

but $\alpha_2+l'_2$. The l_2 identical relations are here relations between composite covariants, and the effect (if any such relation could subsist) would, it appears, be to increase α_2; between syzygies such identical relations do actually exist, and the effect is contrariwise to diminish the α; we may, for instance, for the degree s have irreducible covariants *less* irreducible syzygies $=\alpha_s-l_s$.

331. Assume for a moment that, for a given value of s, α_s is positive; but for the term l_s it would of course follow that there was for the degree in question a certain number of irreducible covariants; and it was in this manner that I was led to infer that the number of the covariants of a quintic was infinite—viz. the transformed expression for the number of asyzygetic covariants is

$$=\text{coeff. } a^\theta \text{ in } (1-a^4)^{-1}(1-a^8)^{-3}(1-a^{12})^{-6}(1-a^{14})^{-4}\ldots,$$

a product which does not terminate, and as to which it is also assumed that the series of negative indices does not terminate.

332. The principle is the same, but the discussion as to the number of the irreducible covariants becomes more precise, if we attend to the covariants as involving not only the coefficients $(a, b, \ldots)$ but also the variables (x, y); we have then to consider the covariants of the form $(a, b, \ldots)^\theta(x, y)^\mu$, or, say, of the form $a^\theta x^\mu$ (degree θ and order μ), and the number of the asyzygetic covariants of this form is given as the coefficient of $a^\theta x^\mu$ in a given function of (a, x), (I write a instead of the z of my Second Memoir in the formulæ which contain x and z): by taking account of the composite covariants and syzygies, we successively determine, from the given number of asyzygetic covariants for each value of θ and μ, the number of the irreducible covariants for the same values of θ and μ. This is, in fact, done for the quintic in my Eighth Memoir up to the covariants and syzygies of the degree 6. But before resuming the discussion for the quintic, I will consider the preceding cases of the quadric, the cubic, and the quartic.

Article Nos. 333 to 336.—New formulæ for the number of Asyzygetic Covariants.

333. For the quadric $(a, b, c)(x, y)^2$, the number of asyzygetic covariants $a^\theta x^\mu$

$$= \text{coeff. } a^\theta x^{\theta-\frac{1}{2}\mu} \text{ in } \frac{1-x}{(1-a)(1-ax)(1-ax^2)},$$

(see Second Memoir, No. 35, observing that q is there $=\theta-\frac{1}{2}\mu$, and that the subtraction of successive coefficients is effected by means of the factor $1-x$ in the numerator. See also Eighth Memoir, No. 251, where a like form is used for the quintic). Writing ax^2 for a, and $\frac{1}{x^2}$ for x, this is

$$= \text{coeff. } a^\theta x^\mu \text{ in } \frac{1-\frac{1}{x^2}}{(1-ax^2)(1-a)\left(1-\frac{a}{x^2}\right)}.$$

The development is

$$\begin{array}{ll|l} 1 & -\frac{1}{x^2} & 1 \\ +ax^2 & & +a\left(\frac{1}{x^2}\right) \\ +a^2(x^4+1) & & +a^2\left(\frac{1}{x^4}+1\right) \\ +a^3(x^6+x^2) & & +a^3\left(\frac{1}{x^6}+\frac{1}{x^2}\right) \\ +a^4(x^8+x^4+1) & & +a^4\left(\frac{1}{x^8}+\frac{1}{x^4}+1\right) \\ \vdots & & \vdots \end{array}$$

which is

$$=\mathrm{A}(x)-\frac{1}{x^2}\mathrm{A}\left(\frac{1}{x}\right),$$

where

$$\mathrm{A}(x)=\frac{1}{(1-ax^2)(1-a^2)};$$

and, since $\frac{1}{x^2}\mathrm{A}\left(\frac{1}{x}\right)$ contains only negative powers, the required number is

$$=\text{coeff. } a^\theta x^\mu \text{ in } \frac{1}{(1-ax^2)(1-a^2)},$$

indicating that the covariants are powers and products of (ax^2 and a^2), the quadric itself, and the discriminant. Compare Second Memoir, No. 49, according to which, writing therein a for x, the number of asyzygetic covariants is

$$=\text{coeff. } a^\theta \text{ in } \frac{1}{(1-a)(1-a^2)}.$$

334. For the cubic $(a, b, c, d\between x, y)^3$ the number of asyzygetic covariants $a^\theta x^\mu$ is

$$=\text{coeff. } a^\theta x^{\theta-\frac{1}{2}\mu} \text{ in } \frac{1-x}{(1-a)(1-ax)(1-ax^2)(1-ax^3)};$$

or transforming as before, this is

$$=\text{coeff. } a^\theta x^\mu \text{ in } \frac{1-\frac{1}{x^2}}{(1-ax^3)(1-ax)(1-ax^{-1})(1-ax^{-3})}:$$

the function is here

$$\mathrm{A}(x)-\frac{1}{x^2}\mathrm{A}\left(\frac{1}{x}\right),$$

where

$$\mathrm{A}(x)=\frac{1-a^6x^6}{(1-ax^3)(1-a^2x^2)(1-a^3x^3)(1-a^4)}$$

(that this is so may be easily verified); and since the second term contains only negative

powers, the required number is $=$coeff. $a^\theta x^\mu$ in $\mathrm{A}(x)$. The formula, in fact, indicates that the covariants are made up of $(ax^3, a^2x^2, a^3x^3, a^4)$, the cubic itself, the Hessian, the cubicovariant, and the discriminant, these being connected by a syzygy (a^6x^6) of the degree 6 and order 6. Compare Second Memoir, No. 50, according to which the number of covariants of degree θ is

$$=\text{coeff. } a^\theta \text{ in } \frac{1-a^6}{(1-a)(1-a^2)(1-a^3)(1-a^4)}.$$

335. For the quartic $(a, b, c, d, e \between x, y)^4$ the number of asyzygetic covariants $a^\theta x^\mu$ is

$$=\text{coeff. } a^\theta x^{\theta-\frac{1}{2}\mu} \text{ in } \frac{1-x}{(1-a)(1-ax)(1-ax^2)(1-ax^3)(1-ax^4)};$$

or transforming as before, this is

$$=\text{coeff. } a^\theta x^\mu \text{ in } \frac{1-x^{-2}}{(1-ax^4)(1-ax^2)(1-a)(1-ax^{-2})(1-ax^{-4})}:$$

the function is here

$$\mathrm{A}(x)-\frac{1}{x^2}\mathrm{A}\left(\frac{1}{x}\right),$$

where

$$\mathrm{A}(x)=\frac{1-a^6x^{12}}{(1-ax^4)(1-a^2x^4)(1-a^2)(1-a^3)(1-a^3x^6)};$$

and the second term containing only negative powers, the required number is $=$coeff. $a^\theta x^\mu$ in $\mathrm{A}(x)$. The formula indicates that the covariants are made up of $(ax^4, a^2x^4, a^2, a^3, a^3x^6)$, the quartic itself, the Hessian, the quadrinvariant, the cubinvariant, and the cubicovariant, these being connected by a syzygy (a^6x^{12}) of the degree 6 and order 12. Compare Second Memoir, No. 51, according to which the number of covariants of degree θ is

$$=\text{coeff. } a^\theta \text{ in } \frac{1-a^6}{(1-a)(1-a^2)^2(1-a^3)^2}.$$

336. For the quintic $(a, b, c, d, e, f \between x, y)^5$ the number of asyzygetic covariants $a^\theta x^\mu$ is

$$=\text{coeff. } a^\theta x^{\theta-\frac{1}{2}\mu} \text{ in } \frac{1-x}{(1-a)(1-ax)(1-ax^2)(1-ax^3)(1-ax^4)(1-ax^5)};$$

or transforming as before, this is

$$=\text{coeff. } a^\theta x^\mu \text{ in } \frac{1-x^{-2}}{(1-ax^5)(1-ax^3)(1-ax)(1-ax^{-1})(1-ax^{-3})(1-ax^{-5})}.$$

The developed expression is

$$\begin{array}{ll|l} 1 & -\frac{1}{x^2} & 1 \\ +ax^5 & & +ax^{-5} \\ +a^2(x^{10}+x^6+x^2) & & +a^2(x^{-10}+x^{-6}+x^{-2}) \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \end{array}$$

but here there is not any *finite* function $A(x)$ such that this development is

$$=A(x) \qquad\qquad -\frac{1}{x^2}A\left(\frac{1}{x}\right).$$

The numerical coefficients are of course the same as those in the development of the untransformed function; viz. they are the numbers given in the third column of Table No. 82 (Eighth Memoir), and also (carried further) in the third column of the following Table, No. 87. And we can, from the discussion of these coefficients, deduce the form of $A(x)$, viz. this is

$$=\frac{\begin{array}{r|r|r|r|l} 1-a^5x^{11} & 1-a^6x^{18} & 1-a^7x^{15} & (1-a^8x^{12})^3 & \ldots \\ & 14 & 13 & (10)^3 & \\ & 12 & 11 & (\ 8)^2 & \\ & 10 & (9)^3 & (\ 6)^2 & \\ & 8 & 7 & & \\ & 6 & & & \end{array}}{\begin{array}{r|r|r|r|r|r|r|r|r|l} 1-ax^5 & 1-a^2x^6 & 1-a^3x^9 & 1-a^4x^6 & 1-a^5x^7 & 1-a^6x^4 & 1-a^7x^3 & 1-a^8x^3 & 1-a^{18} & \ldots \\ & 2 & 5 & 4 & 3 & 2 & 1 & 0 & & \\ & & 3 & 0 & 1 & & & 20 & & \\ & & & & & & & 14 & & \end{array}}$$

where, for shortness, I have written $\begin{array}{c}1-a^2x^6\\ 2\end{array}$ to stand for $(1-a^2x^6)(1-a^2x^2)$, and so in other cases: moreover in the third column of the numerator the $(9)^3$ shows that the factor is $(1-a^7x^9)^3$, and so in other cases: this will be further explained presently. Compare herewith the form, Second Memoir, No. 52, viz. the number of asyzygetic covariants of the degree θ is

$$=\text{coeff. } a^\theta \text{ in } (1-a)^{-1}(1-a^2)^{-2}(1-a^3)^{-3}(1-a^4)^{-3}(1-a^5)^{-2}(1-a^6)^4(1-a^7)^5(1-a^8)^6\ldots.$$

each index being, it will be observed, equal to the number of factors in the numerator, less the number of factors in the denominator, in the corresponding column of the new formula.

Article Nos. 337 to 346.—*The 23 Fundamental Covariants.*

337. GORDAN'S result is that the entire number of the irreducible covariants of the binary quintic is $=23$. I represent these by the letters A, B, C, . . ., W, identifying such of them as were given in my former Memoirs on Quantics with the Tables of these Memoirs, and the new ones, O, P, R, S, T, V, with the Tables Nos. 90, 91, 92, 93, 94, 95 of the present Memoir.

Table No. 87.—Identification of the 23 irreducible covariants of the binary quintic.

			Table No.
A	$(a, b, c, d, e, f \between x, y)^5$	f	13
$\mathrm{B} = \frac{1}{28800}(\mathrm{A}, \mathrm{A})^4$	$(\quad)^2(\quad)^2$	$\iota = (ff)^4$	14
$\mathrm{C} = \frac{1}{800}(\mathrm{A}, \mathrm{A})^2$	$(\quad)^2(\quad)^6$	$\phi = (ff)^2$	15
$\mathrm{D} = -\frac{1}{3}(\mathrm{A}, \mathrm{B})^2$	$(\quad)^3(\quad)^3$	$j = (f\iota)^2$	16
$\mathrm{E} = \frac{1}{5}(\mathrm{A}, \mathrm{B})$	$(\quad)^3(\quad)^5$	$(f\iota)$	17
$\mathrm{F} = \frac{1}{15}(\mathrm{A}, \mathrm{C})$	$(\quad)^3(\quad)^9$	$(f\phi)$	18
$\mathrm{G} = -\frac{1}{2}(\mathrm{B}, \mathrm{B})^2$	$(\quad)^4(\quad)^0$	$(\iota\iota)^2$	19
$\mathrm{H} = -\frac{1}{5}(\mathrm{B}, \mathrm{C})^2 + \frac{2}{5}\mathrm{B}^2$	$(\quad)^4(\quad)^4$	$p = (\phi\iota)^2$	20
$\mathrm{I} = -\frac{1}{6}(\mathrm{B}, \mathrm{C})$	$(\quad)^4(\quad)^6$	$(\phi\iota)$	21
$\mathrm{J} = -\frac{1}{4}(\mathrm{B}, \mathrm{D})^2$	$(\quad)^5(\quad)^1$	$\alpha = (j\iota)^2$	22
$\mathrm{K} = -(\mathrm{B}, \mathrm{D})$	$(\quad)^5(\quad)^3$	$(j\iota)$	23
$\mathrm{L} = -\frac{1}{20}(\mathrm{A}, \mathrm{H}) + \frac{1}{2}\mathrm{BE}$	$(\quad)^5(\quad)^7$	(fp)	24
$\mathrm{M} = -\frac{1}{48}(\mathrm{B}, \mathrm{H})^2 - \frac{1}{6}\mathrm{BG}$	$(\quad)^6(\quad)^2$	$\tau = (p\iota)^2$	83
$\mathrm{N} = \frac{1}{4}(\mathrm{B}, \mathrm{H})$	$(\quad)^6(\quad)^4$	$(p\iota)$	84
$\mathrm{O} = -(\mathrm{B}, \mathrm{J})$	$(\quad)^7(\quad)^1$	$(\iota\alpha)$	*90
$\mathrm{P} = -\frac{1}{5}(\mathrm{A}, \mathrm{M}) - \mathrm{BK}$	$(\quad)^7(\quad)^5$	$(f\tau)$	*91
$\mathrm{Q} = \frac{1}{2}(\mathrm{B}, \mathrm{M})^2$	$(\quad)^8(\quad)^0$	$(\iota\tau)^2$	25
$\mathrm{R} = -\frac{1}{2}(\mathrm{B}, \mathrm{M})$	$(\quad)^8(\quad)^2$	$(\tau\iota)$	*92
$\mathrm{S} = -96(\mathrm{D}, \mathrm{M}) + 16\mathrm{BO} - 7\mathrm{GK}$	$(\quad)^9(\quad)^3$	$(j\tau)$	*93
$\mathrm{T} = -(\mathrm{J}, \mathrm{M})$	$(\quad)^{11}(\quad)^1$	$\gamma = (\tau\alpha)$	*94
$\mathrm{U} = \frac{1}{18}(\mathrm{J}, \mathrm{O}) + \frac{1}{9}\mathrm{GQ}$	$(\quad)^{12}(\quad)^0$	$((\iota\alpha), \alpha)$	29
$\mathrm{V} = -(\mathrm{B}, \mathrm{T})$	$(\quad)^{13}(\quad)^1$	$(\iota\gamma)$	*95
$\mathrm{W} = -\frac{1}{6}(\mathrm{O}, \mathrm{T})$	$(\quad)^{18}(\quad)^0$	$((\iota\alpha), \gamma)$	29A

338. The Table exhibits the generation of the several covariants; viz. (A, B) denotes $\partial_x\mathrm{A}\,.\,\partial_y\mathrm{B} - \partial_y\mathrm{A}\,.\,_x\mathrm{B}$, $(\mathrm{A}, \mathrm{B})^2$ denotes $\partial_x^2\mathrm{A}\,.\,\partial_y^2\mathrm{B} - 2\partial_x\partial_y\mathrm{A}\,.\,\partial_x\partial_y\mathrm{B} + \partial_y^2\mathrm{A}\,.\,\partial_x^2\mathrm{B}$, &c. (see *post*, No. 348). The column f, $\iota = (ff)^4$, &c. shows GORDAN's notation, and the generation of his 23 forms ($(ff)^4$ written as with him for $(f, f)^4$, &c.): it will be observed that the forms are not identical; if the calculations had been made *de novo*, I should have adopted his values, simply omitting numerical factors of the several forms (thus every term of ι, $=(ff)^4$ contains the factor $2\,.\,(120)^2$, $=28800$): of course the presence of these numerical factors renders the f, ι, ϕ, &c. as they stand inconvenient for the expression of results; and the numerical fixation of the values was no part of GORDAN's object. But by reason of the existing Tables the change of notation is in fact more

than this; thus H instead of being a submultiple of $(B, C)^2$, that is, of p, is in fact $=-\frac{1}{5}(B,C)^2+\frac{2}{5}B^2$; and so in other cases. If the occasion for it arises, there is no difficulty in expressing any one of the forms f, ι, ϕ, &c. in terms of the (A, B, C . . V, W); thus in the instance just referred to, $p=(\phi\iota)^2$, we have

$$\phi=(ff)^2=(A, A)^2=800\ C,$$

and

$$\iota=(ff)^4=(A, A)^4=28800B,$$

whence $p=2304000(B, C)^2$; also $(B, C)^2=-5H+2B^2$; and therefore, finally,

$$p=-11520000\ H+4608000\ B^2.$$

339. I remark upon the value $S=-96(D, M)+16BO-7GK$, that S is the complete value of a covariant $(\ )^9(\ )^3$, the leading coefficient of which is given in Table No. 86 of my Eighth Memoir; the form (D, M), omitting a numerical factor (if any), would have had smaller numerical coefficients, but there is in the form actually adopted the advantage that it vanishes for $a=0$, $b=0$, that is, when the quintic has two equal roots.

340. I now form the following Table No. 88, viz. this is the Table No. 82 of my Eighth Memoir, carried as far as a^8, but with the composite covariants expressed by means of the foregoing letters A, B, C, . . . , W; instead of giving the syzygies as in Table No. 82, I transfer them to a separate Table, No. 89. In all other respects the arrangement is as explained, Eighth Memoir, No. 253; but in place of N, S, S′ I have written $*$, Σ, Σ' to denote new covariant, new syzygy, derived syzygy, respectively; and I have, as to the terms a^8x^{14}, a^8x^{20} respectively, introduced the new symbol σ to denote an interconnexion of syzygies, as appearing by the Table No. 89, and as will be further explained.

Table No. 88.

Ind. a.	Ind. x.	Coeff.			
1	5	1	A	.	*
	3	0		.	
	1	0		.	
2	10	1	A^2	.	
	8	0		.	
	6	1	C	.	*
	4	0		.	
	2	1	B	.	*
	0	0		.	
3	15	1	A^3	.	
	13	0		.	
	11	1	AC	.	
	9	1	F	.	*
	7	1	AB	.	
	5	1	E	.	*
	3	1	D	.	*
	1	0		.	

Table No. 88 (continued).

Ind. a.	Ind. x.	Coeff.			
4	20	1	A^5	.	
	18	0		.	
	16	1	A^2C	.	
	14	1	AF	.	
	12	2	$A^2B,\ C^2$	.	
	10	1	AE	.	
	8	2	$AD,\ BC$	.	
	6	1	I	.	*
	4	2	$B^2,\ H$	.	*
	2	0		.	
	0	1	G	.	*
5	25	1	A^5	.	
	23	0		.	
	21	1	A^3C	.	
	19	1	A^2F	.	
	17	2	$A^3B,\ AC^2$	.	
	15	2	$A^2E,\ CF$	.	
	13	2	$A^2D.\ ABC$	.	
	11	2	$AI,\ BF,\ CE$	.	Σ
	9	3	$AB^2,\ AH,\ CD$	.	
	7	2	$BE,\ L$	.	*
	5	2	$AG,\ BD$	.	
	3	1	K	.	*
	1	1	J	.	*
6	30	1	A^6	.	
	28	0		.	
	26	1	A^4C	.	
	24	1	A^3F	.	
	22	2	$A^4B,\ A^2C^2$	.	
	20	2	$A^3E,\ ACF$	.	
	18	3	$A^3D,\ A^2BC,\ C^3F^2$	.	Σ
	16	2	$A^2I,\ ABF.\ ACE$	.	Σ'
	14	4	$A^2B^2,\ A^2H,\ ACD,\ BC^2,\ EF$	.	Σ
	12	3	$ABE,\ AL,\ CI,\ DF$	.	Σ
	10	4	$A^2G,\ ABD,\ B^2C,\ CH,\ E^2$	.	Σ
	8	2	$AK,\ BI,\ DE$	.	Σ
	6	4	$AJ,\ B^3,\ BH,\ CG,\ D^2$	.	Σ
	4	1	N	.	*
	2	2	$BG,\ M$	.	*
	0	0		.	
7	35	1	A^7	.	
	33	0		.	
	31	1	A^5C	.	
	29	1	A^4F	.	
	27	2	$A^5B,\ A^3C^2$	.	
	25	2	$A^4E,\ A^2CF$	.	
	23	3	$A^4D,\ A^3BC,\ AC^3,\ AF^2$	.	Σ'
	21	3	$A^3I,\ A^2BF,\ A^2CE,\ C^2F$	.	Σ'
	19	4	$A^3B^2,\ A^3H,\ A^2CD,\ ABC^2,\ AEF$	.	Σ'
	17	4	$A^2BE,\ A^2L,\ ACI,\ ADF,\ BCF,\ C^2E$	.	$2\Sigma'$
	15	5	$A^3G,\ A^2BD,\ AB^2C,\ ACH,\ AE^2,\ C^2D,\ FI$	.	$\Sigma',\ \Sigma$
	13	4	$A^2K,\ ABI,\ ADE,\ B^2F,\ BCE,\ CL,\ FH$	.	Σ
	11	5	$A^2J,\ AB^3,\ ABH,\ ACG,\ AD^2,\ BCD,\ EI$	.	Σ
	9	4	$AN,\ B^2E,\ BL,\ CK,\ DI,\ EH,\ FG$	.	3Σ
	7	4	$ABG,\ AM,\ B^2D,\ CJ,\ DH$	.	Σ
	5	3	$BK,\ P$	.	*
	3	2	$BJ,\ DG$	.	
	1	1	O	.	*

Table No. 88 (concluded).

Ind. a	Ind. x.	Coeff.			
8	40	1	A^8	.	
	38	0		.	
	36	1	A^6C	.	
	34	1	A^5F	.	
	32	2	$A^6B,\ A^4C^2$	.	
	30	2	$A^5E,\ A^3CF$	.	
	28	3	$A^5D,\ A^4BC,\ A^2C^3,\ A^2F^2$	.	Σ'
	26	3	$A^4I,\ A^3BF,\ A^2CE,\ AC^2F$	.	Σ'
	24	5	$A^4B^3,\ A^4H,\ A^3CD,\ A^2BC^2,\ A^2EF,\ CF^2$	.	Σ'
	22	4	$A^3BE,\ A^3L,\ A^2CI,\ A^2DF,\ ABCF,\ AC^2E$	.	$2\Sigma'$
	20	6	$A^4G,\ A^3BD,\ A^2B^2C,\ A^2CH,\ A^2E^2,\ AC^2D,\ AFI,\ BC^3,\ BF^2,\ CEF$	.	σ
	18	5	$A^3K,\ A^2BI,\ A^2DE,\ AB^2F,\ ABCE,\ ACL,\ AFH,\ C^2I,\ CDF$	.	$4\Sigma'$
	16	7	$A^3J,\ A^2B^3,\ A^2BH,\ A^2CG,\ A^2D^2,\ ABCD,\ AEI,\ B^2C^2,\ BEF,\ C^2H,\ CE^2,\ FI$	.	$5\Sigma'$
	14	5	$A^2N,\ AB^2E,\ ABL,\ ACK,\ ADI,\ AEH,\ AFG,\ BCI,\ BDF,\ CDE$	.	σ
	12	7	$A^2BG,\ A^2M,\ AB^2D,\ ACJ,\ ADH,\ B^3C,\ BCH,\ BE^2,\ C^2G,\ CD^2,\ EL,\ FK,\ I^2$	.	3Σ
	10	5	$ABK,\ AEG,\ AP,\ B^2I,\ BDE,\ CN,\ DL,\ FJ,\ HI$	.	3Σ
	8	6	$ABJ,\ ADG,\ B^4,\ B^2H,\ BCG,\ BD^2,\ CM,\ EK,\ H^2$	.	2Σ
	6	3	$AO,\ BN,\ DK,\ EJ,\ GI$	.	2Σ
	4	4	$B^2G,\ BM,\ DJ,\ DH$	.	
	2	1	R	.	*
	0	2	$G^2,\ Q$	.	*

341. The syzygies and interconnexions of syzygies are given in

Table No. 89.

(5, 11)	$AI+BF-CE=0$	
(6, 18)	$A^3D-A^2BC+4C^3+F^2=0$	
(6, 14)	$A^2H-6ACD-4BC^2-EF=0$	
(6, 12)	$AL-2CI+3DF=0$	
(6, 10)	$A^2G-12ABD-4B^2C-E^2=0$	
(6, 8)	$AK+2BI-3DE=0$	
(6, 6)	$AJ-B^3+2BH-CG-9D^2=0$	
(7, 15)	$A^2BD-ABC^2+ACH-6C^2D-FI=0$	
(7, 13)	$A^2K-ABI-3B^2F+6CL+3FH=0$	
(7, 11)	$A^2J-AB^3+ABH-9AD^2-6BCD-EI=0$	
(7, 9)	$AN-B^2E-6DI+2EH-FG=0$	
	$2BL+6DI-EH+FG=0$	
	$2CK-12DI+EH-FG=0$	
(7, 7)	$AM+2B^2D+CJ-3DH=0$	
σ, (8, 20)	$0.A^2(A^2G-12ABD-4B^2C-E^2)$	*suprà* (6, 10)
	$-A(A^2BD-ABC^2+ACH-6C^2D-FI)$	,, (7, 15)
	$+B(A^3D-A^2BC+4C^3+F^2)$	,, (6, 18)
	$+C(A^2H-6ACD-4BC^2-EF)$	,, (6 14)
	$-F(AI+BF-CE)=0$	,, (5, 11)
σ, (8, 14)	$0.A(AN-B^2E-6DI+2EH-FG)$	*suprà* (7, 9)
	$+A(2BL+6DI-EH+FG)$	,, (,,)
	$+A(2CK-12DI+EH-FG)$	,, (,,)
	$-2B(AL-2CI+3DF)$	,, (6, 12)
	$-2C(AK+2BI-3DE)$	,, (6, 8)
	$+6D(AI+BF-CE)=0$	,, (5, 11)

Table No. 89 (continued).

(8, 12)	$AB^2D-B^3C+2BCH\ -C^2G \qquad +\ I^2=0$ $-3ADH\ \ -2BCH+2C^2G+18CD^2+FK-2I^2=0$ $EL+FK-2I^2=0$
(8, 10)	$ABK-\ CN-6DL-2FJ+\ HI=0$ $AP\ +2CN+\ FJ=0$ $B^2I\ -\ CN+3DL+\ FJ-2HI=0$
(8, 8)	$ABJ-B^4+4B^2H-\ 9BD^2+12CM-EK-3H^2=0$ $ADG\ \ +2B^2H-12BD^2+\ 8CM-EK-2H^2=0$
(8, 6)	$AO+6DK-3EJ+2GI=0$ $BN+3DK-\ EJ+\ GI=0$

342. In illustration take any one of the lines of Table No. 88, for instance the lin

$$(7, 17)\ |\ 4\ |\ A^2BE,\ A^2L,\ ACI,\ ADF,\ BCF,\ C^2E\ \ |\ 2\Sigma'\ |$$

there are here 6 composite covariants, but the number of asyzygetic covariants is = there must therefore be $6-4$, $=2$ syzygies; we have however (see Table No. 89) t derived syzygies of the right form, viz. these are

$$A(AL-2CI+3DF)=0,$$
$$C(AI+\ BF-\ CE)=0,$$

which are designated as $2\Sigma'$, and there is consequently no new syzygy Σ.

But in the line

$$(7, 15)\ |\ 5\ |\ A^3G,\ A^2BD,\ AB^2C,\ ACH,\ AE^2,\ C^2D,\ FI\ \ |\ \Sigma', \Sigma\ |$$

there are 7 composite covariants, but the number of asyzygetic covariants is $=5$; the must therefore be $7-5$, $=2$ syzygies. One of these is the derived syzygy

$$A(A^2G-E^2-12ABD-4B^2C)=0,$$

which is designated by Σ'; *the other is a new syzygy (see Table No. 89),*

$$A^2BD-ABC^2+ACH-6C^2D-FI=0,$$

designated by Σ.

343. Take now the line

$$(8, 20)\ |\ 6\ |\ A^4G,\ A^3BD,\ A^2B^2C,\ A^2CH,\ A^2E^2,\ AC^2D,\ AFI,\ BC^3,\ BF^2,\ CEF\ |\ 5\Sigma',\ \sigma\ |;$$

there are here 10 composite covariants, but the number of irreducible covariants is $=6$; there should therefore be $10-6$, $=4$ syzygies. There are, however, the 5 derived syzygies

$$A^2(A^2G-12ABD-4B^2C-E^2)=0,\ \&c.\ \text{(see Table No. 89)}$$

designated by $5\Sigma'$; *since these are equivalent to 4 syzygies only there must be 1 identical relation between them (designated by* σ*), viz. this* is the equation $0=0$ obtained by adding the several syzygies, multiplied each by the proper numerical factor as shown Table No. 89.

344. Again, for the line

$(8, 14) \mid 5 \mid$ A^{2}N, AB^{2}E, ABL, ACK, ADI, AEH, AFG, BCI, BDF, CDE $\mid 6\Sigma', \sigma \mid$

there are here 10 composite covariants, but only 5 irreducible covariants; there should therefore be $10-5, =5$ syzygies; we have in fact the 6 derived syzygies

$$\mathrm{A}(\mathrm{AN}-\mathrm{B}^2\mathrm{E}-6\mathrm{DI}+2\mathrm{EH}-\mathrm{FG})=0 \text{ \&c. (see Table No. 89)}$$

designated by $6\Sigma'$; these must therefore be connected by 1 identical relation (designated by σ), viz. this is the equation $0=0$ obtained by adding the several syzygies, each multiplied by the proper numerical factor as shown Table No. 89.

345. These two cases (σ) are in fact the instances which present themselves where a correction is required to my original theory. The two identical relations in question were disregarded in my original theory, and this accordingly gave the two non-existent irreducible covariants $(a, ..)^8(x, y)^{14}$ and $(a, ..)^8(x. y)^{20}$. And reverting to No. 336, these give in the denominator of $\mathrm{A}(x)$ the factors $(1-a^8x^{20})(1-a^8x^{14})$. In virtue hereof, writing $x=1$, we have in $\mathrm{A}(x)$ the factor $\frac{(1-a^8)^{10}}{(1-a^8)^4}$, $=(1-a^8)^6$, agreeing with the function $(1-)^{-1}(1-a)^{-2} \ldots . (1-a^8)^6 \ldots .$ And we thus see that the denominator factors of $\mathrm{A}(x)$ do not all of them refer to irreducible covariants; viz. we have

$$ax^5,\ a^2x^6,\ a^2x^2,\ a^3x^9,\ a^3x^5,\ a^3x^3,\ a^4x^6,\ a^4x^4,\ a^4,\ a^5x^7,\ a^5x^3,\ a^5x,\ a^6x^4,\ a^6x^2,\ a^7x^3,\ a^7x,\ a^8x^2,\ a^8,$$

each referring to an irreducible covariant, but a^8x^{20} and a^8x^{14} each referring to an identical relation (σ) or interconnexion of syzygies. And we thus understand how, consistently with the number of the irreducible covariants being finite, the expression for $\mathrm{A}(x)$ may be as above the quotient of two infinite products; viz. there will be in the denominator a finite number of factors each referring to an irreducible covariant, but the remaining infinite series of denominator factors will refer each factor to an identical relation or interconnexion of syzygies. But I do not see how we can by the theory distinguish between the two classes of factors, so as to determine the number of the irreducible covariants, or even to make out affirmatively that the number of them is finite.

346. The new covariants O, P, R, S, T, V are as follows:—

(Remarks added 17th March, 1871.—A. C.)

It will be observed that the Tables are printed in a slightly different form from the preceding ones; this has been done in order to show at a glance in each column the set of terms which contain a given power of a, and in each such set the terms which contain a given power of b.

The numerical verifications are also given, not only for the entire column, but for each set of terms containing the same power of a (viz. the equal sums of the positive and negative coefficients are shown by a number with the prefixed sign $\pm$); and in Table 95 the verification is given in regard to the subsets containing the same powers of a and b; as to these subsets, the sums of the positive and negative coefficients are *not* in all cases equal, but a singular law manifests itself (see p. 44).

Table No. 90 (Covariant O).

a^3	b^0cf^3	$+$	1	a^3	b^0df^3	$-$	1
⋮	def^2	$-$	4	a^3	e^2f^2	$+$	1
a^3	e^3f	$+$	3				
				a^2	$b\,cf^3$	$+$	4
a^2	b^2f^3	$-$	1	⋮	def^2	$+$	3
⋮	$b\,cef^2$	$-$	3		e^3f	$-$	7
	d^2f^2	$+$	16		$b^0c^2ef^2$	$-$	16
	de^2f	$+$	4		cd^2f^2	$+$	6
	e^4	$-$	15		cde^2f	$+$	30
	$b^0c^2df^2$	$-$	6		ce^4	$-$	8
	c^2e^2f	$+$	4	⋮	d^3ef	$-$	18
	cd^2ef	$-$	22	a^2	d^2e^3	$+$	6
	cde^3	$+$	26				
⋮	d^4f	$+$	9	a	b^3f^3	$-$	3
a^2	d^3e^2	$-$	12	⋮	b^2cef^2	$-$	4
					d^2f^2	$-$	4
a	b^3ef^2	$+$	7		de^2f	$-$	1
⋮	b^2cdf^2	$-$	30		e^4	$+$	18
	ce^2f	$+$	1		$b\,c^2df^2$	$+$	22
	d^2ef	$-$	74		c^2e^2f	$+$	74
	de^3	$+$	84		cd^2ef	$-$	160
	$b\,c^3f^2$	$+$	18		cde^3	$-$	32
	c^2def	$+$	160		d^4f	$+$	81
	c^2e^3	$-$	98		d^3e^2	$+$	6
	cd^3f	$-$	20		$b^0c^4f^2$	$-$	9
	cd^2e^2	$-$	94		c^3def	$+$	20
	d^4e	$+$	51		c^3e^3	$-$	112
	b^0c^4ef	$-$	81		c^2d^3f	$-$	18
	c^3d^2f	$+$	18		$c^2d^2e^2$	$+$	284
	c^3de^2	$+$	140	⋮	cd^4e	$-$	216
⋮	c^2d^3e	$-$	100	a	d^6	$+$	54
a	cd^5	$+$	18				
				a^0	b^4ef^2	$+$	15
a^0	b^4df^2	$+$	8	⋮	b^3cdf^2	$-$	26
⋮	e^2f	$-$	18		ce^2f	$-$	84
	$b^3c^2f^2$	$-$	6		d^2ef	$+$	98
	$cdef$	$+$	32		de^3	$-$	45
	ce^3	$+$	45		$b^2c^3f^2$	$+$	12
	d^3f	$+$	112		c^2def	$+$	94
	d^2e^2	$-$	150		c^2e^3	$+$	150
	b^2c^3ef	$-$	6		cd^3f	$-$	140
	c^2d^2f	$-$	284		cd^2e^2	$-$	50
	c^2de^2	$+$	50		d^4e	$+$	15
	cd^3e	$+$	320		$b\,c^4ef$	$-$	51
	d^5	$-$	120		c^3d^2f	$+$	100
	$b\,c^4df$	$+$	216		c^3de^2	$-$	320
	c^4e^2	$-$	15		c^2d^3e	$+$	310
	c^3d^2e	$-$	310		cd^5	$-$	90
	c^2d^4	$+$	130		b^0c^5df	$-$	18
	b^0c^6f	$-$	54		c^5e^2	$+$	120
⋮	c^5de	$+$	90	⋮	c^4d^2e	$-$	130
a^0	c^4d^3	$-$	40	a^0	c^3d^4	$+$	40
		$\pm$	4			$\pm$	1
			59				49
			497				559
			1003				954
		$\pm$	1563			$\pm$	1563

$(\;\;\between (x, y)^1$

Table No. 91 (Covariant P).

(

$a^3\ b^0cdf^2$	$+1$
⋮ ce^2f	-2
⋮ d^2ef	$+2$
$a^3\ de^3$	-1
$a^2\ b^2df^2$	-1
⋮ e^2f	$+2$
$b\,c^2f^2$	-3
$cdef$	-6
ce^3	$+13$
d^3f	-8
d^2e^2	$+2$
b^0c^3ef	$+16$
c^2d^2f	-2
c^2de^2	-38
⋮ cd^3e	$+34$
$a^2\ d^5$	-9
$a\ b^3cf^2$	$+5$
⋮ def	$+2$
e^3	-12
b^2c^2ef	-24
cd^2f	$+52$
cde^2	$+7$
d^3e	-22
$b\,c^3df$	-52
c^3e^2	$+34$
c^2d^2e	$+8$
cd^4	-1
b^0c^5f	$+18$
⋮ c^4de	-25
$a\ c^3d^3$	$+10$
$a^0\ b^5f^2$	-2
⋮ b^4cef	$+10$
d^2f	-28
de^2	$+30$
b^3c^2df	$+32$
c^2e^2	-35
cd^2e	-50
d^4	$+30$
b^2c^4f	-12
c^3de	$+70$
c^2d^3	-40
⋮ $b\,c^5e$	-15
$a^0\ c^4d^2$	$+10$

$\pm$ 3
67
136
182
± 388

$a^3\ b^0cef^2$	-2
⋮ d^2f^2	$+5$
de^2f	-1
$a^3\ e^4$	-2
$a^2\ b^2ef^2$	$+2$
$b\,cdf^2$	-17
ce^2f	$+13$
d^2ef	-32
de^3	$+32$
$b^0c^3f^2$	$+4$
c^2def	$+36$
c^2e^3	-24
cd^3f	-10
⋮ cd^2e^2	-16
$a^2\ d^4e$	$+12$
$a\ b^3df^2$	$+7$
⋮ e^2f	-12
$b^2c^2f^2$	$+6$
$cdef$	$+42$
d^3f	$+54$
d^2e^2	-91
$b\,c^3ef$	-68
c^2d^2f	-64
c^2de^2	$+14$
cd^3e	$+204$
d^5	-93
b^0c^4df	$+37$
c^4e^2	$+86$
⋮ c^3d^2e	-208
$a\ c^2d^4$	$+86$
$a^0\ b^4cf^2$	-5
⋮ def	-12
b^3c^2ef	$+34$
cd^2f	-46
cde^2	$+105$
d^3e	-20
b^2c^3df	$+50$
c^3e^2	-110
c^2d^2e	-170
cd^4	$+115$
$b\,c^5f$	-21
c^4de	$+250$
c^3d^3	-150
⋮ b^0c^6e	-60
$a^0\ c^5d^2$	$+40$

$\pm$ 5
99
536
594
± 1234

$a^3\ b^0cf^3$	-1
⋮ def^2	$+6$
$a^3\ e^3f$	-5
$a^2\ b^2f^3$	$+1$
⋮ $b\,cef^2$	-11
d^2f^2	-4
de^2f	-4
e^4	$+17$
$b^0c^2df^2$	$+2$
c^2e^2f	$+26$
cd^2ef	-2
cde^3	-40
⋮ d^4f	-9
$a^2\ d^3e^2$	$+24$
$a\ b^3ef^2$	$+5$
⋮ b^2cdf^2	-4
ce^2f	$+35$
d^2ef	-26
de^3	-22
$b\,c^3f^2$	$+10$
c^2def	-72
c^2e^3	-106
cd^3f	$+76$
cd^2e^2	$+210$
d^4e	-99
b^0c^4ef	-13
c^3d^2f	-10
c^3de^2	$+128$
⋮ c^2d^3e	-184
$a\ cd^5$	$+72$
$a^0\ b^4df^2$	$+4$
⋮ e^2f	-42
$b^3c^2f^2$	-8
$cdef$	$+124$
ce^3	$+105$
d^3f	$+56$
d^2e^2	-130
b^2c^3ef	-26
c^2d^2f	-296
c^2de^2	-170
cd^3e	$+340$
d^5	-60
$b\,c^4df$	$+260$
c^4e^2	$+25$
c^3d^2e	-150
⋮ $a^0\ c^5de$	$+40$

$\pm$ 6
70
536
954
± 1566

$a^3\ b^0df^3$	$+1$
$a^3\ e^2f^2$	-1
$a^2\ b\,cf^3$	-6
⋮ def^2	$+11$
e^3f	-5
$b^0c^2ef^2$	$+4$
cd^2f^2	-2
cde^2f	$+4$
ce^4	-4
⋮ d^3ef	-10
$a^2\ d^2e^3$	$+8$
$a\ b^3f^3$	$+5$
⋮ b^2cef^2	$+4$
d^2f^2	-26
de^2f	-35
e^4	$+42$
$b\,c^2df^2$	$+2$
c^2e^2f	$+26$
cd^2ef	$+72$
cde^3	-124
d^4f	$+13$
d^3e^2	$+26$
$b^0c^4f^2$	$+9$
c^3def	-76
c^3e^2	-56
c^2d^3f	$+10$
$c^2d^2e^2$	$+296$
⋮ cd^4e	-260
$a\ d^6$	$+72$
$a^0\ b^4ef^2$	-17
⋮ b^3cdf^2	$+40$
ce^2f	$+22$
d^2ef	$+106$
de^3	-105
$b^2c^3f^2$	-24
c^2def	-210
c^2e^3	$+130$
cd^3f	-128
cd^2e^2	$+170$
d^4e	-25
$b\,c^4ef$	$+99$
c^3d^2f	$+184$
c^3de^2	-340
c^2d^3e	$+150$
cd^5	-40
⋮ $a^0\ c^5e^2$	$+60$

$\pm$ 1
27
577
961
± 1566

$a^2\ b\,df^3$	$+2$
⋮ e^2f^2	-2
$b^0c^2f^3$	-5
$cdef^2$	$+17$
ce^3f	-7
d^3f^2	-4
⋮ d^2e^2f	-6
$a^2\ de^4$	$+5$
$a\ b^2cf^3$	$+1$
⋮ def^2	-13
e^3f	$+12$
$b\,c^2ef^2$	$+32$
cd^2f^2	-36
cde^2f	-42
ce^4	$+24$
d^3ef	$+56$
d^2e^3	-34
$b^0c^3df^2$	$+10$
c^3e^2f	-54
c^2d^2ef	$+64$
c^2de^3	$+46$
cd^4f	-37
⋮ cd^3e^2	-50
$a\ d^5c$	$+21$
$a^0\ b^4f^3$	$+2$
⋮ b^3cef^2	-32
d^2f^2	$+24$
$b^2c^2df^2$	$+16$
c^2e^2f	$+91$
cd^2ef	-14
cde^3	-105
d^4f	-86
d^3e^2	$+110$
$b\,c^4f^2$	-12
c^3def	-204
c^3e^3	$+20$
c^2d^3f	$+208$
$c^2d^2e^2$	$+170$
cd^4e	-250
d^6	$+60$
b^0c^5ef	$+93$
c^4d^2f	-86
c^4de^2	-115
⋮ c^3d^3e	$+150$
$a^0\ c^2d^5$	-40

$\pm$ 24
266
944
± 1234

$a^2\ b^0cdf^3$	-1
⋮ ce^2f^2	$+1$
d^2ef^2	$+3$
⋮ de^3f	-5
$a^2\ e^5$	$+2$
$a\ b^2df^3$	$+2$
⋮ e^2f^2	-2
$b\,c^2f^3$	-2
$cdef^2$	$+6$
ce^3f	-2
d^3f^2	-16
d^2e^2f	$+24$
de^4	-10
$b^0c^3ef^2$	$+8$
$c^2d^2f^2$	$+2$
c^2de^2f	-52
c^2e^4	$+28$
cd^3ef	$+52$
cd^2e^3	-32
⋮ d^5f	-18
$a\ d^4c^2$	$+12$
$a^0\ b^3cf^3$	$+1$
⋮ def^2	-13
e^3f	$+12$
$b^2c^2ef^2$	-2
cd^2f^2	$+38$
cde^2f	-7
ce^4	-30
d^3ef	-34
d^2e^3	$+35$
$b\,c^3df^2$	-34
c^3e^2f	$+22$
c^2d^2ef	-8
c^2de^3	$+50$
cd^4f	$+25$
cd^3e^2	-70
d^5e	$+15$
$b^0c^5f^2$	$+9$
c^4def	$+1$
c^4e^3	-30
c^3d^3f	-10
⋮ $c^3d^2e^2$	$+40$
$a^0\ c^2d^4e$	-10

$\pm$ 6
134
248
± 388

$)(x, y)$

Table No. 92 (Covariant R).

$(\quad)(x, y)^2$

a^3	$b^0c^2f^3$	−	1
⋮	$cdef^2$	+	6
	ce^3f	−	4
	d^3f^2	−	3
⋮	d^2e^2f	+	1
a^3	de^4	+	1
a^2	b^2cf^3	+	2
⋮	def^2	−	6
	e^3f	+	4
	$b\,c^2ef^2$	−	3
	cd^2f^2	+	3
	cde^2f	−	18
	ce^4	+	17
	d^3ef	+	22
	d^2e^3	−	21
	$b^0c^3e^2f$	+	13
	c^2d^2ef	−	12
	c^2de^3	−	21
	cd^4f	−	3
⋮	cd^3e^2	+	32
a^2	d^5e	−	9
a	b^4f^3	−	1
⋮	$b^3d^2f^2$	+	6
	de^2f	+	16
	e^4	−	18
	$b^2c^2df^2$	−	3
	c^2e^2f	+	3
	cd^2ef	−	18
	cde^3	+	14
	d^4f	−	41
	d^3e^2	+	39
	$b\,c^3def$	−	32
	c^3e^3	−	2
	c^2d^3f	+	84
	$c^2d^2e^2$	+	24
	cd^4e	−	106
	d^6	+	36
	b^0c^5ef	+	18
	c^4d^2f	−	33
	c^4de^2	−	25
⋮	c^3d^3e	+	60
a	c^2d^5	−	21
a^0	b^5ef^2	+	3
⋮	b^4cdf^2	−	6
	ce^2f	−	15
	d^2ef	−	38
	de^3	+	45
	$b^3c^3f^2$	+	3
	c^2def	+	102
	c^2e^3	−	15
	cd^3f	+	76
	cd^2e^2	−	175
	d^4e	+	35
	b^2c^4ef	−	42
	c^3d^2f	−	182
	c^3de^2	+	120
	c^2d^3e	+	150
	cd^5	−	70
	$b\,c^5df$	+	126
	c^5e^2	−	15
	c^4d^2e	−	175
	c^3d^4	+	75
	b^0c^7f	−	27
⋮	c^6de	+	45
a^0	c^5d^3	−	20

± 8
93
300
780
±1181

a^3	$b^0d^2ef^2$	+	2
⋮	de^3f	−	4
a^3	e^5	+	2
a^2	$b\,c^2f^3$	−	2
⋮	ce^3f	+	4
	d^3f^2	−	14
	d^2e^2f	+	30
	de^4	−	18
	$b^0c^3ef^2$	+	14
	c^2de^2f	−	66
	c^2e^4	+	26
	cd^3ef	+	56
	cd^2e^3	−	18
⋮	d^5f	−	18
a^2	d^4e^2	+	6
a	b^3cf^3	+	4
⋮	def^2	−	4
	$b^2c^2ef^2$	−	30
	cd^2f^2	+	66
	ce^4	−	18
	d^3ef	−	84
	d^2e^3	+	66
	$b\,c^3df^2$	−	56
	c^3e^2f	+	84
	c^2de^3	−	20
⋮	cd^4f	+	40
a	cd^3e^2	−	72
a^0	b^5f^3	−	2
⋮	b^4cef^2	+	18
	d^2f^2	−	26
	de^2f	+	18
	$b^3c^2df^2$	+	18
	c^2e^2f	−	66
	cd^2ef	+	20
	d^4f	+	58
	d^3e^2	−	50
	$b^2c^4f^2$	−	6
	c^3def	+	72
	c^3e^2	+	50
	c^2d^3f	−	156
	cd^4e	+	90
	d^6	−	30
	$b\,c^5ef$	−	24
	c^4d^2f	+	94
	c^4de^2	−	90
	c^2d^5	+	10
	b^0c^6df	−	18
⋮	c^6e^2	+	30
a^0	c^5d^2e	−	10

± 4
136
476
478
±1094

a^3	$b^0d^2f^3$	+	1
⋮	de^2f^2	−	2
a^3	e^4f	+	1
a^2	$b\,cdf^3$	−	6
⋮	ce^2f^2	+	6
	d^2ef^2	+	3
	e^5	−	3
	$b^0c^3f^3$	+	3
	c^2def^2	−	3
	c^2e^3f	−	6
	cd^2e^2f	+	3
⋮	cde^4	+	6
a^2	d^3e^3	−	3
a	b^3df^3	+	4
⋮	e^2f^2	−	4
	$b^2c^2f^3$	−	1
	$cdef^2$	+	18
	ce^3f	−	16
	d^3f^2	−	13
	d^2e^2f	−	3
	de^4	+	15
	$b\,c^3ef^2$	−	22
	$c^2d^2f^2$	+	12
	c^2de^2f	+	18
	c^2e^4	+	38
	cd^3ef	+	32
	cd^2e^3	−	102
	d^5f	−	18
	d^4e^2	+	42
	$b^0c^4df^2$	+	3
	c^4e^2f	+	41
	c^3d^2ef	−	84
	c^3de^3	−	76
	c^2d^4f	+	33
	$c^2d^3e^2$	+	182
⋮	cd^5e	−	126
a	d^7	+	27
a^0	b^4cf^3	−	1
⋮	def^2	−	17
	e^3f	+	18
	$b^3c^2ef^2$	+	21
	cd^2f^2	+	21
	cde^2f	−	14
	ce^4	−	45
	d^3ef	+	2
	d^2e^3	+	15
	$b^2c^3df^2$	−	32
	c^3e^2f	−	39
	c^2d^2ef	−	24
	c^2de^3	+	175
	cd^4f	+	25
	cd^3e^2	−	120
	d^5e	+	15
	$b\,c^5f^2$	+	9
	c^4def	+	106
	c^4e^3	−	35
	c^3d^3f	−	60
	$c^3d^2e^2$	−	150
	c^2d^4e	+	175
	cd^6	−	45
	b^0c^6ef	−	36
	c^5d^2f	+	21
	c^5de^2	+	70
⋮	c^4d^3e	−	75
a^0	c^3d^5	+	20

± 2
21
465
693
±1181

Table No. 93 (Covariant S, $=(a,\ldots)^9(x, y)^3$).

Coefficient of x^3.

a^4	cef^3	$+\ \ 9$
⋮	d^2f^3	$+\ \ 21$
⋮	de^2f^2	$-\ \ 78$
a^4	e^4f	$+\ \ 48$
a^3	b^2ef^3	$-\ \ 9$
⋮	cdf^3	$-\ \ 162$
	ce^2f^2	$+\ \ 99$
	d^2ef^2	$+\ \ 309$
	de^3f	$+\ \ 12$
	e^5	$-\ \ 240$
	c^3f^3	$-\ \ 81$
	c^2def^2	$+\ \ 1026$
	c^2e^3f	$-\ \ 768$
	cd^3f^2	$-\ \ 738$
	cd^2e^2f	$-\ \ 564$
	cde^4	$+\ \ 1056$
⋮	d^4ef	$+\ \ 756$
a^3	d^3e^3	$-\ \ 696$
a^2	b^3df^3	$+\ \ 120$
⋮	e^2f^2	$-\ \ 21$
	$b^2c^2f^3$	$+\ \ 486$
	$cdef^2$	$-\ \ 2160$
	ce^3f	$+\ \ 1023$
	d^3f^2	$+\ \ 120$
	d^2e^2f	$-\ \ 1053$
	de^4	$+\ \ 1314$
	$b\,c^3ef^2$	$-\ \ 1863$
	$c^2d^2f^2$	$+\ \ 2538$
	c^2de^2f	$+\ \ 2340$
	c^2e^4	$+\ \ 672$
	cd^3ef	$+\ \ 2820$
	cd^2e^3	$-\ \ 7812$
	d^5f	$-\ \ 3024$
	d^4e^2	$+\ \ 4572$
	b^0cdf^2	$-\ \ 324$
	c^4e^2f	$+\ \ 3888$
	c^3d^2ef	$-\ \ 8748$
	c^3de^3	$-\ \ 4800$
	c^2d^4f	$+\ \ 4248$
	$c^2d^3e^2$	$+14520$
⋮	cd^5e	-11448
a^2	d^7	$+\ \ 2592$
a	b^4cf^3	$-\ \ 576$
⋮	def^2	$+\ \ 672$
	e^3f	$-\ \ 459$
	$b^3c^2ef^2$	$+\ \ 3456$
	cd^2f^2	$-\ \ 864$
	cde^2f	$+\ \ 2094$
	ce^4	$-\ \ 3915$
	d^3ef	$+\ \ 528$
⋮	d^2e^3	$-\ \ 45$
⋮	$b^2c^3df^2$	$-\ \ 2592$
⋮	$b^2c^3e^2f$	$-\ \ 9747$
⋮	c^2d^2ef	$-\ \ 8496$
	c^2de^3	$+26610$
	cd^4f	$+\ \ 8544$
	cd^3e^2	-16650
	d^5e	$+\ \ 720$
	c^5f^2	$+\ \ 972$
	$b\,c^4def$	$+24624$
	c^4e^3	$-\ \ 5040$
	c^3d^3f	-15984
	$c^3d^2e^2$	-29340
	c^2d^4e	$+34320$
	cd^6	$-\ \ 8640$
	c^6ef	$-\ \ 7776$
	c^5d^2f	$+\ \ 5184$
	c^5de^2	$+12960$
⋮	c^4d^3e	-14400
a	c^3d^5	$+\ \ 3840$
a^0	b^6f^3	$+\ \ 192$
⋮	b^5cef^2	$-\ \ 1440$
	d^2f^2	$-\ \ 192$
	de^2f	$-\ \ 1080$
	e^4	$+\ \ 2025$
	$b^4c^2df^2$	$+\ \ 1728$
	c^2e^2f	$+\ \ 4410$
	cd^2ef	$+\ \ 5280$
	cde^3	-13500
	d^4f	$-\ \ 4800$
	d^3e^2	$+\ \ 7800$
	$b^3c^4f^2$	$-\ \ 648$
	c^3def	-14040
	c^3e^3	$+\ \ 3075$
	c^2d^3f	$+\ \ 9120$
	$c^2d^2e^2$	$+16350$
	cd^4e	-19200
	d^6	$+\ \ 4800$
	b^2c^5ef	$+\ \ 4860$
	c^4d^2f	$-\ \ 3240$
	c^4de^2	$-\ \ 8100$
⋮	c^3d^3e	$+\ \ 9000$
a^0	c^2d^5	$-\ \ 2400$

$\pm$ 78
3258
41253
124524
68640
± 237753

Coefficient of x^2y.

a^4	cf^4	$+\ \ 9$
⋮	def^3	$-\ \ 45$
a^4	e^3f^2	$+\ \ 36$
a^3	b^2f^4	$-\ \ 9$
⋮	$b\,cef^3$	$-\ \ 18$
	d^2f^3	$+\ \ 243$
	de^2f^2	$+\ \ 9$
	e^4f	$-\ \ 216$
	c^2df^3	$-\ \ 351$
	$c^2e^2f^2$	$+\ \ 144$
	cd^2ef^2	$+\ \ 1836$
	cde^3f	$-\ \ 2592$
	ce^5	$+\ \ 1152$
	d^4f^2	$-\ \ 1458$
⋮	d^3e^2f	$+\ \ 2268$
a^3	d^2e^4	$-\ \ 1008$
a^2	b^3ef^3	$+\ \ 63$
⋮	b^2cdf^3	$-\ \ 234$
	ce^2f^2	$-\ \ 18$
	d^2ef^2	$-\ \ 3231$
	de^3f	$+\ \ 4293$
	e^5	$-\ \ 972$
	$b\,c^3f^3$	$+\ \ 810$
	c^2def^2	$-\ \ 3825$
	ce^3f	$+\ \ 4032$
	cd^3f^2	$+\ \ 7938$
	cd^2e^2f	$-\ \ 9360$
	cde^4	$-\ \ 864$
	d^4ef	$-\ \ 1296$
	d^3e^3	$+\ \ 2700$
	c^4ef^2	$-\ \ 324$
	$c^3d^2f^2$	$-\ \ 2484$
	c^3de^2f	$+\ \ 6624$
	c^3e^4	$-\ \ 6912$
	c^2d^3ef	$-\ \ 4428$
	$c^2d^2e^3$	$+12672$
	cd^5f	$+\ \ 1944$
⋮	cd^4e^2	$-\ \ 9072$
a^2	d^6e	$+\ \ 1944$
a	b^4df^3	$+\ \ 144$
⋮	e^2f^2	$-\ \ 243$
	$b^3c^2f^3$	$-\ \ 900$
	$cdef^2$	$+10620$
	ce^3f	$-\ \ 8586$
	d^3f^2	$-\ \ 864$
	d^2e^2f	$-\ \ 1215$
	de^4	$+\ \ 1215$
	$b^2c^3ef^2$	$-\ \ 1836$
	$c^2d^2f^2$	-16812
	c^2de^2f	$+\ \ 6651$
⋮	c^2e^4	$+12960$
⋮	b^2cd^3ef	$+18612$
⋮	cd^2e^3	-18900
	d^5f	$-\ \ 3888$
	d^4e^2	$+\ \ 2970$
	$b\,c^4df^2$	$+15228$
	c^4e^2f	$-\ \ 4968$
	c^3d^2ef	-14544
	c^3de^3	-12960
	c^2d^4f	$+\ \ 1296$
	$c^2d^3e^2$	$+22500$
	cd^5e	$-\ \ 6480$
	c^6f^2	$-\ \ 3888$
	c^5def	$+\ \ 5184$
	c^5e^3	$+\ \ 5760$
	c^4d^3f	$-\ \ 576$
⋮	$c^4d^2e^2$	$-\ \ 9360$
a	c^3d^4e	$+\ \ 2880$
a^0	b^5cf^3	$+\ \ 288$
⋮	def^2	$-\ \ 3888$
	e^3f	$+\ \ 3645$
	$b^4c^2ef^2$	$+\ \ 756$
	cd^2f	$+\ \ 7488$
	cde^2f	$-\ \ 4050$
	ce^4	$-\ \ 6075$
	d^3ef	$-\ \ 4320$
	d^2e^3	$+\ \ 6075$
	$b^3c^3df^2$	$-\ \ 7128$
	c^3e^2f	$+\ \ 2970$
	c^2d^2ef	$+\ \ 3060$
	c^2de^3	$+10125$
	cd^4f	$+\ \ 1440$
	cd^3e^2	-13950
	d^5e	$+\ \ 3600$
	$b^2c^5f^2$	$+\ \ 1944$
	c^4def	$-\ \ 1620$
	c^4e^3	$-\ \ 4500$
	c^3d^3f	$-\ \ 360$
⋮	$c^3d^2e^2$	$+\ \ 6300$
a^0	c^2d^4e	$-\ \ 1800$

$\pm$ 45
5652
43020
106020
47691
± 202428

Table No. 93 (continued).

Coefficient of xy^2.

a^4	df^4	$-$	9
a^4	e^2f^3	$+$	9
a^3	$b\,cf^4$	$+$	45
⋮	def^3	$+$	18
	e^3f^2	$-$	63
	c^2ef^3	$-$	243
	cd^2f^3	$+$	351
	cde^2f^2	$+$	234
	ce^4f	$-$	144
	d^3ef^2	$-$	810
⋮	d^2e^3f	$+$	900
a^3	de^5	$-$	288
a^2	b^3f^4	$-$	36
⋮	b^2cef^3	$-$	9
	d^2f^3	$-$	144
	de^2f^2	$+$	18
	e^4f	$+$	243
	$b\,c^2df^3$	$-$	1836
	$c^2e^2f^2$	$+$	3231
	cd^2ef^2	$+$	3825
	cde^3f	$-$	10620
	ce^5	$+$	3888
	d^4f^2	$+$	324
	d^3e^2f	$+$	1836
	d^2e^4	$-$	756
	c^4f^3	$+$	1458
	c^3def^2	$-$	7938
	c^3e^3f	$+$	864
	$c^2d^3f^2$	$+$	2484
	$c^2d^2e^2f$	$+$	16812
	c^2de^4	$-$	7488
	cd^4ef	$-$	15228
	cd^3e^3	$+$	7128
⋮	d^6f	$+$	3888
a^2	d^5e^2	$-$	1944
a	b^4ef^3	$+$	216
⋮	b^3cdf^3	$+$	2592
	ce^2f^2	$-$	4293
	d^2ef^2	$-$	4032
	de^3f	$+$	8586
	e^5	$-$	3645
	$b^2c^3f^3$	$-$	2268
	c^2def^2	$+$	9360
	c^2e^3f	$+$	1215
	cd^3f^2	$-$	6624
	cd^2ef	$-$	6651
	cde^4	$+$	4050
	d^4ef	$+$	4968
	d^3e^3	$-$	2970
⋮	$b\,c^4ef^2$	$+$	1296
⋮	$c^3d^2f^2$	$+$	4428
⋮	$b\,c^3de^2f$	$-$	18612
⋮	c^3e^4	$+$	4320
	c^2d^3ef	$+$	14544
	$c^2d^2e^3$	$-$	3060
	cd^5f	$-$	5184
	cd^4e^2	$+$	1620
	$c^5d^2f^2$	$-$	1944
	c^5e^2f	$+$	3888
	c^4d^2ef	$-$	1296
	c^4de^3	$-$	1440
⋮	c^3d^4f	$+$	576
a	$c^3d^3e^2$	$+$	360
a^0	b^5df^3	$-$	1152
	e^2f^2	$+$	972
	$b^4c^2f^3$	$+$	1008
	$cdef^2$	$+$	864
	ce^3f	$-$	1215
	d^3f^2	$+$	6912
	d^2e^2f	$-$	12960
	de^4	$+$	6075
	$b^3c^3ef^2$	$-$	2700
	$c^2d^2f^2$	$-$	12672
	c^2de^2f	$+$	18900
	c^2e^4	$-$	6075
	cd^3ef	$+$	12960
	cd^2e^3	$-$	10125
	b^3d^5f	$-$	5760
	d^4e^2	$+$	4500
	$b^2c^4df^2$	$+$	9072
	c^4e^2f	$-$	2970
	c^3d^2ef	$-$	22500
	c^3de^3	$+$	13950
	c^2d^4f	$+$	9360
	$c^2d^3e^2$	$-$	6300
	$b\,c^6f^2$	$-$	1944
	c^5def	$+$	6480
	c^5e^3	$-$	3600
⋮	c^4d^3f	$-$	2880
a^0	$c^4d^2e^2$	$+$	1800

$\pm$ 9
1548
45999
62019
92853
$\pm$202428

Coefficient of y^3.

a^3	$b\,df^4$	$-$	9
⋮	e^2f^3	$+$	9
	c^2f^4	$-$	21
	$cdef^3$	$+$	162
	ce^3f^2	$-$	120
	d^3f^3	$+$	81
	$d^2e^2f^2$	$-$	486
⋮	de^4f	$+$	576
a^3	e^6	$-$	192
a^2	b^2cf^4	$+$	78
⋮	def^3	$-$	99
	e^3f^2	$+$	21
	$b\,c^2ef^3$	$-$	309
	cd^2f^3	$-$	1026
	cde^2f^2	$+$	2160
	ce^4f	$-$	672
	d^3ef^2	$+$	1863
	d^2e^3f	$-$	3456
	de^5	$+$	1440
	c^3df^3	$+$	738
	$c^3e^2f^2$	$-$	120
	$c^2d^2ef^2$	$-$	2538
	c^2de^3f	$+$	864
	c^2e^5	$+$	192
	cd^4f^2	$+$	324
	cd^3e^2f	$+$	2592
	cd^2e^4	$-$	1728
⋮	d^5ef	$-$	972
a^2	d^4e^3	$+$	648
a	b^4f^4	$-$	48
⋮	b^3cef^3	$-$	12
	d^2f^3	$+$	768
	de^2f^2	$-$	1023
	e^4f	$+$	459
	$b^2c^2df^3$	$+$	564
	$c^2e^2f^2$	$+$	1053
	cd^2ef^2	$-$	2340
	cde^3f	$-$	2094
	ce^5	$+$	1080
	d^4f^2	$-$	3888
	d^3e^2f	$+$	9747
	d^2e^4	$-$	4410
	$b\,c^4f^3$	$-$	756
	c^3def^2	$-$	2820
	c^3e^3f	$-$	528
	$c^2d^3f^2$	$+$	8748
	$c^2d^2e^2f$	$+$	8496
	c^2de^4	$-$	5280
	cd^4ef	$-$	24624
	cd^3e^3	$+$	14040
⋮	d^6f	$+$	7776
⋮	d^5e^2	$-$	4860
⋮	c^5ef^2	$+$	3024
⋮	$c^4d^2f^2$	$-$	4248
	c^4de^2f	$-$	8544
	c^4e^4	$+$	4800
	c^3d^3ef	$+$	15984
	$c^3d^2e^3$	$-$	9120
⋮	c^2d^5f	$-$	5184
a	$c^2d^4e^2$	$+$	3240
a^0	b^5ef^3	$+$	240
⋮	b^4cdf^3	$-$	1056
	ce^2f^2	$-$	1314
	d^2ef^2	$-$	672
	de^3f	$+$	3915
	e^5	$-$	2025
	$b^3c^3f^3$	$+$	696
	c^2def^2	$+$	7812
	c^2e^3f	$+$	45
	cd^3f^2	$+$	4800
	cd^2e^2f	$-$	26610
	cde^4	$+$	13500
	d^4ef	$+$	5040
	d^3e^3	$-$	3075
	$b^2c^4ef^2$	$-$	4572
	$c^3d^2f^2$	$-$	14520
	c^3de^2f	$+$	16650
	c^3e^4	$-$	7800
	c^2d^3ef	$+$	29340
	$c^2d^2e^3$	$-$	16350
	cd^5f	$-$	12960
	cd^4e^2	$+$	8100
	$b\,c^5df^2$	$+$	11448
	c^5e^2f	$-$	720
	c^4d^2ef	$-$	34320
	c^4de^3	$+$	19200
	c^3d^4f	$+$	14400
	$c^3d^3e^2$	$-$	9000
	c^7f^2	$-$	2592
	c^6def	$+$	8640
	c^6e^3	$-$	4800
⋮	c^5d^3f	$-$	3840
a^0	$c^5d^2e^2$	$+$	2400

$\pm$ 828
10920
79779
146226
$\pm$237753

Table No. 94 (Covariant T, $=(a, \ldots)^{11}(x, y)^{1}$).

Coefficient of x. Coefficient of x.

a^4	c^2df^4	$-\ 1$
⋮	$c^2e^2f^3$	$+\ 1$
	cd^2ef^3	$+\ 7$
	cde^3f^2	$-\ 12$
	ce^5f	$+\ 5$
	d^4f^3	$-\ 6$
	$d^3e^2f^2$	$+\ 12$
⋮	d^2e^4f	$-\ 7$
a^4	de^6	$+\ 1$
a^3	b^2cdf^4	$+\ 2$
⋮	ce^2f^3	$-\ 2$
	d^2ef^3	$-\ 7$
	de^3f^2	$+\ 12$
	e^5f	$-\ 5$
	$b\,c^3f^4$	$+\ 3$
	c^2def^3	$-\ 30$
	$c^2e^3f^2$	$+\ 21$
	cd^3f^3	$+\ 44$
	$cd^2e^2f^2$	$-\ 69$
	cde^4f	$+\ 62$
	ce^6	$-\ 28$
	d^4ef^2	$-\ 6$
	d^3e^3f	$-\ 8$
	d^2e^5	$+\ 11$
	$b^0c^4ef^3$	$-\ 6$
	$c^3d^2f^3$	$-\ 11$
	$c^3de^2f^2$	$+\ 96$
	c^3e^4f	$-\ 64$
	$c^2d^3ef^2$	$-\ 66$
	$c^2d^2e^3f$	$-\ 29$
	c^2de^5	$+\ 68$
	cd^5f^2	$+\ 18$
	cd^4e^2f	$+\ 75$
	cd^3e^4	$-\ 78$
⋮	d^6ef	$-\ 27$
a^3	d^5e^3	$+\ 24$
a^2	b^4df^4	$-\ 1$
⋮	e^2f^3	$+\ 1$
	[illegible]	[illegible]
	[illegible]	[illegible]
	$cdef^4$	$+\ 46$
	ce^3f^2	$-\ 30$
	d^3f^3	$-\ 20$
	$d^2e^2f^2$	$+\ 33$
	de^4f	$-\ 48$
	e^6	$+\ 27$
	$b^2c^3ef^3$	$+\ 39$
	$c^2d^2f^3$	$-\ 105$
	$c^2de^2f^2$	$+\ 18$
	c^2e^4f	$-\ 6$
	cd^3ef^2	$+\ 114$
	cd^2e^3f	$-\ 57$
	cde^5	$+\ 12$
	d^5f^2	$-\ 6$
⋮	d^4e^2f	$+\ 3$
	d^3e^4	$-\ 12$

⋮	$b\,c^4df^3$	$+\ 90$
	$c^4e^2f^2$	$-\ 198$
	$c^3d^2ef^2$	$-\ 9$
	c^3de^3f	$+\ 238$
	c^3e^5	$+\ 116$
	$c^2d^4f^2$	$-\ 6$
	$c^2d^3e^2f$	$+\ 108$
	$c^2d^2e^4$	$-\ 513$
	cd^5ef	$-\ 294$
	cd^4e^3	$+\ 513$
	d^7f	$+\ 108$
	d^6e^2	$-\ 153$
	$b^0c^6f^3$	$-\ 27$
	c^5def^2	$+\ 108$
	c^5e^3f	$+\ 194$
	$c^4d^3f^2$	$-\ 42$
	$c^4d^2e^2f$	$-\ 663$
	c^4de^4	$-\ 274$
	c^3d^4ef	$+\ 570$
	$c^3d^3e^3$	$+\ 914$
	c^2d^6f	$-\ 153$
	$c^2d^5e^2$	-1032
⋮	cd^7e	$+\ 486$
a^2	d^9	$-\ 81$
a	b^5cf^4	$+\ 7$
⋮	def^3	$-\ 16$
	e^3f^2	$+\ 9$
	$b^4c^2ef^3$	$-\ 53$
	cd^2f^3	$+\ 104$
	cde^2f^2	$-\ 150$
	ce^4f	$-\ 117$
	d^3ef^2	$-\ 48$
	d^2e^3f	$+\ 138$
	de^5	$-\ 108$
	$b^3c^3df^3$	$-\ 82$
	$c^3e^2f^2$	$+\ 315$
	$c^2d^2ef^2$	$+\ 153$
	c^2de^3f	$-\ 390$
	c^2e^5	$-\ 234$
	[illegible]	[illegible]
	cd^4e^2f	$-\ 308$
	cd^3e^4	$+\ 735$
	d^6ef	$+\ 208$
	d^5e^3	$-\ 283$
	$b^2c^5f^3$	$+\ 27$
	c^4def^2	$-\ 396$
	c^4e^3f	$-\ 337$
	$c^3d^3f^2$	$+\ 222$
	$c^3d^2e^2f$	$+\ 783$
	c^3de^4	$+\ 880$
	c^2d^4ef	$+\ 93$
	$c^2d^3e^3$	-1986
	cd^6f	$-\ 240$
⋮	cd^5e^2	$+1098$
	d^7e	$-\ 144$

⋮	$b\,c^6cf^2$	$+\ 81$
	$c^5d^2f^2$	$-\ 54$
	c^5de^2f	$+\ 570$
	c^5e^4	$-\ 148$
	c^4d^3ef	-1116
	$c^4d^2e^3$	$-\ 527$
	c^3d^5f	$+\ 474$
	$c^3d^4e^2$	$+1662$
	c^2d^6e	-1185
	cd^8	$+\ 243$
	$b^0c^7e^2f$	$-\ 216$
	c^6d^2ef	$+\ 369$
	c^6de^3	$+\ 340$
	c^5d^4f	$-\ 149$
	$c^5d^3e^2$	$-\ 730$
⋮	c^4d^5e	$+\ 488$
a	c^3d^7	$-\ 102$
a^0	b^7f^4	$-\ 2$
⋮	b^6cef^3	$+\ 20$
	d^2f^3	$-\ 24$
	de^2f^2	$+\ 72$
	e^4f	$-\ 54$
	$b^5c^2df^3$	$+\ 16$
	$c^2e^2f^2$	$-\ 129$
	cd^2ef^2	$-\ 108$
	cde^3f	$+\ 72$
	ce^5	$+\ 135$
	d^4f^2	$+\ 84$
	d^3e^2f	$-\ 112$
	$b^4c^4f^3$	$-\ 6$
	c^3def^2	$+\ 240$
	c^3e^3f	$+\ 179$
	$c^2d^3f^2$	$-\ 144$
	$c^2d^2e^2f$	$+\ 306$
	c^2de^4	$-\ 765$
	cd^4ef	$+\ 28$
	cd^3e^3	$+\ 280$
	d^6f	$-\ 88$
	d^5e^2	$+\ 40$
	[illegible]	$-\ 63$
	$c^4d^2f^2$	$+\ 42$
	c^4de^2f	$-\ 798$
	c^4e^4	$+\ 175$
	c^3d^3ef	$-\ 224$
	$c^3d^2e^3$	$+1365$
	$b^3c^2d^5f$	$+\ 368$
	$c^2d^4e^2$	-1025
⋮	cd^6e	$+\ 60$
	d^8	$+\ 30$

⋮	$b^2c^6e^2f$	$+\ 252$
	c^5d^2ef	$+\ 798$
	c^5de^3	$-\ 700$
	c^4d^4f	$-\ 578$
	$c^4d^3e^2$	$-\ 370$
	c^3d^5e	$+\ 880$
	c^2d^7	$-\ 240$
	$b\,c^7def$	$-\ 486$
	c^7e^3	$+\ 60$
	c^6d^3f	$+\ 312$
	$c^6d^2e^2$	$+\ 645$
	c^5d^4e	$-\ 735$
	c^4d^6	$+\ 190$
	b^0c^9cf	$+\ 81$
	c^8d^2f	$-\ 54$
	c^8de^2	$-\ 135$
⋮	c^7d^3e	$+\ 150$
a^0	c^6d^5f	$-\ 40$

$\pm\ 26$
436
3738
9116
6880
± 20196

Table No. 94 (continued).

Coefficient of y. Coefficient of y.

a^4	cd^2f^4	$+\ 1$
⋮	cde^2f^3	$-\ 2$
	ce^4f^2	$+\ 1$
	d^3ef^3	$-\ 3$
	$d^2e^3f^2$	$+\ 8$
⋮	de^5f	$-\ 7$
a^4	e^7	$+\ 2$
a^3	$b^2d^2f^4$	$-\ 1$
⋮	de^2f^3	$+\ 2$
	e^4f^2	$-\ 1$
	$b\,c^2df^4$	$-\ 7$
	$c^2e^2f^3$	$+\ 7$
	cd^2ef^3	$+\ 30$
	cde^3f^2	$-\ 46$
	ce^5f	$+\ 16$
	d^4f^3	$+\ 6$
	$d^3e^2f^2$	$-\ 39$
	d^2e^4f	$+\ 53$
	de^6	$-\ 20$
	$b^0c^4f^4$	$+\ 6$
	c^3def^3	$-\ 44$
	$c^3e^3f^2$	$+\ 20$
	$c^2d^3f^3$	$+\ 11$
	$c^2d^2e^2f^2$	$+\ 105$
	c^2de^4f	$-\ 104$
	c^2e^6	$+\ 24$
	cd^4ef^2	$-\ 90$
	cd^3e^3f	$+\ 82$
	cd^2e^5	$-\ 16$
	d^6f^2	$+\ 27$
⋮	d^5e^2f	$-\ 27$
a^3	d^4e^4	$+\ 6$
a^2	b^3cdf^4	$+\ 12$
⋮	ce^2f^3	$-\ 12$
	d^2ef^3	$-\ 21$
	de^3f^2	$+\ 30$
	e^5f	$-\ 9$
	$b^2c^3f^4$	$-\ 12$
	c^2def^3	$+\ 69$
	$c^2e^3f^2$	$-\ 33$
	cd^3f^3	$-\ 96$
	$cd^2e^2f^2$	$-\ 18$
	cde^4f	$+\ 150$
	ce^6	$-\ 72$
	d^4ef^2	$+\ 198$
	d^3e^3f	$-\ 315$
	d^2e^5	$+\ 129$
	$b\,c^4ef^3$	$+\ 6$
	$c^3d^2f^3$	$+\ 66$
	$c^3de^2f^2$	$-\ 114$
	c^3e^4f	$+\ 48$
	$c^2d^3ef^2$	$+\ 9$
	$c^2d^2e^3f$	$-\ 153$
⋮	c^2de^5	$+\ 108$
⋮	cd^5f	$-\ 108$

⋮	cd^4e^2f	$+\ 396$
⋮	cd^3e^4	$-\ 240$
	d^6ef	$-\ 81$
	d^5e^3	$+\ 63$
	$b^0c^5df^3$	$-\ 18$
	$c^5e^2f^2$	$+\ 6$
	$c^4d^2ef^2$	$+\ 6$
	c^4de^3f	$+\ 114$
	c^4e^5	$-\ 84$
	$c^3d^4f^2$	$+\ 42$
	$c^3d^3e^2f$	$-\ 222$
	$c^3d^2e^4$	$+\ 144$
⋮	c^2d^5ef	$+\ 54$
a^2	$c^2d^4e^2$	$-\ 42$
a	b^5df^4	$-\ 5$
⋮	e^2f^3	$+\ 5$
	$b^4c^2f^4$	$+\ 7$
	$cdef^3$	$-\ 62$
	ce^3f^2	$+\ 48$
	d^3f^3	$+\ 64$
	$d^2e^2f^2$	$+\ 6$
	de^4f	$-\ 117$
	e^6	$+\ 54$
	$b^3c^3ef^3$	$+\ 8$
	$c^2d^2f^3$	$+\ 29$
	de^2f^2	$+\ 57$
	e^4f	$-\ 138$
	cd^3ef^2	$-\ 238$
	cd^2e^3f	$+\ 390$
	cde^5	$-\ 72$
	d^5f^2	$-\ 194$
	d^4e^2f	$+\ 337$
	d^3e^4	$-\ 179$
	$b^2c^4df^3$	$-\ 75$
	$c^4e^2f^2$	$-\ 3$
	$c^3d^2ef^2$	$-\ 108$
	c^3de^3f	$+\ 308$
	c^3e^5	$+\ 112$
	$c^2d^4f^2$	$+\ 663$
	$c^2d^3e^2f$	$-\ 783$
	$c^2d^2e^4$	$-\ 306$
	cd^5ef	$-\ 570$
	cd^4e^3	$+\ 798$
	d^7f	$+\ 216$
	d^6e^2	$-\ 252$
	$b\,c^6f^3$	$+\ 27$
	c^5def^2	$+\ 294$
	c^5e^3	$-\ 208$
	$c^4d^2e^2f$	$-\ 93$
	$c^4d^3f^2$	$-\ 570$
	c^4de^4	$-\ 28$
	c^3d^4ef	$+1116$
	$c^3d^3e^3$	$+\ 224$
	c^2d^6f	$-\ 369$
	cd^5e^2	$-\ 798$
⋮	cd^7e	$+\ 486$
⋮	d^9	$-\ 81$

⋮	$b^0c^7ef^2$	$-\ 108$
⋮	$c^6d^2f^2$	$+\ 153$
	c^6de^2f	$+\ 240$
	c^6e^4	$+\ 88$
	c^5d^3ef	$-\ 474$
	$c^5d^2e^3$	$-\ 368$
	c^4d^5f	$+\ 149$
	$c^4d^4e^2$	$+\ 578$
⋮	c^3d^6e	$-\ 312$
a	c^2d^8	$+\ 54$
a^0	b^6cf^4	$-\ 1$
⋮	def^3	$+\ 28$
	e^3f^2	$-\ 27$
	$b^5c^2ef^3$	$-\ 11$
	cd^2f^3	$-\ 68$
	cde^2f^2	$-\ 12$
	ce^4f	$+\ 108$
	d^3ef^2	$-\ 116$
	d^2e^3f	$+\ 234$
	de^5	$-\ 135$
	$b^4c^3df^3$	$+\ 78$
	$c^3e^2f^2$	$+\ 12$
	$c^2d^2ef^2$	$+\ 513$
	c^2de^3f	$-\ 735$
	cd^4f^2	$+\ 274$
	cd^3e^2f	$-\ 880$
	cd^2e^4	$+\ 765$
	d^5ef	$+\ 148$
	d^4e^3	$-\ 175$
	$b^3c^5f^3$	$-\ 24$
	c^4def^2	$-\ 513$
	c^4e^3f	$+\ 283$
	$c^3d^3f^2$	$-\ 914$
	$c^3d^2e^2f$	$+1986$
	c^3de^4	$-\ 280$
	c^2d^4ef	$+\ 527$
	$c^2d^3e^3$	-1365
	cd^6f	$-\ 340$
	cd^5e^2	$+\ 700$
	d^7e	$-\ 60$
	$b^2c^6ef^2$	$+\ 153$
	$c^5d^2f^2$	$+1032$
	c^5de^2f	-1098
	c^5e^4	$-\ 40$
	c^4d^3ef	-1662
	$c^4d^2e^3$	$+1025$
	c^3d^5f	$+\ 730$
	$c^3d^4e^2$	$+\ 370$
⋮	c^2d^6e	$-\ 645$
⋮	cd^8	$+\ 135$

⋮	$b\,c^7df^2$	$-\ 486$
⋮	c^7e^2f	$+\ 144$
	c^6d^2ef	$+1185$
	c^6de^3	$-\ 60$
	c^5d^4f	$-\ 488$
	$c^5d^3e^2$	$-\ 880$
	c^4d^6e	$+\ 735$
	c^3d^7	$-\ 150$
	$b^0c^9f^2$	$+\ 81$
	c^8def	$-\ 243$
	c^8e^3	$-\ 30$
	c^7d^3f	$+\ 102$
	$c^7d^2e^2$	$+\ 240$
⋮	c^6d^4e	$-\ 190$
a^0	c^5d^6	$+\ 40$

$\pm$ 12
395
1650
6511
11628

$\pm$20196

Table No. 95 (Covariant V, $=(a, \ldots)^{13}(x, y)^{1}$).

x coefficient.

$a^5\ b^0c^2df^5$	$-$	2
e^2f^4	$+$	2
$c\,d^2ef^4$	$+$	10
de^3f^3	$-$	16
e^5f^2	$+$	6
$c^0d^4f^4$	$-$	6
$d^3e^2f^3$	$+$	12
$d^2e^4f^2$	$-$	10
de^6f	$+$	6
$a^5\ e^8$	$-$	2
$a^4\ b^2c\,df^5$	$+$	4
e^2f^4	$-$	4
$c^0d^2ef^4$	$-$	10
de^3f^3	$+$	16
e^5f^2	$-$	6
$b\,c^3f^5$	$+$	6
c^2def^4	$-$	26
e^3f^3	$+$	8
$c\,d^3f^4$	$+$	32
$d^2e^2f^3$	$-$	116
de^4f^2	$+$	180
e^6f	$-$	78
$c^0d^4ef^3$	$+$	24
$d^3e^3f^2$	$-$	20
d^2e^5f	$-$	44
de^7	$+$	34
$b^0c^4ef^4$	$-$	30
$c^3d^2f^4$	$+$	4
de^2f^3	$+$	240
e^4f^2	$-$	130
$c^2d^3ef^3$	$-$	160
$d^2e^3f^2$	$-$	280
de^5f	$+$	332
e^7	$-$	54
$c\,d^5f^3$	$+$	24
$d^4e^2f^2$	$+$	360
d^3e^4f	$-$	320
d^2e^6	$+$	38
$c^0d^6ef^2$	$-$	108
d^5e^3f	$+$	96
$a^4\ d^4e^5$	$-$	12
$a^3\ b^4c^0df^5$	$-$	2
e^2f^4	$+$	2
$b^3c^2f^5$	$-$	16
$c\,def^4$	$+$	32
[illegible]	$-$	[illegible]
$d^2e^2f^3$	$+$	80
de^4f^2	$-$	160
e^6f	$+$	72
	$\pm$	36
		20
		284
		1094
		2
	$\pm$	184

y coefficient.

$a^5\ b^0c\,d^2f^5$	$-$	2
de^2f^4	$+$	4
e^4f^3	$-$	2
$c^0d^3ef^4$	$+$	6
$d^2e^3f^3$	$-$	16
de^5f^2	$+$	14
$a^5\ e^7f$	$-$	4
$a^4\ b^2\ d^2f^5$	$+$	2
de^2f^4	$-$	4
e^4f^3	$+$	2
$b\,c^2df^5$	$+$	10
e^2f^4	$-$	10
$c\,d^2ef^4$	$-$	26
de^3f^3	$+$	32
e^5f^2	$-$	6
$c^0d^4f^4$	$-$	30
$d^3e^2f^3$	$+$	84
$d^2e^4f^2$	$-$	50
de^6f	$-$	22
e^8	$+$	18
$b^0c^4f^5$	$-$	6
c^3def^4	$+$	32
e^3f^3	$-$	8
$c^2d^3f^4$	$+$	4
$d^2e^2f^3$	$-$	104
de^4f^2	$+$	90
e^6f	$-$	26
$c\,d^4ef^3$	$+$	96
$d^3e^4f^2$	$-$	160
d^2e^5f	$+$	124
de^7	$-$	36
$c^0d^6f^3$	$-$	36
$d^5e^2f^2$	$+$	72
d^4e^4f	$-$	60
$a^4\ d^3e^6$	$+$	18
$a^3\ b^3c\,df^5$	$-$	16
e^2f^4	$+$	16
$c^0d^2ef^4$	$+$	8
e^5f^2	$-$	8
$b^2c^3f^5$	$+$	12
c^2def^4	$-$	116
e^3f^3	$+$	80
$c\,d^3f^4$	$+$	240
$d^2e^2f^3$	$-$	160
de^4f^2	$-$	120
[illegible]	$+$	[illegible]
$c^0d^4ef^3$	$-$	120
$d^3e^3f^2$	$-$	80
d^2e^5f	$+$	368
de^7	$-$	180
	$\pm$	24
		4
		144
		436
		24
	$\pm$	776

Table No 95 (continued)

x coefficient		y coefficient	
$a^3\ b^2c^3ef^4$	$+\ 84$	$a^3\ b\,c^4ef^4$	$+\ 24$
$c^2d^2f^4$	$-\ 104$	$c^3d^2f^4$	$-\ 160$
de^2f^3	$-\ 160$	de^2f^3	$+\ 320$
e^4f^2	$+\ 60$	e^4f^2	$-\ 280$
$c\,d^3ef^3$	$+\ 320$	$c^2d^3ef^3$	$-\ 560$
$d\,e^3f^2$	$+\ 80$	$d\,e^3f$	$+1280$
de^5f	$-\ 496$	de^5f	$-\ 688$
e^7	$+\ 252$	e^7	$+\ 184$
$c^0d^5f^3$	$-\ 72$	$c\,d^5f^3$	$+\ 288$
$d^4e^2f^2$	$-\ 420$	d^4e^2f	$-\ 240$
d^3e^4f	$+\ 860$	d^3e^4f	$-\ 480$
$d\,e^6$	$-\ 404$	d^2e^6	$+\ 264$
$b\,c^4df^4$	$+\ 96$	$c^0d^6ef^2$	$-\ 144$
e^2f^3	$-\ 120$	d^5e^3f	$+\ 336$
$c^3d\,ef^3$	$-\ 560$	d^4e^5	$-\ 144$
de^3f^2	$+\ 160$	$b^0c^5df^4$	$+\ 24$
e^5f	$+\ 304$	e^2f^3	$-\ 72$
$c^2d^4f^3$	$+\ 280$	$c^4d^2ef^3$	$+\ 280$
$d^3e\,f^2$	$+1440$	de^3f^2	$-\ 440$
$d\,e^4f$	$-\ 960$	e^5f	$+\ 400$
de^6	$-\ 376$	$c^3d^4f^3$	$-\ 140$
$c\,d\,ef^2$	-1296	$d\,e^4f$	$+\ 40$
d^4e^3f	$+\ 80$	de^6	$-\ 368$
d^3e^5	$+\ 832$	$c^2d\,ef$	$+\ 108$
$c^0d\,f^2$	$+\ 432$	$d\,e^3f$	$-\ 40$
$d^6e\,f$	$-\ 72$	d^3e^5	$+\ 376$
d^5e^4	$-\ 240$	$c\,d^6e\,f$	$-\ 36$
$b^0c^6f^4$	$-\ 36$	d^5e^4	$-\ 168$
c^5def^3	$+\ 288$	$\iota^3\ \ c^0d^7e^3$	$+\ 36$
$e\,f$	$-\ 56$		
$c^4d^3f^3$	$-\ 140$		
$d\,e\,f^2$	$-\ 480$		
de^4f	$+\ 420$		
e^6	$-\ 276$		
$c^3d^4ef^2$	$+\ 420$		
$d^3e\,f$	-1120		
$d\,e$	$+1112$		
$c\,d^5f$	$-\ 144$		
d^5e^2f	$+1620$	$b\,c^0df^5$	$+\ 6$
$d\,e^4$	-1620	e^2f^4	$-\ 6$
$c\,d\,ef$	$-\ 864$	b^4c^2f	$-\ 10$
d^6e^3	$+\ 876$	$c\,def^4$	$+\ 180$
c^0d^9f	$+\ 162$	e^3f^3	$-\ 160$
$a^3\ \ d^8e^2$	$\pm\ 162$	$c^0\,l\,f^4$	$-\ 130$
		$d\,e\,f$	$+\ 60$
		de^4f	$+\ 60$
$a^2\ b^2c\,f^5$	$+\ 14$	$b^3c\,ef^4$	$-\ 20$
c^0def^4	$-\ 6$	$c^2d\,f^4$	$-\ 280$
e^3f^3	$-\ 8$	de^2f^3	$+\ 80$
$b^4c^2ef^4$	$-\ 50$	e^4f^2	$+\ 300$
$c\,d\,f^4$	$+\ 90$	$c\,d\,ef^3$	$+\ 160$
de^2f^3	$-\ 120$	de^3f	$-\ 192$
e^4f^2	$+\ 60$	e	$-\ 108$
	± 1656		± 2696
	3624		1264
	4898		6
	$\pm\ 14$		$\pm\ 300$

Table No. 95 (continued).

x coefficient.

a^2	$b^4c^0d^3ef^3$	$-$	280
$\vdots$	$d^2e^3f^2$	$+$	300
	de^5f	$+$	216
	e^7	$-$	216
	$b^3c^3df^4$	$-$	160
	e^2f^3	$-$	80
	$c^2d^2ef^3$	$+$	1280
	e^5f	$-$	312
	$c\,d^4f^3$	$-$	440
	$d^3e^2f^2$	$-$	2160
	d^2e^4f	$+$	1740
	de^6	$-$	216
	$c^0d^5ef^2$	$+$	2344
	d^4e^3f	$-$	3240
	d^3e^5	$+$	1244
	$b^2c^5f^4$	$+$	72
	c^4def^3	$-$	240
	e^3f^2	$+$	940
	$c^3d^2e^2f^2$	$-$	1320
	de^4f	$-$	2640
	e^6	$+$	908
	$c^2d^4ef^2$	$+$	600
	d^3e^3f	$+$	3360
	d^2e^5	$-$	168
	$c\,d^6f^2$	$-$	1656
	d^5e^2f	$+$	3408
	d^4e^4	$-$	3480
	c^0d^7ef	$-$	1008
	d^6e^3	$+$	1224
	$b\,c^6ef^3$	$-$	144
	$c^5d^2f^3$	$+$	108
	de^2f^2	$-$	768
	e^4f	$-$	700
	$c^4d^3ef^2$	$+$	900
	d^2e^3f	$+$	8160
	de^5	$-$	2148
	$c^3d^5f^2$	$+$	912
	d^4e^2f	$-$	15060
	d^3e^4	$+$	2800
	c^2d^6ef	$+$	6624
	d^5e^3	$+$	2052
	$c\,d^8f$	$-$	918
	d^7e^2	$-$	2304
	c^0d^9e	$+$	486
	$b^0c^7e^2f^2$	$+$	504
	$c^6d^2ef^2$	$-$	576
	de^3f	$-$	2288
	e^5	$+$	1172
	$c^5d^4f^2$	$-$	124
	d^3e^2f	$+$	4336
	d^2e^4	$-$	2540
	c^4d^5ef	$-$	1912
	d^4e^3	$+$	2100
	c^3d^7f	$+$	240
	d^6e^2	$-$	1560
$\vdots$	c^2d^8e	$+$	810
a^2	$c\,d^{10}$	$-$	162
		$\pm$	666
			6608
			10512
			22042
		$\pm$	9162

y coefficient.

a^2	$b^3c^0d^5f^3$	$-$	56
$\vdots$	$d^4e^2f^2$	$+$	940
	d^3e^4f	$-$	1580
	d^2e^6	$+$	756
	$b^2c^4df^4$	$+$	360
	e^2f^3	$-$	420
	$c^3d^2ef^3$	$+$	1440
	de^3f^2	$-$	2160
	e^5f	$+$	984
	$c^2d^4f^3$	$-$	480
	$d^3e^2f^2$	$-$	1320
	d^2e^4f	$+$	2040
	de^6	$-$	732
	$c\,d^5ef^2$	$-$	768
	d^4e^3f	$+$	2640
	d^3e^5	$-$	1440
	$c^0d^7f^2$	$+$	504
	d^6e^2f	$-$	1296
	d^5e^4	$+$	648
	$b\,c^6f^4$	$-$	108
	c^5def^3	$-$	1296
	e^3f^2	$+$	2344
	$c^4d^3f^3$	$+$	420
	$d^2e^2f^2$	$+$	600
	de^4f	$-$	3420
	e^6	$-$	1172
	$c^3d^4ef^2$	$+$	900
	d^3e^3f	$-$	1280
	d^2e^5	$+$	6360
	$c^2d^6f^2$	$-$	576
	d^5e^2f	$+$	1668
	d^4e^4	$-$	6420
	$c\,d^7ef$	$-$	576
	d^6e^3	$+$	2988
	c^0d^9f	$+$	162
	d^8e^2	$-$	594
	$b^0c^7ef^3$	$+$	432
	$c^6d^2f^3$	$-$	144
	de^2f^2	$-$	1656
	e^4f	$-$	1516
	$c^5d^3ef^2$	$+$	912
	d^2e^3f	$+$	7312
	de^5	$+$	2344
	$c^4d^5f^2$	$-$	124
	d^4e^2f	$-$	8020
	d^3e^4	$-$	10100
	c^3d^6ef	$+$	3792
	d^5e^3	$+$	14648
	c^2d^8f	$-$	702
	d^7e^2	$-$	10296
$\vdots$	$c\,d^9e$	$+$	3564
a^2	c^0d^{11}	$-$	486
		$\pm$	2236
			8616
			15442
		$\pm$	33044

Table No. 95 (continued).

x coefficient.		y coefficient.	
$a\ b^7c^0f^5$	$-\ 4$	$a\ b^6c\,f^5$	$+\ 6$
$b^6c\,ef^4$	$-\ 22$	c^0def^4	$-\ 78$
$c^0d^2f^4$	$-\ 26$	e^3f^3	$+\ 72$
de^2f^3	$+\ 76$	$b^5c^2ef^4$	$-\ 44$
$b^5c^2df^4$	$+\ 124$	$c\,d^2f^4$	$+\ 332$
e^2f^3	$+\ 368$	de^2f^3	$-\ 496$
$c\,d^2ef^3$	$-\ 688$	e^4f^2	$+\ 216$
de^3f^2	$-\ 192$	$c^0d^3ef^3$	$+\ 304$
$c^0d^4f^3$	$+\ 400$	$d^2e^3f^2$	$-\ 312$
$d^3e^2f^2$	$+\ 984$	$b^4c^3df^4$	$-\ 320$
d^2e^4f	$-\ 2160$	e^2f^3	$+\ 860$
de^6	$+\ 1080$	$c^2d^2ef^3$	$-\ 960$
$b^4c^4f^4$	$-\ 60$	de^3f^2	$+\ 1740$
c^3def^3	$-\ 480$	e^5f	$-\ 2160$
e^3f^2	$-\ 1580$	$c\,d^4f^3$	$+\ 420$
$c^2d^3f^3$	$+\ 40$	$d^3e^2f^2$	$-\ 2640$
$d^2e^2f^2$	$+\ 2040$	d^2e^4f	$+\ 2910$
de^4f	$+\ 2910$	de^6	$+\ 540$
e^6	$-\ 810$	$c^0d^5ef^2$	$-\ 700$
$c\,d^4ef^2$	$-\ 3420$	d^4e^3f	$+\ 1840$
d^3e^3f	$+\ 4800$	d^3e^5	$-\ 1530$
d^2e^5	$-\ 3510$	$b^3c^5f^4$	$+\ 96$
$c^0d^6f^2$	$-\ 1516$	c^4def^3	$+\ 80$
d^5e^2f	$+\ 2156$	e^3f^2	$-\ 3240$
d^4e^4	$-\ 430$	$c^3d^3f^3$	$-\ 1120$
$b^3c^5ef^3$	$+\ 336$	$d^2e^2f^2$	$+\ 3360$
$c^4d^2f^3$	$-\ 40$	de^4f	$+\ 4800$
de^2f^2	$+\ 2640$	e^6	$+\ 2520$
e^4f	$+\ 1840$	$c^2d^4ef^2$	$+\ 8160$
$c^3d^3ef^2$	$-\ 1280$	d^3e^3f	-13360
d^2e^3f	-13360	d^2e^5	$-\ 6000$
de^5	$+\ 3200$	$c\,d^6f^2$	$-\ 2288$
$c^2d^5f^2$	$+\ 7312$	d^5e^2f	$-\ 1312$
d^4e^2f	$-\ 2360$	d^4e^4	$+\ 9360$
d^3e^4	$+\ 3840$	c^0d^7ef	$+\ 1824$
$c\,d^6ef$	$-\ 5344$	d^6e^3	$-\ 2880$
d^5e^3	$+\ 2800$	$b^2c^6ef^3$	$-\ 72$
c^0d^8f	$+\ 1956$	$c^5d^2f^3$	$+\ 1620$
d^7e^2	$-\ 1680$	de^2f^2	$+\ 3408$
$b^2c^6df^3$	$-\ 36$	e^4f	$+\ 2156$
e^2f^2	$-\ 1296$	$c^4d^3ef^2$	-15060
$c^5d^2ef^2$	$+\ 1668$	d^2e^3f	$-\ 2360$
de^3f	$-\ 1312$	de^5	$-\ 9260$
e^5	$-\ 2060$	$c^3d^5f^2$	$+\ 4336$
$c^4d^4f^2$	$-\ 8020$	d^4e^2f	$+15220$
d^3e^2f	$+15220$	d^3e^4	$+19920$
d^2e^4	$+\ 1180$	c^2d^6ef	$-\ 5808$
c^3d^5ef	$+\ 3712$	d^5e^3	-22740
d^4e^3	$-\ 8540$	cd^8f	$-\ 90$
c^2d^7f	$-\ 2952$	d^7e^2	$+10080$
$c\,d^8e$	$+\ 3330$	c^0d^9e	$-\ 1350$
c^0d^{10}	$-\ 810$		
	$-\ 4$		$\pm\ 78$
	$\pm\ 48+\ 28$		852
	$2956-\ 84$		8310
	$11806+140$		30200
	$23924-140$		± 56740
	$\pm 25026+\ 84$		

Table No. 95 (continued).

x coefficient.			y coefficient.		
$a\ b c^7def^2$	$-$	576	$a\ b^1c^7df^3$	$-$	864
e^3f	$+$	1824	e^2f^2	$-$	1008
$c^6d^3f^2$	$+$	3792	$c^6d^2ef^2$	$+$	6624
d^2e^2f	$-$	5808	de^3f	$-$	5344
de^4	$+$	3240	e^5	$+$	1720
c^5d^4ef	$-$	4768	$c^5d^4f^2$	$-$	1912
d^3e^3	$-$	6240	d^3e^2f	$+$	3712
c^4d^6f	$+$	2608	d^2e^4	$+$	4920
d^5e^2	$+$	12440	c^4d^5ef	$-$	4768
c^3d^7e	$-$	8160	d^4e^3	$-$	16520
c^2d^9	$+$	1620	c^3d^7f	$+$	1920
$b^0c^9ef^2$	$+$	162	d^6e^2	$+$	19440
$c^8d^2f^2$	$-$	702	c^2d^8e	$-$	9540
de^2f	$-$	90	cd^{10}	$+$	1620
e^4	$-$	1290	$b^0c^9f^3$	$+$	162
c^7d^3ef	$+$	1920	c^8def^2	$-$	918
d^3e^3	$+$	3640	e^3f	$+$	1956
c^6d^5f	$-$	796	$c^7d^3f^2$	$+$	240
d^4e^2	$-$	5340	d^2e^2f	$-$	2952
c^5d^6e	$+$	3100	de^4	$-$	3440
$a\ c^4d^8$	$-$	600	c^6d^4ef	$+$	2608
			d^3e^3	$+$	8760
			c^5d^6f	$-$	796
			d^5e^2	$-$	9160
			c^4d^7e	$+$	4260
			$a\ c^3d^9$	$-$	720
$a^0\ b^8c^0d^0ef^4$	$+$	18	$a^0\ b^8f^5$	$-$	2
$b^7c\,d\,f^4$	$-$	36	$b^7c\,ef^4$	$+$	34
e^2f^3	$-$	180	$c^0d^2f^4$	$-$	54
$c^0d^2ef^3$	$+$	184	de^2f^3	$+$	252
de^3f^2	$-$	108	e^4f^2	$-$	216
$b^6c^3f^4$	$+$	18	$b^6c^2df^4$	$+$	38
c^2def^3	$+$	264	e^2f^3	$-$	404
e^3f^2	$+$	756	$cdef^3$	$-$	376
$c\,d^3f^3$	$-$	368	de^3f^2	$-$	216
$d^2e^2f^2$	$-$	732	e^5f	$+$	1080
de^4f	$+$	540	$c^0d^4f^3$	$-$	276
$c^0d^4ef^2$	$-$	1172	$d^3e^2f^2$	$+$	908
d^3e^3f	$+$	2520	$d\ e^4f$	$-$	810
d^2e^5	$-$	1350	$b^5c^4f^4$	$-$	12
$b^5c^4ef^3$	$-$	144	c^3def^3	$+$	832
$c^3d^2f^3$	$+$	376	e^3f^2	$+$	1244
de^2f^2	$-$	1440	$c^2d^3f^3$	$+$	1112
e^4f	$-$	1530	$d^2e^2f^2$	$-$	168
$c^2d^3ef^2$	$+$	6360	de^4f	$-$	3510
d^2e^3f	$-$	6000	e^5	$-$	1350
de^5	$+$	1350	cd^4ef^2	$-$	2148
cd^5f^2	$+$	2344	d^3e^3f	$+$	3200
d^4e^2f	$-$	9260	d^2e^5	$+$	1350
d^3e^4	$+$	7200	$c^0d^6f^2$	$+$	1172
c^0d^6ef	$+$	1720	d^5e^2f	$-$	2060
d^5e^3	$-$	1900	d^4e^4	$+$	450
$b^4c^5df^3$	$-$	168	$b^4c^5ef^3$	$-$	240
e^2f^2	$+$	648	$c^4d^2f^3$	$-$	1620
$\pm 25524 - 28$			$\pm$ 39956		
$8818 + 4$			17986		
$+ 18$			$- 2$		
$184 - 140$			$270 + 16$		
$3622 + 476$			$2026 - 56$		
$\pm 19350 - 924$			$\pm 9248 + 112$		

Table No. 95 (continued).

x coefficient.		y coefficient.	
$a^0b^4c^4d^2ef^2$	$-\ 6420$	$a^0b^4de^2f^2$	$-\ 3480$
⋮ de^3f	$+\ 9360$	⋮ e^4f	$-\ 430$
e^5	$+\ 450$	$c^3d^3ef^2$	$+\ 2800$
$c^3d^4f^2$	-10100	d^2e^3f	$+\ 3840$
d^3e^2f	$+19920$	de^5	$+\ 7200$
d^2e^4	-10300	$c^2d^5f^2$	$-\ 2540$
c^2d^5ef	$+\ 4920$	d^4e^2f	$+\ 1180$
d^4e^3	-10100	d^3e^4	-10300
$c\,d^7f$	$-\ 3440$	$c\,d^6ef$	$+\ 3240$
d^6e^2	$+\ 7100$	d^5e^3	$+\ 600$
c^0d^8e	$-\ 750$	c^0d^8f	$-\ 1290$
$b^3c^7f^3$	$+\ 36$	d^7e^2	$+\ 900$
c^6def^2	$+\ 2988$	$b^3c^6df^3$	$+\ 876$
e^3f	$-\ 2880$	e^2f^2	$+\ 1224$
$c^5d^3f^2$	$+14688$	$c^5d^2ef^2$	$+\ 2052$
d^2e^2f	-22740	de^3f	$+\ 2800$
de^4	$+\ 600$	e^5	$-\ 1900$
c^4d^4ef	-16520	$c^4d^4f^2$	$+\ 2100$
d^3e^3	$+23300$	d^3e^2f	$-\ 8540$
c^3d^6f	$+\ 8760$	d^2e^4	-10100
d^5e^2	$-\ 5200$	c^3d^5ef	$-\ 6240$
c^2d^7e	$-\ 5400$	d^4e^3	$+23300$
$c\,d^9$	$+\ 1500$	c^2d^7f	$+\ 3640$
$b^2c^8ef^2$	$-\ 594$	d^6e^2	$-\ 8800$
$c^7d^2f^2$	-10296	$c\,d^8e$	$-\ 750$
de^2f	$+10080$	d^{10}	$+\ 450$
e^4	$+\ 900$	$b^2c^8f^3$	$-\ 162$
c^6d^3ef	$+19440$	c^7def^2	$-\ 2304$
d^2e^3	$-\ 8800$	e^3f	$-\ 1680$
c^5d^5f	$-\ 9160$	$c^6d^3f^2$	$-\ 1560$
d^4e^2	-11900	de^4	$+\ 7100$
c^4d^6e	$+13900$	c^5d^4ef	$+12440$
c^3d^8	$-\ 3150$	d^3e^3	$-\ 5200$
$b\,c^9df^2$	$+\ 3564$	c^4d^6f	$-\ 5340$
e^2f	$-\ 1350$	d^5e^2	-11900
c^8d^2ef	$-\ 9540$	c^3d^7e	$+10800$
de^3	$-\ 750$	c^2d^9	$-\ 2250$
c^7d^4f	$+\ 4260$	$b\,c^9ef^2$	$+\ 486$
d^3e^2	$+10800$	$c^8d^2f^2$	$+\ 810$
c^6d^5e	$-\ 9100$	de^2f	$+\ 3330$
c^5d^7	$+\ 2000$	e^4	$-\ 750$
$b^0c^{11}f^2$	$-\ 486$	c^7d^3ef	$-\ 8160$
$c^{10}def$	$+\ 1620$	d^2e^3	$-\ 5400$
e^3	$+\ 450$	c^6d^5f	$+\ 3100$
c^9d^3f	$-\ 720$	d^4e^2	$+13900$
d^2e^2	$-\ 2250$	c^5d^6e	$-\ 9100$
⋮ c^8d^4e	$+\ 1800$	c^4d^8	$+\ 1800$
a^0 c^7d^6	$-\ 400$	$b^0c^{10}df^2$	$-\ 162$
		e^2f	$-\ 810$
		c^9d^2ef	$+\ 1620$
		de^3	$+\ 1500$
		c^8d^4f	$-\ 600$
		d^3e^2	$-\ 3150$
		⋮ c^7d^5e	$+\ 2000$
		a^0 c^6d^7	$-\ 400$
$\pm 41278+1120$		$\pm 19760-140$	
$51872-\ 868$		$36330+112$	
$43900+\ 420$		$30340-\ 56$	
$20624-\ 116$		$23410-\ 16$	
$\pm\ 3856+\ 14$		$\pm\ 5120-\ 2$	

It may be noticed in regard to the numerical coefficients that we have as follows:—

x coefficient.

$a^5 b^0$		$\pm$ 36	
			$\pm$ 36
$a^4 b^2$		$\pm$ 20	
b^1		284	
b^0		1094	
			$\pm$ 1398
$a^3 b^4$		2	
b^3		184	
b^2		1656	
b		3624	
b^0		4898	
			$\pm$ 10364
$a^2 b^5$		$\pm$ 14	
b^4		666	
b^3		6608	
b^2		10512	
b		22042	
b^0		9162	
			$\pm$ 49004
$a b^7$	$-$ 4	$\pm$	
b^6	$+$ 28	48	
b^5	$-$ 84	2956	
b^4	$+$ 140	11806	
b^3	$-$ 140	23924	
b^2	$+$ 84	25026	
b	$-$ 28	25524	
b^0	$+$ 4	8818	
	$\pm$ 256	$\pm$ 98102	
			$\pm$ 98358
$a^0 b^8$	$+$ 18	$\pm$	
b^7	$-$ 140	184	
b^6	$+$ 476	3622	
b^5	$-$ 924	19350	
b^4	$+$ 1120	41278	
b^3	$-$ 868	51872	
b^2	$+$ 420	43900	
b^1	$-$ 116	20624	
b^0	$+$ 14	3856	
	$\pm$ 2048	$\pm$ 184686	
			$\pm$ 186734
			$\pm$ 345894

y coefficient.

$a^5 b^0$		$\pm$ 24	
			$\pm$ 24
$a^4 b^2$		$\pm$ 4	
b		144	
b^0		436	
			$\pm$ 584
$a^3 b^3$		$\pm$ 24	
b^2		776	
b		2696	
b^0		1264	
			$\pm$ 4760
$a^2 b^5$		$\pm$ 6	
b^4		300	
b^3		2236	
b^2		8616	
b		15442	
b^0		33044	
			$\pm$ 59644
$a b^6$		$\pm$ 78	
b^5		852	
b^4		8310	
b^3		30200	
b^2		56740	
b		39956	
b^0		17986	
			$\pm$ 154122
$a^0 b^8$	$-$ 2		
b^7	$+$ 16	$\pm$ 270	
b^6	$-$ 56	2026	
b^5	$+$ 112	9248	
b^4	$-$ 140	19760	
b^3	$+$ 112	36330	
b^2	$-$ 56	30340	
b	$+$ 16	23410	
b^0	$-$ 2	5120	
	$\pm$ 256	$\pm$ 126504	
			$\pm$ 126760
			$\pm$ 345894

viz. in the x coefficient, the coefficients of a^5b^0 are ± 36, that is the sum of the positive coefficients is $=+36$, and the sum of the negative coefficients is $=-36$. But in ab^6 the coefficients are $+28\pm 48$, that is, the sum of the positive coefficients is $=+76$, and the sum of the negative coefficients is $=-48$; and so in other cases. The total sum is ± 345894, viz. the sum of the positive coefficients and that of the negative coefficients (taken as a positive number) are each $=345894$, and so in the y coefficient there is the same total sum ± 345894; which is as it should be, since there are in a different order the same numerical coefficients.

Article Nos. 347 to 365.—*Sketch of Professor* GORDAN'S *proof for the finite Number,* $=23$, *of the Covariants of a Binary Quintic.*

347. I propose to reproduce the leading points of Professor GORDAN'S proof that the binary quintic $(a, b, c, d, e, f \between x, y)^5$ has a finite system of 23 covariants, viz. a system such that every other covariant whatever is a rational and integral function of these 23 covariants.

348. *Derivation.*—Consider for a moment any two binary quantics ϕ, ψ of the same or different orders, and which may be either independent quantics, or they may be both or one of them covariants, or a covariant, of a binary quantic f. We may form the series of *derivatives*

$$(\phi, \psi)^0=\phi\psi,$$

$$(\phi, \psi)^1=\overline{12}\,\phi_1\psi_2=\partial_x\phi\,.\,\partial_y\psi-\partial_y\phi\,.\,\partial_x\psi,$$

$$(\phi, \psi)^2=\overline{12}^2\phi_1\psi_2=\partial_x^2\phi\,.\,\partial_y^2\psi-2\partial_x\partial_y\phi\,.\,\partial_x\partial_y\psi+\partial_y^2\phi\,.\,\partial_x^2\psi,$$

where, however, there is no occasion to use the notation $(\phi, \psi)^0$ (as this is simply the product $\phi\psi$), and the succeeding derivatives may (when there is no risk of ambiguity) be written more shortly $(\phi\psi)$, $(\phi\psi)^2$, $(\phi\psi)^3$, &c.; in all that follows the word "derivative" (GORDAN'S *Uebereinanderschiebung*) is to be understood in this special sense.

349. The degree of the derivative $(\phi\psi)^k$ is the sum of the degrees of the constituents ϕ, ψ; the order of the derivative is the sum of the orders *less* $2k$; it being understood throughout that the word degree refers to the coefficients, and the word order to the variables. In speaking generally of the covariants or of all the covariants of a quantic f, or of the covariants or all the covariants of a given degree or order, we of course exclude from consideration covariants linearly connected with other covariants (for otherwise the number of terms would be infinite); but unless it is expressly so stated, we do not carry this out rigorously so as to make the system to consist of asyzygetic covariants; viz. it is assumed that the system is complete, but not that it is divested of superfluous terms.

350. *Theorem* A.—The covariants of a quantic f of a given degree m can be all of them obtained by derivation from f and the covariants of the next inferior degree $(m-1)$.

In particular for the degree 1 the only covariant is the quantic f itself; for the degree 2 the covariants are $(ff)^0$, $(ff)^2$, $(ff)^4$, ...: using for a moment β to denote each of these in succession, the covariants of the third degree are $(\beta f)^0$, $(\beta f)^1$, $(\beta f)^2$, ...; and so on.

351. Suppose that the covariants of the second degree $(ff)^0$, $(ff)^2$, $(ff)^4$... are in this order represented by β_1, β_2, β_3 ... then the covariants of the third degree written in the order

$$(\beta_1 f)^0,\ (\beta_1 f),\ (\beta_1 f)^2 \ldots\ (\beta_2 f)^0,\ (\beta_2 f),\ (\beta_2 f)^2 \ldots\ (\beta_3 f)^0,\ (\beta_3 f),\ \beta_3 f)^2 \ldots.$$

may be represented by $\gamma_1, \gamma_2, \gamma_3, \ldots$, the covariants of the fourth degree written in the order

$$(\gamma_1 f)^0, (\gamma_1 f), (\gamma_1 f)^2 \ldots (\gamma_2 f)^0, (\gamma_2 f), (\gamma_2 f)^2 \ldots (\gamma_3 f)^0, (\gamma_3 f), (\gamma_3 f)^2 \ldots$$

may be represented by $\delta_1, \delta_2, \delta_3 \ldots$, and so on: we thus obtain in a definite order the covariants of a given degree m; say, these are $\mu_1, \mu_2, \mu_3, \mu_4, \ldots$: any term μ_3 is said to be a *later* term than the preceding terms μ_1, μ_2, and an *earlier* term than the following ones, μ_5, μ_6, &c.

Observe that each term μ_r is a derivative $(\lambda_q f)^k$, the derivatives of an earlier λ are earlier than those of a later λ; and as regards the derivatives of the same λ, the derivative with a less index of derivation is earlier than that with a greater index of derivation, or, what is the same thing, those are earlier which are of the higher order.

352. The series $\mu_1, \mu_2, \mu_3, \mu_4 \ldots$ is not asyzygetic; we make it so, by considering in succession whether the several terms $\mu_2, \mu_3, \ldots$ respectively are expressible as linear functions of the earlier terms, and by omitting every term which is so expressible. The reduced series thus obtained is called $T_1, T_2, T_3, \ldots$ Observe that not every μ is a T, but that every T is a μ; every T therefore arises from a derivation upon f and a certain term λ; which term λ (supposing the λ series reduced in like manner to $S_1, S_2, S_3, \ldots$) is a linear function of certain of the S's. Each later T is derived from later S's, or it may be from the same S's as an earlier T; viz. if the later T is derived from $(S_1, S_2, \ldots S_\theta)$, then the earlier T is derived, it may be, from $(S_1, S_2, \ldots S_\theta)$, or from $(S_1, S_2, \ldots S_{\theta-k})$, but so that there is not in the series any term later than S_θ.

And if, considering any *T as thus derived from certain of the S's*, and in like manner each of these S's as derived from certain of the R's, and so on, we descend to any preceding series,

$$M_1, M_2, M_3 \ldots\ldots$$

it will appear that the T is derived from a certain number $(M_1, M_2, \ldots M_\phi)$ of the terms of this series.

353. The quadricovariants $(ff)^0, (ff)^2, (ff)^4, \ldots$ are of different orders, and consequently asyzygetic. They form therefore a series such as the T-series, and they may be represented by

$$B_1, B_2, B_3, \ldots\ldots$$

Supposing f to be of the order n, B_1 is of the order $2n$, B_2 of the order $2n-4$, B_3 of the order $2n-8$, and so on. Those terms which are of an order greater than n, are said to be of the form W (agreeing with a subsequent more general definition of W); those which are of an order equal to or less than n, are said to be of the form χ; so that the earlier terms of the B series are W, and the later terms are χ; viz. the χ terms taken in order, beginning with the earliest, are $\chi_1, \chi_2, \chi_3, \ldots$

354. By what precedes any particular T is derived from certain terms $B_1, B_2, \ldots B_\theta$, of the B series. This series, $B_1, B_2, \ldots B_\theta$, may stop short of the terms χ, or it may include a certain number of them, say $\chi_1, \chi_2, \ldots \chi_r$. The terms derived from the χ's are in the sequel denoted by P_χ.

355. Every covariant whatever is a form or sum of forms such as

$$\overline{12}^\alpha\overline{13}^\beta\overline{23}^\gamma \ldots f_1 f_2, \ldots f_m;$$

writing in regard to any such expression

$$\Sigma \text{ ind. } 1=i, \ \Sigma \text{ ind. } 2=j, \ldots$$

(viz. i is the sum of all those indices α, β, &c. which belong to a term containing the symbolic number 1, j the sum of all the indices α, γ, &c. which belong to a term containing the symbolic number 2, and so on) then each of the numbers $i, j, \ldots$ is at most $=n$, that is $n-i$, $n-j$, ... may be any of them $=0$, but they cannot be any of them negative; the degree of the function is $=m$, and its order is $=mn-i-j\ldots$ It is to be further observed that the form is a function of the differential coefficients of f of the orders $n-i$, $n-j$, &c. respectively. It follows that if $n-i$, $n-j$, ... are none of them $=0$, the form in question may be obtained from a like form belonging to a quantic f' of the next inferior order $n-1$ by replacing therein the coefficients $a', b', \ldots$ by $ax+by$, $bx+cy$, &c. respectively: for example, if f denote the cubic function $(a, b, c, d\between x, y)^3$, then the Hessian hereof is $\overline{12}^2 f_1 f_2$; the like form in regard to the quadric $f'=(a', b', c'\between x, y)^2$ is $\overline{12}^2 f'_1 f'_2$, which is $=a'c'-b'^2$; and substituting herein $ax+by$, $bx+cy$, $cx+dy$ for a', b', c' respectively, we have the Hessian $\overline{12}^2 f_1 f_2$ of the cubic. A covariant of f derivable in this manner from a covariant of the next inferior quantic f' is said to be a special covariant.

356. Reverting to the form

$$\overline{12}^\alpha\overline{13}^\beta\overline{23}^\gamma \ldots f_1 f_2 \ldots f_m;$$

if, as before, $n-i$, $n-j$, &c. are each of them >0; if there is at least one index i which is $=$ or $<\frac{1}{2}n$ (that is, for which $n-i>\frac{1}{2}n$), and if the order $mn-i-j\ldots$ be $>n$, then the form, or any sum of such forms, is said to be a form or covariant W. Every covariant W is thus a special covariant, but not conversely. In the particular case $m=2$, the form is

$$\overline{12}^\alpha f_1 f_2,$$

which will be a form W if $n-\alpha>\frac{1}{2}n$, or, what is the same thing, $2n-2\alpha>n$, that is if the order be $>n$. Hence, as already mentioned, the covariants T of the degree 2 are W, or else χ, according as the order is greater than n, or as it is equal to or less than n.

357. *Theorem* B.—If any covariant T be expressible as the sum of a form W and of earlier T's than itself, then forming the derivative $(\mathrm{T}f)^k$, either this is not a form T, or being a form T, it is expressible as the sum of a form W and of earlier T's than itself; or, what is the same thing, $(\mathrm{T}f)^k$, if it be a form T, is (like the original T) the sum of a form W and of earlier T's than itself.

Hence also every form T is the sum of a form W, and of forms derived from the functions $\chi_1, \chi_2, \ldots$, say

$$\mathrm{T}=\mathrm{W}+\mathrm{P}_\chi,$$

or, what is the same thing, every covariant whatever is of the form $\mathrm{W}+\mathrm{P}_\chi$.

358. The proof that for a form f of the order n the number of covariants is finite, depends on the assumption that the number is finite for a form f' of the next inferior order $n-1$: this being so, the number of the special covariants of f will be finite; say these are $A_1, A_2, A_3 \ldots$ (f is itself one of the series, but we may separate it, and speak of the form f and its special covariants): the forms W are functions of the special covariants, and hence every covariant whatever of f is of the form $F(A)+P_\chi$; but it requires still a long investigation to pass from this to the theorem of the existence of a finite number of forms V such that every covariant whatever is F(V). I pass this over, and reproduce only the investigation for the case of the quintic.

359. Starting from the assumed system of forms,

$$f,\ \phi=(ff)^2,\ i=(ff)^4,\ j=(fi)^2,\ \alpha=(ji)^2,\ p=(\phi i)^2,\ \tau=(pi)^2,\ \gamma=(\tau\alpha),$$
$$(f\phi),\ (fp),\ (f\tau),\ (j\tau),$$
$$(fi),\ (\phi i),\ (ji),\ (pi),\ (\tau i),$$
$$(i\alpha),\ (i\gamma),\ (ii)^2,\ ((i\alpha),\ \alpha),\ (i\tau)^2,\ ((i\alpha),\ \gamma),$$

say, the 23 forms U, it is to be shown that every other covariant whatever of the quintic is of the form F(U).

The special covariants are f, ϕ, $(f\phi)$, i, j, which are forms U; the only form χ is i, so that instead of P_χ writing P_i, every covariant whatever of f is

$$=F(U)+P_i\,;$$

so that it remains to show that every form P_i is F(U); or, what is the same thing, that if H be any form F(U) whatever, then that (Hi) and $(Hi)^2$ are each of them F(U).

360. In order to show that every covariant of a degree not exceeding m is F(U), it will be sufficient to show that the several forms (Hi) and $(Hi)^2$ of a degree not exceeding m are each of them F(U); and if for this purpose we assume that it is shown that every covariant of a degree not exceeding $m-1$ is F(U), then in regard to the forms (Hi) and $(Hi)^2$ of the degree m, it will be sufficient to show that any such form is a function of *covariants of a degree inferior to m.*

361. First for the form (Hi): we have $(PQ,\ i)=P(Qi)+Q(Pi)$; and hence we see *that (Hi) will be F(U) if only (Ui) is always F(U).*

In forming the derivative of i with the several covariants U, we may omit i itself, and also the four invariants $(ii)^2$, $(i\tau)^2$, $((i\alpha),\ \alpha)$, $((i\alpha),\ \gamma)$, since in each of these cases the derivative is $=0$. We have therefore to consider the derivative of i with

$$f,\ \phi,\ j,\ \alpha,\ p,\ \tau,\ \gamma,\ (f\phi),\ (fp),\ (f\tau),\ (j\tau),\ (fi),\ (\phi i),\ (ji),\ (pi),\ (\tau i),\ (i\alpha),\ (i\gamma),$$

respectively: the first seven of these are each of them U; the remaining eleven are each of them of the form $((PQ),\ i)$. Now $((PQ),\ i)$ is a linear function of $P(Qi)^2$, $Q(Pi)^2$, and $i(PQ)^2$, that is $((PQ),\ i)$ is a function of covariants of a lower degree than itself.

362. Next for the form $(Hi)^2$, we have $(PQ,\ i)^2$, a linear function of $P(Qi)^2$, $Q(Pi)^2$, $i(PQ)^2$; and we hence see that $(Hi)^2$ will be F(U) if only $(Ui)^2$ is always F(U).

In forming the second derivative of i with the several covariants U, we may omit as before the four invariants, and also omit the four linear covariants α, $i\alpha$, γ, $i\gamma$; we have therefore to consider the second derivatives of i with

$$f,\ \phi,\ i, j,\ p,\ \tau,\ (f\phi),\ (fp),\ (f\tau),\ (j\tau),\ (fi),\ (\phi i),\ (ji),\ (pi),\ (\tau i)$$

respectively: the first six of these are each of them U; the remaining nine are each of the form $((\mathrm{PQ}), i)^2$. Now $((\mathrm{PQ}), i)^2$ is a linear function of $((\mathrm{P}i)^2, \mathrm{Q})$, $((\mathrm{Q}i)^2, \mathrm{P})$, $\mathrm{P}(\mathrm{Q}i)^2$, and $\mathrm{Q}(\mathrm{P}i)^2$. The first two of these are terms of the same form; $(\mathrm{P}i)^2$, as a covariant of a lower degree than $((\mathrm{PQ}), i)^2$, is F(U), and hence $((\mathrm{P}i)^2, \mathrm{Q})$ will be F(U) if only (U, Q) is F(U); Q being here any one of the functions f, ϕ, i, j, p, τ, and U being any one of the functions

$$f,\ \phi,\ i, j,\ p,\ \tau,\ \alpha,\ \gamma,\ (f\phi),\ (fp),\ (f\tau),\ (j\tau),\ (fi),\ (\phi i)\ (ji)\ (pi)\ (\tau i)\ (i\alpha)\ (i\gamma).$$

363. For U equal to any one of the last eleven values, the form is (Q, RS), which is $=\mathrm{R}(\mathrm{QS})+\mathrm{S}(\mathrm{QR})$, and is thus a function of covariants of a lower degree; there remains only the derivatives formed with two of the functions f, ϕ, i, j, p, τ, or of one of these with α or γ. But these are all U other than the derivatives

$$(fj),\ (\phi j),\ (\phi p),\ (\phi\tau),\ (p\tau);\ (f\alpha),\ (\phi\alpha),\ (j\alpha),\ (p\alpha);\ (f\gamma)\ (\phi\gamma)\ (j\gamma),\ (p\gamma),\ (\tau\gamma),$$

and since $\gamma=(\tau\alpha)$, the derivatives containing γ will depend upon covariants of a lower degree; there remain therefore only (fj), (ϕj), (ϕp), $(\phi\tau)$, (pr); $(f\alpha)$, $(\phi\alpha)$, $(j\alpha)$, $(p\alpha)$: each of these can be actually calculated in the form F(U).

Hence finally, assuming that every covariant of a degree inferior to m is F(U), it follows that every covariant of the degree m is F(U); whence every covariant whatever is F(U), viz. it is a rational and integral function of the 23 covariants U.

364. It will be observed that, writing A, B, C for P, Q, i, the proof depends on the theorems

((AB), C), a linear function of			$\mathrm{A}(\mathrm{BC})^2$,	$\mathrm{B}(\mathrm{CA})^2$,	$\mathrm{C}(\mathrm{AB})^2$,
$(\mathrm{AB}, \mathrm{C})^2$	,,	,,	do.	do.	do.
$((\mathrm{AB}), \mathrm{C})^2$	,,	,,	$((\mathrm{AC})^2, \mathrm{B})$,	$((\mathrm{BC})^2, \mathrm{A})$,	$\mathrm{B}(\mathrm{AC})^3$, $\mathrm{C}(\mathrm{AB})^3$,

which are theorems relating to any three functions A, B, C whatever.

365. I remark upon the proof that the really fundamental theorem seems to be that which I have called theorem A. As to the forms W it is difficult to see *à priori* why such forms are to be considered, or what the essential property involved in their definition is; and in fact in a more recent paper, "Die Simultanen Systeme binären Formen" (Clebsch and Neumann, t. 2 (1869), see p. 256), Professor GORDAN has modified the definition of the forms W by omitting the condition that the order of the function shall exceed n; if it were possible further to omit the condition of at least one index being $=$ or $<\frac{1}{2}n$, and so only retain the conditions $n-i$, $n-j$, &c., each of them >0, then the essential property of the forms W would be that any such form was a rational and integral function of the special covariants formed, as above, by means of the quantic of the next inferior order. And moreover, as regards the theorem B, there seems something

indirect and artificial in the employment of such a property; one sees no reason why, when a system of irreducible covariants is once written down, it should not be possible to show that the derivatives of F(U) with the original quantic f are each of them F(U), instead of having to show this in regard to the derivatives of F(U) with the several covariants χ: as regards the quintic, where there is a single covariant χ, the *quadric* function i, there is obviously a great abbreviation in this employment of i in place of f; but for the higher orders, assuming that the proof could be conducted by means of the quantic f itself, it does not appear that there would be even an abbreviation in the employment in its stead of the several covariants χ. The like remarks apply to the proof in the last-mentioned paper. I cannot but hope that a more simple proof of Professor GORDAN'S theorem will be obtained—a theorem the importance of which, in reference to the whole theory of forms, it is impossible to estimate too highly.

III. *On Supersaturated Saline Solutions.*—Part II. *By* CHARLES TOMLINSON, *F.R.S.*

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I HAVE already in former papers considered the conditions under which gas[1] or steam[2] or salt[3] is separated from its supersaturated solution, and have endeavoured to show that a body is *active* or *inactive* as a nucleus, according as it is chemically *unclean* or *clean*. An objection to these terms has been started, on the ground that a stick of tallow, for example, may be as chemically clean as a catharized glass rod. In the first Section of this paper an attempt is made to define with rigour the terms *clean* and *unclean*, and to settle the conditions on which nuclei really act. In the second place, an attempt is made to confirm the conclusion arrived at in Part I., that supersaturation depends mainly on the absence of a nucleus, by a number of examples in which highly supersaturated saline solutions, when reduced to temperatures at and below the zero of FAHRENHEIT's scale, rather solidify than crystallize, and in melting reassume the condition of clear, bright, supersaturated solutions.

SECTION I.—*On the Functions of Nuclei.*

I have already endeavoured to show that the obscure and often contradictory behaviour of solids as nuclei in separating gas or vapour or salt from their supersaturated solutions, becomes clear by considering whether the solids used as nuclei were or were not *chemically clean* as to surface at the moment of contact with the solution into which they were placed.

A *nucleus* was defined as a body that has a stronger attraction for the gas or the vapour or the salt of a solution than for the liquid that holds it in solution.

A body is *chemically clean* the surface of which is entirely free from any substance foreign to its own composition.

It is to be observed that I speak of *surface* only, and shall hereafter omit the expression of surface in referring to nuclei. I call, for instance, a glass rod chemically clean, although a particle of carbon or of oxide of iron or other matter be enclosed and shut up within it; but not so if that particle reach and form a portion of the surface itself. So, also, I call a stick of tallow, stearine, paraffine, &c. chemically clean, so long as its surface falls under the definition just given.

In like manner, liquid oils, both fixed and volatile, and other liquids are chemically clean, provided they are chemically pure, and contain no substance, mixed or dissolved,

[1] Phil. Mag., August and September, 1867. [2] Proceedings of the Royal Society, No. 108, 1869.

[3] Transactions of the Royal Society, 1868, p. 659.

that is foreign to their composition. But in considering such bodies, there is a distinction to be made with respect to their nuclear functions, whether they exist in the mass, such as a lens or globule, or in the form of thin films.

Catharization[4] is the act of clearing the surface of bodies from all alien matter; and the substance is said to be *catharized* when its surface is so cleared.

As every thing exposed to the air or to the touch takes more or less a deposit or film of foreign matter, substances may be conveniently classed as *catharized* or *uncatharized* according as they have been, or not, so freed from foreign matter.

And it is perhaps not taking too much licence with language to extend the term *catharized* (denoting, as it does, the condition of pure surface) to those substances whose surface has not required the process. Thus a flint stone in the rough has an uncatharized surface; but split it and the inner surface of the pieces will, for a time, be clean. There can be no impropriety in speaking of the new surfaces as in a catharized or chemically clean state.

Referring to the definition of a nucleus, substances may be divided into *nuclear* and *non-nuclear*.

The nuclear are those that may, *per se*, become nuclei; the non-nuclear are those that have not that quality.

The nuclear substances would seem to be comparatively few, the larger number of natural substances ranking under the other division.

Under nuclear substances are included those vapours and oily or other liquids that form thin films on the surfaces of liquids and solids; and generally all substances in the form of film, and only in that form. Thus a stick of tallow, chemically clean, will not act, but a film of it will act powerfully; and, again, a globule of castor-oil will not act if chemically clean, but in the form of a film, whether chemically clean or not, it will act powerfully.

If a drop of a liquid be placed on the surface of another liquid, it may mingle with it; or it will either spread out into a film or remain in a lenticular shape, according to the general proposition that if a drop of a liquid B, whose surface-tension is b, be placed on the surface of another liquid A, whose surface-tension is a, the drop will spread into a film, if $a>b+c$ (c being the tension of the common surface of the liquids A and B); but if, on the contrary, $a=<b+c$, the drop will remain in the form of a lens. Hence if B spread on A, A will not spread on the surface of B. c has no value whenever the liquids A and B mingle in all proportions, as in the case of water and alcohol.

In the case of supersaturated saline solutions, the spreading of the drop may in some cases be slow, on account of the *superficial viscosity*, or the greater or less difficulty of the superficial molecules to be displaced.

A glass rod drawn through the hand becomes covered with a thin film, or the same rod by exposure to the air contracts a film by the condensation of floating vapour, dust, &c., and in either case is brought into the nuclear condition.

[4] From καθαρίζω, to purge, purify, or clean—from καθαρὸς, pure, clean.

A second class of nuclear bodies are permanently *porous* substances, such as charcoal, coke, pumice, meerschaum, &c. The action of these is chiefly confined to vaporous solutions, and, if catharized, they have no power of separating salts from their supersaturated solutions.

Under the non-nuclear, forming by far the larger class of substances, are glass, the metals, &c., while their surfaces are chemically clean; and by chemically clean is meant, as already noted, wholly free from any substance foreign to their own composition.

Among the non-nuclear will be found air; for its ascribed nuclear character is due, not to itself, but to the nuclear particles of which it is the vehicle. Thus, as stated in former papers, if air be filtered through cotton-wool, it loses its apparent nuclear action; so also if heated.

I have already published numerous examples from the recorded experiments of OERSTED, SCHÖNBEIN, LIEBIG, LÖWEL, and others; and I have also shown by new experiments[5] that Class II., namely the non-nuclear, are really so, and have only been thought otherwise for want of adverting to the condition to which I first called attention, namely their being in a catharized state or not. To run through all the substances of which this may be predicated were impossible. To treat of a few will answer the purpose if we show that those which have been thought to have the quality of nucleus do not possess it, but only seem to do so, by having their surfaces soiled with foreign matter or films that really do possess the power. The only answer that such a view admits of as to those substances is manifestly to show that they had been catharized previously to experiment, and yet were found to be active as nuclei.

My former papers contain sufficient details as to the general action of nuclei in separating gases, vapours, and salts from their supersaturated solutions. When the nucleus is catharized or clean, it has no such separating action, because the solution adheres to it as a whole, that is, there is the same adhesion between the gas or vapour or salt and the nucleus as between the liquid part of the solution and the nucleus; but when the nucleus is non-catharized or unclean, the gas or vapour or salt adheres to the surface more strongly than the liquid part of the solution, and hence there is a separation of gas or vapour or salt from solution. My present purpose is to show that an active nucleus, or unclean or uncatharized surface, is contaminated with a film of foreign matter, which filmy condition is necessary to that close adhesion which brings about the nuclear action; for it can be shown that a liquid, such as an oil, is non-nuclear in its action when in the form of a lens or globule, but powerfully nuclear when in the form of a film.

There are certain liquids which form films, and act as nuclei by separating water instead of salt from supersaturated solutions. Absolute alcohol acts in this way. Other liquids, such as glycerine, diffuse through the solution without acting as nuclei. Saline solutions may also saponify fatty oils, or otherwise act chemically upon them,

[5] Journal of the Chemical Society for April 1869. Chemical News, 1868, 1869, 1870. See also notes (1), (2), (3).

without any separation of salt due to nuclear action. Cases of this sort will be illustrated experimentally.

In preparing the flasks and other apparatus for the exhibition of the properties referred to, attention must be paid to cleansing, by means of caustic alkali, sulphuric acid, or spirits of wine, as insisted on in former papers; but in the experiments on the action of nuclei the flasks are not to be plugged with cotton-wool, but merely covered loosely with watch-glasses. Solutions may thus be kept supersaturated in a quiet room during weeks, and even months. Cotton-wool answers admirably where the flasks have to be kept, as it were, in store, and then opened only once for the purpose of determining crystallization; but when the flask is to be opened with a view to the insertion of a nucleus, cotton-wool is objectionable for several reasons, such as the liability to detach a few fragments in taking out the plug; and this is certain to happen if in filtering the solution a little of it get into the neck and become entangled and crystallized with the fibres; moreover, in taking out the cotton-wool, an equivalent volume of air enters the flask, and this is almost certain to deposit a nucleus on the solution. In the following experiments a solution of one, two, or three parts of GLAUBER's salt to one of water was made in a large flask, and filtered while boiling into a number of three-ounce flasks made clean. Each flask received about two ounces of the solution, and was covered with a watch-glass, and left to cool during some hours, or until the next day.

Experiment 1. Four ounces of GLAUBER's salt in four ounces of water was boiled and filtered into four flasks, covered with watch-glasses and left till cold. A clean glass rod was dipped into a bottle of clear pale seal-oil, and the watch-glass being gently removed from off one flask, a drop of the oil was carefully delivered to the surface of the solution; the glass rod was withdrawn, and the watch-glass restored to its place. The drop of oil expanded into a well-shaped film, with a display of iridescent rings; and immediately from the lower surface of the film there fell large well-shaped flat prisms with dihedral summits of the ten-atom sodic sulphate, being the normal salt. The prisms were an inch or an inch and a half in length, and three eighths of an inch across. The crystallization proceeded from every part of the lower surface of the film, and as one set of crystals fell off another set was formed, until the whole solution became a mass of fine crystals in a small quantity of liquid. This effect is entirely different from the usual crytallization which takes place when a supersaturated solution of GLAUBER's salt is subjected to the action of a nucleus at one or two points in its surface, as when motes of dust enter from the air, or the surface is touched with a nuclear body. In such cases small crystalline needles diverge from the point touched, and proceed rapidly in well-packed lines to the bottom, the whole being too crowded and too rapid to allow of the formation of regular crystals. But in the case before us, where the whole surface of the solution, and the surface only, is subjected to nuclear action by the spreading of the oil-film, the action is not so rapid as in the former case, because it is not downwards but parallel with the surface; the crystals mould themselves, as it were, upon the oil-

film, and falling off, allow fresh portions of the solution to come into contact with the film, until at length a very fine crop of crystals is produced. This experiment was repeated on the solutions in the other three flasks with the same result.

Similar experiments were tried on solutions of GLAUBER's salt of different strengths, with drops of ether, absolute alcohol, naphtha, benzole, oil of cajeput, oil of turpentine, and other volatile oils, herring-oil, sperm-oil, olive-oil, castor-oil, and some other fixed oils of animal and vegetable origin, with this general result, that, whenever the oil spread out into a film, it acted as a powerful nucleus.

Experiment 2. A solution of two parts of GLAUBER's salt to one part of water[6] was boiled and filtered into three flasks, which were covered with watch-glasses and left until the next day. A drop of castor-oil was then placed upon the surface of each: it formed a lens which gradually flattened; but there was no separation of salt, even when the flasks were shaken so as to break up the oil into small globules.

The necks of some clean flasks were oiled inside with castor-oil, and the boiling solution of GLAUBER's salt filtered into them: the oil formed into globules, and some of it ran into the solution. Next day there was no separation of salt. The flask was turned round quickly so as to form a whirlpool, in which the globules of oil descended to the bottom and were broken up and diffused so as to form a kind of emulsion; but by repose the solution became clear again, the oil collected into larger globules, and there was no separation of salt.

If, while the flask is being turned round, a sudden jerk be given to it, so as to flatten some of the globules against the side into films, the whole solution instantly becomes solid; or if by strong pressure with the finger against the interior of the neck a smear of oil be made, the solution crystallizes as soon as it is brought into contact with such smear. But if it be objected that the finger may introduce nuclear matter, then a catharized wire, which dipped into the solution is inactive, will, if rubbed against the side so as to flatten some of the small oil-globules, leave a number of minute crystals of a chalky white colour filling up the lines traced by the wire, and these serve as nuclei to the rest of the solution.

Experiment 3. Stearine from sheep-tallow produced immediate crystallization in supersaturated solutions of GLAUBER's salt. The solutions were gently heated and gradually raised to boiling. In cooling, the stearine, now catharized, collected into disks; but it was no longer nuclear, even when the vessel was shaken. When the flask was uncovered, a nucleus from the air produced immediate crystallization.

Of course, unless the oils be chemically clean, according to the definition, they induce crystallization, whether they form a lens or a film on the surface of the solution. Thus a specimen of sperm-oil placed on the centre of the surface of a supersaturated

[6] This solution, it will be seen, contained twice as much salt as the solution used in Experiment 1, and hence was much more sensitive to the action of nuclei. This sensitiveness also increases as the temperature falls; and at low temperatures the crystals are more rapidly formed, more crowded and hence less distinct, than when formed from weaker solutions and in summer weather, as in Experiment 1.

solution of GLAUBER's salt produced a well-shaped lens, from the under convex surface of which crystals of the salt radiated rapidly downwards. The flask was heated and gradually raised to near the boiling-point, and in cooling the oil formed a number of lenticular disks on the surface; but they produced no separation of the salt, even when the flask was shaken.

Also in the case of volatile oils containing products of oxidation, dust, &c., they may form lenses on the surface of the solutions and yet act powerfully as nuclei. If such oils be redistilled they still form lenses on the surface, but are no longer nuclear.

Experiment 4. A dark-coloured oil of bitter almonds formed a lens on the surface of a solution of GLAUBER's salt, and the lens immediately became bristled with crystals. The crystallization was so vigorous that crystals projected above the surface, supporting the oil, as it were, on a platform. The oil was distilled into a limpid colourless liquid, of which about twenty drops were allowed to fall in succession on the surface of a supersaturated solution of GLAUBER's salt. The drops coalesced into a beautiful, lustrous lens; but there was no longer any separation of salt. After half an hour or so the lens became surrounded with an opaque white disk or halo of benzoic acid. On shaking up the solution the oil was diffused in globules through it, and in the course of some days became converted into flakes of benzoic acid, which rotated vigorously on the surface of chemically clean water, after the manner of camphor. There was no separation of salt until the flakes were taken out with a non-catharized spatula, the contact of which with the solution caused it to solidify.

Newly distilled turpentine formed a lens on the surface of a supersaturated solution of GLAUBER's salt, and remained so during some days. A drop of the old oil, from which the new oil had been distilled, was placed on the surface; it spread into a film, displacing the lens and producing immediate crystallization.

Experiment 5. Carbonic disulphide and also chloroform formed well-shaped lenses on the surface of the supersaturated solution of GLAUBER's salt, and slowly evaporated without producing crystallization.

Glycerine and wood-naphtha mixed with the solution, rendering it dull, without separating the salt. Of course, on touching the solutions thus treated with a wire they immediately crystallized.

Experiment 6. A solution of $1\frac{1}{2}$ part of potash-alum and one of water was boiled and filtered into a number of flasks. When cold the solutions showed great viscosity. A drop of a limpid pale seal-oil spread out into a film on the surface of the alum solution. Crystallization set in from every part, and the junction of the planes of crystallization gave the surface the appearance of a central point and six radii forming six angles, some of 30° and some of 60°.

Experiment 7. Oleine of tallow, the oleine of fish-oil, and oleic acid from palm-oil formed well-shaped lenses on the surface of the alum solution without any separation of salt.

Experiment 8. Benzole, naphtha, oleum anethi, turpentine, and some other liquids

formed films, on the under surface of which octahedral crystals of alum were formed.

Experiments were also tried with supersaturated solutions of ammonia alum, of the magnesic sulphate, and the sodic acetate. Although these are not so well adapted as GLAUBER's salt for the display of the phenomena in question, yet by carefully separating anomalous cases, as where a lens of old oil is not chemically clean, or where a mote from the air, in loosely covered vessels, produces crystallization, the result of a large number of experiments with a variety of liquids leads to the conclusion that liquids are nuclear in the form of films, and non-nuclear in the form of lenses, globules, or drops.

In the form of films, as when a drop placed on one of the solutions spreads out and covers the surface with a display of colour, the surface-tension of the solution is so far diminished as to allow the film to come into contact, when that differential kind of action takes place whereby, the salt of the solution adhering more strongly to the film than the water of the solution does, the action of separation and crystallization thus once begun is continued and propagated. A similar action takes place with solid bodies that have contracted filmy nuclei by being drawn through the hand or touched or merely exposed to the air; they are active, or nuclear, by virtue of the films of matter which more or less cover them.

On the other hand, when a drop of oil, or many drops, is placed on the surface of a supersaturated solution, and it assumes the lenticular form, or even flattens down into disks, which more or less cover the surface, such lenses or disks retain their surface-tension, and do not greatly interfere with the tension of the solution on which they rest. The adhesion is, indeed, very different from that of a film, as may be made manifest by pouring, say, thirty or forty drops of freshly distilled oil of turpentine on the chemically clean surface of water in a shallow glass vessel about $3\frac{1}{2}$ inches in diameter. The oil will nearly cover the surface of the water without being in contact, so to speak, with it; for contact is prevented by the surface-tension at the boundary of the two liquids. If, now, upon the oil thus resting on the surface of the water some fragments of camphor be scraped, these will be immediately wetted by the turpentine and be covered with a solution of camphor in that oil. This solution of camphor will form a film with iridescent colours on the surface of the water; a fragment of camphor will sail about, and being bounded on each side by nearly symmetrical films of iridescent colour, it has the appearance of a tropical butterfly. As it moves over the surface of the water it displaces the turpentine, and cuts it up into numerous lenses, until the surface-tension of the water is so far reduced, that no force is left to give rotation to the camphor.

The lens of oil &c. is not then sufficiently in contact with the surface of the supersaturated solution to allow of that differential kind of action taking place whereby salt is separated. Even when, by shaking, the oil is broken up into globules, and these are submerged, the conditions of the case are not greatly altered, since a submerged globule and the solution moulded upon it are separated by surface-tension, which prevents actual

contact. When, however, by a sudden jerk the globules are flattened against the wall of the flask and form films, the solution instantly becomes solid.

A body may also be said to act as a nucleus when the cohesion of its own particles is weaker than its affinity for one or other of the constituents of the solution. In a solid or in a liquid globule the cohesion of its own particles is greater than that affinity; but in a film (which has extension only, length and breadth, the thickness being small enough to be disregarded) cohesion acts but laterally and not in every direction, and therefore the substance, coming into actual contact with the solution, yields to the attraction of one or other of the constituents of the solution for which it has the greater affinity, that attraction acting at right angles to the plane of the cohesive action of the film, and consequently meeting little or no resistance. Separation of the constituents of the solution then sets in. For example, the affinity in the case of a film of alcohol is for the water of the solution, and in the case of an oil-film for its salt, &c.

But if the film have no preferential affinity for one or other of the constituents of the solution no separation results, as when glycerine diffuses through the solution[7].

This would suggest that *form* and affinity were the elements in nuclear action. If a film of glass, for example, could be reduced to a sufficient state of tenuity, would it act as a nucleus? But at any rate it may be supposed that *contact* and affinity are the sole elements.

It will be gathered from the foregoing details that the distinction between *unclean* and *clean* resolves itself into the condition of *filmy* and *globular* in the case of liquids; and when solids act as nuclei, they are more or less contaminated with *films* of matter *foreign to their composition.*

Although in general the surface-tension of those liquids that form lenses, instead of

bles and evaporating to dryness, the same quantity of salt was found to exist at four different parts of a column, 11 inches high, of a supersaturated solution of GLAUBER'S salt.

The internal viscosity of such a solution will probably account for a case mentioned in the next section, where a highly supersaturated saline solution of GLAUBER'S salt was reduced to about 20° FAHR. without any deposit of the 7-atom salt.

SECTION II.—*On the Action of low Temperatures on Supersaturated Saline Solutions.*

I propose in this place to show that the solutions of certain salts which remain liquid and supersaturated at and about the freezing-point of water, by a further reduction in temperature to about the zero of FAHRENHEIT'S thermometer, become solid, but on being restored to the temperature of 32° recover their liquid state without any separation of salt.

Ferrous sulphate ($Fe\,SO_4$, 7Aq) was dissolved in hot water, 1·25 of the salt to 1 of water, and was cooled rapidly, first, by placing the tube containing the filtered solution in water at 50°, then in snow-water, and lastly in a mixture of salt and snow at 0°. Beautiful tetrahedral crystals formed at the surface of the solution, and were propagated downwards, until the contents of the tube became solid. In snow-water at 32° the solid mass shrank from the sides of the tube, formed into a smooth rounded mass, and gradually melted, leaving the solution clear and bright without any deposit. On removing the cotton-wool from the mouth of the tube, small but well-shaped rhomboidal crystals soon filled the solution.

The double salt formed by mixing in atomic proportions solutions of the *zincic* and *magnesic sulphates* retains all the water of crystallization of the constituent salts, namely, 14 equivalents, and hence is well adapted to display the phenomena of supersaturation; since, in general, the more highly hydrated a salt is, the more readily does it become supersaturated. One reason for this is that the water of crystallization adds to the volume of the menstruum and so assists solution. So much is this the case, that while the salt readily undergoes the watery fusion, the addition of a very small quantity of water is sufficient to cause the solution to remain entirely liquid in closed vessels, even under considerable reductions of temperature. On the other hand, if too much water be added to the salt in making the solution, saturation, and not supersaturation, is produced, so that on lowering the temperature the behaviour of the solution is quite different in the two cases. For example, 246·3 grains of the magnesic sulphate, 287 grains of the zincic sulphate, and one ounce of water, boiled and filtered into clean tubes, forms only a saturated solution at ordinary atmospheric temperatures. If such a solution be reduced to 0°, or from that to $-5°$ or $-8°$, a portion of the solution solidifies. It first forms an opaque growing mass of snowy whiteness, adapting itself to the smooth curved surface of the tube, but within, directed towards the axis, covered with well-shaped tetrahedral crystals. If at this low temperature the tube be transferred to a mixture of snow and water, it will of course be thickly encrusted with ice, the mass of

crystals within the tube will become deformed, and rising to the surface, quickly melt; but if before plunging the tube into snow and water it be held for a short time in air, say, at about 50°, the crystals break up, and throw down a quantity of anhydrous salt, which, occupying the bottom of the tube, begins to combine with water and to heat, so that, on putting the tube into snow-water, the middle portion becomes quickly encrusted with ice, as before, while the bottom of the tube for about $\frac{1}{4}$ to $\frac{3}{8}$ of an inch is protected from ice.

By detaining the tube long enough in the freezing-mixture the solution becomes entirely solid, and on transferring it to ice-cold water it rapidly melts into a very bright, highly refractive liquid, without any separation of salt; and the same liquid may be solidified and melted any number of times with the same results, provided the solution be protected from the action of nuclei. If the cotton-wool be taken out of the tube, even when the contents are solid, and be restored again to its place, there will be a separation of salt during the melting, in consequence of a nuclear particle being admitted from the air.

Such is the behaviour of the solution made with one ounce of water. Although not supersaturated at ordinary temperatures, and not sensitive to the action of nuclei, yet it soon reaches a temperature at which it becomes supersaturated, and then it is very sensitive to such action. For example, in the case in hand, when the solution is at 50° or thereabouts, it may be touched, or the tube below the solution rubbed with a wire or a glass rod, and show no disposition to crystallize; but at the temperature of 20° the same nuclei are powerfully active in inducing crystallization, crystals instantly attach themselves to the nucleus, or the part of the tube rubbed becomes at once chalky white, from which mark crystallization spreads[8].

Next, as to the behaviour of the solution made with half an ounce instead of an ounce of water. Such a solution, after being boiled and filtered, remains clear and bright, but retains its viscosity after twenty or thirty hours. Unlike the former solution, made with one ounce of water, this begins to crystallize at the temperature of the room, as soon as the cotton-wool is removed. *The crystallization sets in from the surface in the form of minute needles spreading out like tufts of thistle-down and soon closing in the surface, so that the tube can be inverted, and the heat-currents generated in the viscous fluid consequent on change of state can be seen.*

But if, instead of subjecting such a solution to the action of the aërial nuclei, it be

[8] Supersaturated solutions of GLAUBER's salt &c. are much less sensitive to the action of nuclei in warm than in cold weather. This has led to a good deal of misapprehension as to the real functions of nuclei, and has caused some observers to produce lists of bodies which will or will not induce crystallization; whereas the same body, under the same conditions, only varying with the temperature of the solution, will induce crystallization at one time and not at another. This is strikingly shown in the case of calcic chloride (Ca Cl, 6 aq); the same nuclei which are inactive when the solution is at the temperature of the room act at once when the solution is reduced to from 24° to 34°. A strong solution of this salt in an open evaporating-dish, which under ordinary circumstances becomes weaker by absorbing moisture, will, if the dish be put into a freezing-mixture, give a splendid crop of well-formed crystals.

reduced in temperature to about 16° FAHR., there is a deposit of anhydrous salt, as in the case of GLAUBER'S salt and one or two others, and the subsequent behaviour is similar. Upon this deposit are built rhomboidal plates of a lower degree of hydration than the double salt of the solution. If, now, the cotton-wool be removed and crystallization sets in from the surface, the fully hydrated salt, on coming into contact with the modified salt, renders it opaque without altering its form. If before crystallization sets in the upper part of the solution[9] be poured off, and the crystals be taken out and dried between folds of filtering-paper, with a view to ascertain their state of hydration, they become hot from the fixation of an additional quantity of water, and pass at once into the condition of hydration of the normal salt.

All these phenomena closely resemble the behaviour of a supersaturated solution of GLAUBER'S salt on a proper reduction of temperature. The anhydrous salt passes into solution, a portion of it is thrown down when the solution cools sufficiently (and this varies with the solubility of the salt and the strength of the solution), and it partially enters into solution at the bottom of the vessel; the heat thus generated further assists the solution, until at length a dense solution is formed occupying the lowest stratum, and, in the case of sodic sulphate, containing only water enough to form the 7-watered instead of the usual 10-watered salt. The crystals of the modified salt thus formed occupy the bottom of the vessel, while the solution above is not its mother liquor, as is commonly supposed, but still remains a solution of the anhydrous salt, as may be proved by again lowering the temperature of the solution, when octahedral crystals of the anhydrous salt will again be thrown down. If at any time after the modified salt has been formed the solution be exposed to free air by taking out the cotton-wool plug, crystallization sets in from the surface, and proceeding downwards carries with it sufficient water to convert the 7-atom into the 10-atom salt; and in doing so it effects such a minute state of division as to change the transparent salt into an opaque white.

It is commonly stated that a supersaturated solution of GLAUBER'S salt forms the modified salt at temperatures between 60° and 40° F.; but if the solution be clean and contained in a clean vessel, a highly supersaturated solution, formed of two or three parts salt to one of water, may be reduced to 20° and under without throwing down any salt. Such a solution contained in an 8-oz. globular flask, the globe of which was quite full, was exposed all night to a temperature which fell to a minimum of about 20° F. In the morning the solution was perfectly clear and bright, and there was no deposit of the modified salt; but on shaking the flask it instantly became opaque, from the multitude of small octahedral crystals that filled the solution; these quickly subsided, and within a quarter of an hour a fine crop of the 7-atom crystals was erected upon them, while the solution above still remained that of the anhydrous salt.

The sulphate of zinc and magnesia is rich in water of crystallization, and hence its behaviour is different from that of a double salt containing only a small proportion of combined water, such as the double salt formed by dissolving the *zincic sulphate* in a

[9] This, it should be observed, is not the mother liquor, but a solution of the anhydrous salt.

saturated solution of the *sodic sulphate.* As this double salt contains only four proportionals of water of crystallization, its supersaturated solutions are not very stable; but by mingling the salts in atomic proportions in a small quantity of water (such as 287 grains of zinc sulphate, 322 of the sodic sulphate, and half an ounce of water) so as to ensure supersaturation, raising the temperature to about 100° F. until the solution becomes clear, then boiling and filtering into clean tubes and plugging with cotton-wool, such a solution may be kept and even reduced to 10° F. without any separation of the salt, although on removing the plug for an instant crystallization sets in from the surface in tufts.

But by repose, even in clean tubes and in the absence of nuclei, long crystals of the separated salts occupy the length of the tube; but they are invisible on account of having the same refractive index as that of the solution in which they are immersed. By inverting the tube so as to drain off the liquor, the crystals become visible, air enters the interstices, and on restoring the tube to its original position the bubbles of air have a very solid appearance, and may be taken for amorphous masses of crystals. These air-bubbles may be driven away by gently heating the tube. By further repose of some weeks a portion of the water of the solution evaporates through the cotton-wool, and as the solution becomes more dense, the crystals become more and more visible. These crystals are easily fusible at about 90°.

It may happen that the anhydrous powder thrown down by the zincic sulphate during the cooling is not taken up again. If in this state the tube be suddenly heated, the sodic sulphate will throw down an anhydrous salt, and on attempting to boil the salt there will be much kicking of the tube. But as the temperature rises this kicking ceases, and the mixed anhydrous powders pass rapidly into solution, the one apparently assisting the solubility of the other; for in the case of strong sodic-sulphate solutions alone, when once the anhydrous powder is deposited by heat, the continued application of heat increases the quantity of the deposit and aggravates the *soubresauts.*

If a saturated solution of the double salt be reduced to a low temperature, such as that of 0° F., the anhydrous powder *of GLAUBER'S salt is thrown down; but in very clean* tubes the solution passes into a transparent jelly-like mass that has the appearance of a liquid. A platinum spatula plunged into it while the tube is inverted does not produce any flow of liquid. By repose the pulpy mass becomes crystalline, the mother liquor separates, and the crystals become invisible in it. If the tube be put into warm water at 75° or 80°, the crystals liquefy in a few minutes, but they form again in a freezing-mixture at 14° or 16° F.

The double salt fuses at about 140° into an opaque mass; if raised to about 180° there is much kicking of the vessel, the anhydrous powder separates, and the clear liquor collects above. The powder consists chiefly of anhydrous sodic sulphate. If a little water be added the solution may be boiled, but only a portion of the powder is taken up. By reduction of temperature, this powder, influenced by the solidification of the zinc salt, forms cauliflower masses which creep a little way up the tube. On opening the tube

crystallization sets in from above and proceeds downwards, and the cauliflower masses become chalky white. Thus the presence of the two salts in solution affords a number of new phenomena which vary with the strength of the solution and the amount of reduction in temperature, and also the old phenomena belonging to GLAUBER's salt modified by the presence of the zinc salt.

The *ammonia* and *zinc sulphates* mixed in atomic proportions, boiled with an ounce or two of water, and filtered into an evaporating-dish, form a definite compound with six equivalents of water. It forms supersaturated solutions of moderate strength, but very sensitive to the action of nuclei, a shower of brilliant crystals falling as soon as the tube is opened. Eighty grains of the salt in three drachms of water, boiled and filtered into a clean tube, and exposed to the temperature of 4° F., formed large feathery crystals with a central quill formed by the junction of two planes of crystallization, the effect being very lovely and striking. In snow and water at 32° the crystals balled, ascended, and quickly melted. These alternate effects of solidifying and melting were repeated three times, when the tube was opened and inspected to see if the whole contents were solid. On reinserting the cotton-wool and placing the tube in water at 32°, the solid quickly melted with immediate separation of the salt, the act of opening the tube having introduced a nucleus from the air.

Nickel sulphate with 7 equivalents of water forms supersaturated solutions which resist a cold of 6°. Mixed with an equivalent weight of *cupric sulphate* with 5 equivalents of water of crystallization and one ounce of water, the mixture when hot is not to be distinguished from the solution of nickel sulphate, but when cold a faint shade of blue comes over it. The salts separate in cooling in the air; but the supersaturated solution is solid at 0° F., and produces the beautiful feathery crystals noticed in the case of the solution of the ammonia and zinc sulphates, and, like it, rapidly melts at 32° without any separation of salt, if nuclei be excluded.

A very beautiful display of the leaf-like crystals, and also of the so-called tetrahedral crystals, is furnished by a solution of *zinc sulphate* in one of *potash-alum*. The mixed solution, which contained the salts in atomic proportions, was evaporated to dryness, and 200 grains of the salt was boiled with 5 drachms of water and filtered into a clean tube. The solution was bright and viscous, but, as is usual in such cases, the viscosity became more equally diffused throughout the solution in the course of a few days. At 4° F. tetrahedral crystals formed on the surface, and were propagated downwards until the whole was solid; in snow and water it rapidly melted. The tube was left to repose during a week, when on examining it again it had thrown down a dense white powder, apparently of anhydrous zinc sulphate. On placing the tube in a freezing-mixture at 0°, the powder at the bottom increased in volume and formed acicular crystals, while in the solution above a series of large leaves resembling ivy gradually ascended to the surface of the solution, and even projected above it. In snow-water the ivy-leaves rapidly melted, and threw down small bright crystalline grains of alum, while the powder at the bottom remained.

The leaf-like crystals seem to be deformations of the so-called tetrahedral crystals produced by the action of cold on these solutions, highly supersaturated as they were at the low temperatures to which they were reduced. From the recurrence of similar forms during the solidification of solutions of different salts, it is reasonable to suppose that the water of the solution determines the form, and that the presence of the saline molecules modifies it. From the abundance of these saline molecules in comparatively so small a quantity of water, and in so confined a space as a glass tube, the water as a preparatory step to solidification is not able to eject the saline molecules; they therefore remain and influence the forms of the solid which results from the low temperature. My business, however, is only incidentally mixed up with the solidification of supersaturated solutions. What I wish to insist on is the fact that, in the absence of a nucleus, these solutions prefer rather to become solid than to crystallize; and that there is no real freezing (if by freezing is meant, as a necessary condition, the separation of salt during the act) is proved, I think, by those cases where the result of a moderate elevation of temperature is to produce a bright, clear, and still highly supersaturated solution[10]; while in other cases, where there is a separation of salt, it is the anhydrous or a salt of a lower degree of hydration than that employed to make the solution. But if, at any time during the process of solidifying or of melting, the solution be subjected to the action of a nucleus, crystallization is sure to set in, and the original fully hydrated salts are reproduced.

A solution of the *cupric* and *magnesic sulphates* at 4° F. produced the crystals already described, and when solid the outside of the tube displayed some elegant forms in different shades of blue.

The *sodic* and *magnesic sulphates* in atomic proportions form, with water, a supersaturated solution; but on reducing it to low temperatures, the salts are apt to separate. When such is not the case, an anhydrous salt is thrown down on greatly reducing the temperature; and as this salt takes up a portion of water, acicular crystals shoot out from it. Other interesting phenomena may be observed with solutions of this double salt at various degrees of supersaturation and at various temperatures; as also with the sulphates of other bases, such as those of *cadmium* and of *nickel*, either alone or in conjunction with other sulphates. The addition of *potassic sulphate* to other sulphates in atomic proportions leads to the formation of double salts, which, so far as I have examined them, do not form supersaturated solutions.

[10] RÜDORFF (*Jahresber. der Chemie*, 1862, p. 20) showed that by employing saturated solutions of single salts and reducing them to low temperatures, ice is formed, together with a hydrated compound of the salt in question. In my experiments with supersaturated solutions in close vessels, chemically clean, the conditions are, of course, quite different, and no ice is formed. In some cases the solutions become viscid, like syrup.

Addenda.

[18th February, 1871.] Since the reading of the foregoing paper and the publication of the Abstract thereof in the Proceedings (vol. xviii. p. 533), some points connected with Section I. seemed to require further elucidation, which I am permitted to offer in this place by way of addenda.

(1) *On the definition of a nucleus.*—In the case of a supersaturated saline solution, a nucleus is defined as "a body that has a stronger attraction for the salt of the solution than for the liquid which holds it in solution." This is true in the great majority of cases; but in certain instances, such as those of absolute alcohol and washed ether, the nucleus has an attraction for the aqueous rather than the saline molecules of the solution. In order therefore to make the definition include such cases, it may be necessary to add to the definition, as given above, the following words:—"or a stronger attraction for the liquid than for the salt." In the case of supersaturated gaseous solutions, such as soda-water, Seltzer-water, champagne, &c., it has been shown[11] that a glass rod, or other body, covered with a film of foreign matter is active in liberating gas, while the same body in a catharized state is inactive. Similar phenomena are exhibited in the case of liquids at or near the boiling-point, which I regard as supersaturated solutions of their own vapour[12]. A catharized body is inactive; but if such body be drawn through the hand or in any other way covered with a film, it immediately produces a burst of vapour when placed in the vaporous solution. Hence in saline, gaseous, and vaporous supersaturated solutions the action of nuclei is the same, and may fairly come under the same definition, subject only, in the case of saline solutions, to the modification above given, and, in the case of vaporous solutions, to the additional fact that porous bodies, whether catharized or not, are energetic nuclei. Gathering up these various details, a nucleus may be defined as a body that has a preferential attraction for the salt or the gas or the vapour of a supersaturated solution, or for the liquid which holds any one of the three in solution, separation of the constituents of the solution being in either case the result; only in the case of vaporous solutions, porous bodies act as permanent nuclei.

(2) *On the pure substances capable of causing immediate crystallization.*—Conclusions cannot, obviously, be drawn from liquids that have contracted dust or products of oxidation &c., since such liquids are generally active as nuclei, whether in a lenticular or a filmy form. Fixed oils require to be refined or filtered, volatile oils to be distilled, and solid fats to be fused and filtered or passed through flame, alcohol or naphtha to be distilled, ether to be washed, and so on. Whatever the body that forms films, it must be treated in some way or other that shall ensure the deposit of a pure film on the supersaturated saline solution, and not a film containing a small speck of foreign matter that is likely to act as a nucleus. Moreover, in trying the action of a film of any one substance, the experiment should not be limited to one flask of solution, but to

[11] Phil. Mag. July 1867.

[12] Proceedings of the Royal Society, vol. xvii. p. 240.

many solutions of the same, and of various degrees of supersaturation. There is also a difference in the mode of action of a speck, compared with that of a film. A speck or mote or point acts by determining the crystallization downwards at right angles to the surface, or radially, in closely packed crystalline lines; whereas in the case of a pure film crystallization proceeds from every part of its lower surface simultaneously, so as often to produce large well-shaped crystals parallel with the film.

(3) *Why should a film act as a nucleus, and a lens of the same oil not so act?*—It can be shown experimentally that a clean inactive rod immediately becomes active simply by drawing it through the hand; so also a clean surface, such as the inside of a flask, smeared with a little oil becomes powerfully nuclear; but if the smear on the surface of the rod or on the inner wall of the flask can gather itself up into globules, these are not nuclear. Phenomena of this kind seem to be explained by reference to the surface-tension of liquids. In the case of a film formed by depositing a drop of an oil on the surface of a supersaturated saline solution, the surface-tension of such solution being diminished at the point touched by the oil, the surface of the solution surrounding the point touched exerts a tractive force on the oil, and spreads it into a film, with adhesion to the surface and a nuclear action on the solution; because being brought into contact with the saline particles, but not with the aqueous (seeing that oil and water do not combine, while adhesion readily takes place between oil and salt), the saline molecules are separated from the aqueous, and the action once begun is propagated. When, on the other hand, an oil deposited on the surface of the solution assumes the form of a well-shaped double convex lens, or even a disk, there is no adhesion properly so called, and no nuclear action, since the lens is separated from the solution by surface-tension, and is not nuclear because it is really not in contact with the solution. Even when the solution is shaken so as to break up the lens into globules and to submerge them, each globule is as completely separated by surface-tension from contact with the solution as if it were outside the flask, at least so far as any nuclear action is concerned. Whether a drop of oil form a film or a lens depends on several variable circumstances, such as the temperature of the solution, its degree of supersaturation, its surface viscosity, and so on.

(4) *Is it possible to deposit a film on a supersaturated saline solution without its acting as a nucleus?*—In a recently published account of a series of experiments, it is stated that an oil, such as citronella, was dissolved in ether (approximately 1 part oil to 20 of ether), and a drop of this solution was allowed to fall upon a supersaturated solution of GLAUBER's salt; as the ether evaporated, a film of the oil was left on the surface of the solution, and it did not act as a nucleus. In repeating this experiment, I found that in most cases the ether itself, or its vapour acting at a distance, formed a film on the surface of the solution, and produced crystallization; but when this effect was not brought about, and the ether evaporated, the oil was left in the form of a multitude of minute globules or lenses, which, as already stated, are not nuclear. But granting the result stated above to have been obtained, and that a film was actually deposited on the sur-

face of the solution without any nuclear action, the answer appears to me to be found in a highly ingenious experiment by Professor VAN DER MENSBRUGGHE, of the University of Ghent, contained in a Memoir on the Surface-tension of Liquids[13]. Some filaments of silkworm's cocoon were tied together at the ends so as to form an irregular circle, and being made chemically clean and pressed flat, the circlet was placed on the surface of water in such a way as to be exactly in contact with it, without being below the level. If, now, "a drop of ether be held above that portion of the surface limited by the coil of filaments, this coil immediately undergoes lively trepidations, and tends to assume the true circular form, evidently because the vapour of ether diminishes the tension of the surface within the silken boundary, and this, in its turn, yields to the superior traction of the portion external to it. The moment the drop of ether touches the surface within the flexible contour, the silk expands into a circular form; but it as quickly contracts, since the evaporation of the ether cools the surface and so restores the contractile force. When, on the other hand, the ether is deposited outside the silken boundary, this immediately becomes reduced in size, but expands again as the cold produced by evaporation augments the contractile force of the exterior portion"[14].

Now, supposing a drop of the ethereal solution of oil to spread on the surface of a supersaturated solution of GLAUBER'S salt, the first effect would be, according to my view, to lower the surface-tension of the solution, and in the absence of the ether to spread out the oil-film with nuclear action; but the presence of the ether (especially in so large a proportion as 20 to 1) prevents this; for it immediately begins to evaporate, and in doing so restores the surface-tension of the solution, so much so, that by the time the oil-film is released from its ethereal chain, the surface is in a condition to resist adhesion.

In the case of a drop of oil free from ether being deposited on the surface, it is the superior tractive force of the surface surrounding the drop that spreads it into a film, and in the very act of doing so degrades its own contractile force to that of the oil; in the case of the ethereal solution the oil is not in contact with the surface, but when the oil is spread by the surface itself it is: the distance in the one case, as compared with that of the other, may be less than the millionth of an inch, but it is not the less real.

(5) *Are we to understand that the minute microscopic particles floating in the air act as nuclei, not of themselves, but in consequence of some film of other matter with which they are contaminated?*—Such is my view; for if the fine dust gathered from the floor or from shelves or the tops of books &c., which is powerfully nuclear, be washed in a weak solution of caustic alkali and dried out of contact with air, it is no longer nuclear. Flasks containing supersaturated saline solutions may be kept uncovered during a long time in an open space in the country, in calm weather, without any separation of salt;

[13] *Sur la Tension superficielle des Liquides. Mémoires Couronnés de l'Acad. Roy. des Sciences de Belgique*, 1869. An abstract of this paper is given in the Phil. Mag. for Dec. 1869 and Jan. 1870.

[14] Phil. Mag. Dec. 1869, § 32. See also another remarkable example in Phil. Mag. Jan. 1870, § 22.

but when crystallization does take place it will be found that a small fly, a speck of soot, &c. has entered the flask and acted as a nucleus. The finger, if cleaned in caustic alkali or alcohol, or by drawing it through the flame of a spirit-lamp, may be held in a supersaturated saline solution for some time without inducing crystallization; but if the finger while in this condition be drawn along the side of the glass so as to produce a smear, such smear is immediately active, the skin supplying the matter of the active film.

(6) *Is it true that a fully hydrated salt, chemically clean, is not a nucleus to its supersaturated solution?*—In my first paper on Supersaturated Saline Solutions[15] it was shown that a crystal is not *necessarily* a nucleus to a saturated saline solution of its own kind. For example, a supersaturated solution of GLAUBER'S salt, by long keeping, parts with a portion of its water by evaporation through the cotton-wool plug, and crystals are formed on the glass above the solution; on washing these down they do not act as nuclei: so also a cold supersaturated solution of magnesic sulphate *in vacuo* over sulphuric acid forms, also by evaporation, crystalline crusts on the surface, and these crusts are not nuclear. Cases of other salts have been urged where such a crystal becomes a nucleus, and the old fact has been restated that a salt of a lower degree of hydration is not a nucleus to a supersaturated solution of a salt of a higher degree of hydration. What is wanted is to show that the sodic sulphate and the magnesic sulphate, formed as above described, are not the normal salts, but salts of a lower degree of hydration; and this has not yet been done.

[15] Philosophical Transactions, for 1868, p. 665.

IV. *Magnetic Observations made at Stonyhurst College Observatory, from April* 1863 *to March* 1870. *By the Rev.* S. J. Perry. *Communicated by the President.*

Received January 23,—Read March 9, 1871.

Results of Seven Years' Observations of the Dip and Horizontal Force.

In a paper published in the Philosophical Transactions for 1863, the President of the Royal Society enters into a discussion of a six years' series of the Kew Magnetic Observations, with the view of ascertaining whether the sun's position with reference to the earth really produces a sensible semiannual inequality in the terrestrial magnetic elements. The probability of the existence of such a disturbing cause, founded on a comparison of the monthly determinations of the Dip and Horizontal Force taken at Hobarton and at Toronto, was made much more evident by Sir Edward Sabine's discussion of the Kew observations; and it was from a desire of bringing more data to bear upon this important question that the reduction of the seven years' observations, just completed at this observatory, has been undertaken. With this object in view, it was considered of great importance to adhere closely to the plan laid down by Sir Edward Sabine, and to present the results, for the sake of comparison, in three Tables. The first contains the mean monthly determinations of the elements, with their deduced mean values and secular variation, the second presents a view of the semiannual inequality, and the third gives the residual errors and the consequent most probable errors of any single observation and of the deduced mean values of the elements.

Magnetic observations were first taken at Stonyhurst in 1858, but the continuous series of monthly determinations of the Dip, Declination, and Intensity were only commenced in March 1863. The same instruments, *i. e.* a dip-circle by Barrow and a Jones unifilar, have been used throughout the whole seven years. These instruments were both tested at Kew before being sent to this observatory, and the dip-circle was again examined there last January with most satisfactory results. The constants of the vibration-magnet were determined by Mr. Welsh, and are given in the Proceedings of the Royal Society for February 1865. In calculating the value of the Horizontal Force, it has never been found necessary to apply any correction for the arc of vibration, which has always been small, nor for the Frodsham chronometer, whose rate has never exceeded 2^s per day.

The yearly adopted values of P are the following:—

for 1863	−0·00217	for 1867	−0·00254
1864	−0·00219	1868	−0·00271
1865	−0·00254	1869	−0·00210
1866	−0·00275		

Each of these values is the mean of the monthly determinations obtained by observations of deflections at 1·0 and 1·3 feet.

For the dip three needles have been employed, Nos. 1 and 2 from 1863 to August 1868, and Nos. 1 and 3 since the latter date. A correction has been applied since August 1868 for the change of needle; this correction $=-0' \, 20''$ for the mean value.

The observations have always been taken by the Director of the Observatory, viz. by the Rev. W. SIDGREAVES, from 1863 to September 1868, and since that date by myself.

Unfortunately the station of observation has not been always the same, nor perfectly free from disturbing influences. The first station was a stone pillar erected in 1858 in the botanical garden of the college. In June 1864 the N.W. wall of the garden was replaced by an iron railing, whose nearest distance from the magnetic pillar is 91 feet. In January of the following year a three-quarter inch iron pipe was laid down on the S.E. side of the pillar, at a distance of 38 feet 9 inches; and in September 1867 a second iron pipe, $1\frac{1}{4}$ inch in diameter, was placed parallel to the former at $2\frac{1}{2}$ inches greater distance from the pillar.

At the beginning of 1868 a wooden hut, three of whose sides are in great part glass, was erected in a retired part of the garden, and so placed that the shade of overhanging trees might protect it from the sun in summer, whilst it remained fully exposed to the sun's rays when at low altitudes. No iron was in the vicinity at the time, but a 2-inch iron pipe was laid down in October 1869 at a distance of 51 feet 9 inches from the stone pier on which the observations are taken.

A series of observations of the Dip, Declination, and Horizontal Force have been made on the same days at the old and new stations for the purpose of determining the corrections to be applied for change of locality. The corrections thus found for the first station are $-0·0051$ in British units for the Horizontal Force, and $+3' \, 34''$ for the Dip. It is now impossible to determine absolutely what part of these corrections is due to the several masses of iron near the old pillar; but since it seems probable that the two iron pipes would have produced at most but a very slight disturbance, the whole correction has been applied from the time of the placing of the iron railings, *i. e.* from July 1864 to March 1868, both inclusively.

The following Tables contain the corrected values.

The Horizontal Force.

TABLE I.

Monthly mean values of the Horizontal Force.

April to September.	1863.	1864.	1865.	1866.	1867.	1868.	1869.	Mean of seven years.
April	3·5912	3·5950	3·5958	3·5978	3·6094	3·6052	3·6157	3·6014
May	3·5920	3·5995	3·6016	3·5985	3·6088	3·6099	3·6250	3·6050
June	3·5850	3·5999	3·5997	3·6004	3·6090	3·6100	3·6195	3·6034
July	3·5988	3·6004	3·5920	3·6063	3·6070	3·6055	3·6138	3·6034
August	3·5956	3·5924	3·5987	3·5980	3·6104	3·6060	3·6177	3·6027
September	3·5924	3·5876	3·5996	3·5973	3·6067	3·6129	3·6077	3·6006
Means	3·5925	3·5958	3·5979	3·5997	3·6085	3·6083	3·6166	3·6028
October to March.	1863–64.	1864–65.	1865–66.	1866–67.	1867–68.	1868–69.	1869–70.	
October	3·5909	3·5891	3·5912	3·5992	3·6101	3·6173	3·6190	3·6024
November	3·5894	3·5953	3·5966	3·6036	3·6079	3·6173	3·6204	3·6044
December	3·5962	3·5912	3·5978	3·6036	3·6073	3·6194	3·6249	3·6058
January	3·5944	3·5959	3·5986	3·6043	3·5871	3·6194	3·6236	3·6033
February	3·5971	3·6001	3·5990	3·6061	3·6060	3·6223	3·6166	3·6067
March	3·5877	3·5969	3·5958	3·6060	3·6066	3·6100	3·6099	3·6018
Means	3·5926	3·5948	3·5965	3·6038	3·6042	3·6176	3·6191	3·6041
Yearly means	3·5926	3·5953	3·5972	3·6018	3·6064	3·6129	3·6178	3·6034

The values for August and October 1863 are interpolations.

The value in January 1868 is evidently too small; this arose from a sudden change in the time of vibration of the magnet.

The above Table gives for the epoch October 1st, 1866,

The mean Horizontal Force $=3·6034$.
With a secular acceleration $=0·0042$.

These quantities enable us directly to calculate the probable value of the force at any epoch, and we thus form the next Table.

TABLE II.

Semiannual inequality of the Horizontal Force.

Date.	Correction for secular variation.	Mean ± secular variation.	Observed values.	Observed − Computed.	
				April to September.	October to March
July 1, 1863	−0·0137	3·5897	3·5925	+0·0028	
Jan. 1, 1864	−0·0116	·5918	·5926	... .	+0·0008
July 1, 1864	−0·0095	·5939	·5958	+0·0019	
Jan. 1, 1865	−0·0074	·5960	·5948	....	−0·0012
July 1, 1865	−0·0053	·5981	·5979	−0·0002	
Jan. 1, 1866	−0·0032	·6002	·5965	....	−0·0037
July 1, 1866	−0·0011	·6023	·5997	−0·0026	
Jan. 1, 1867	+0·0011	·6045	·6038	....	−0·0007
July 1, 1867	+0·0032	·6066	·6085	+0·0019	
Jan. 1, 1868	+0·0053	·6087	·6042	....	−0·0045
July 1, 1868	+0·0074	·6108	·6083	−0·0025	
Jan. 1, 1869	+0·0095	·6129	·6176	....	+0·0047
July 1, 1869	+0·0116	·6150	·6166	+0·0016	
Jan. 1, 1870	+0·0137	·6171	·6191	....	+0·0021
Mean differences in the semiannual periods				+0·00014	−0·00036

Hence we may conclude that there exists an annual variation whose mean value is 0·0005; but the great difference between the figures for the semiannual periods shows that the variation in this particular case is not wholly due to the disturbing action of the sun.

We can now test the accuracy of our assumed values of the secular and semiannual variation, and of the observations themselves, by the formation of the following Table.

TABLE III.

Residual errors in the monthly mean values of the Horizontal Force.

	1863–64	1864–65.	1865–66	1866–67	1867–68	1868–69	1869–70	Mean	Semiannual mean.
April	+21	+17	−17	−39	+ 35	−49	+ 14	−0·00026	
May	+25	+58	+37	−36	+ 25	− 6	+103	+0·00294	
June	−48	+59	+15	−20	+ 24	− 8	+ 45	+0·00096	
July	+86	+60	−66	+35	0	−57	− 16	+0·00060	+0·00016
August ...	+51	−23	− 2	−51	+ 31	−55	+ 20	−0·00041	
September	+15	−75	+ 3	−62	− 10	+10	− 84	−0·00290	
October ...	+ 2	−58	−79	−41	+ 26	+56	+ 31	−0·00061	
November	−17	0	−29	− 1	0	+52	+ 41	+0·00066	
December	+48	−44	−20	− 4	− 9	+70	+ 83	+0·00177	
January ...	+26	− 1	−16	− 1	−215	+66	+ 66	−0·00107	+0·00014
February	+50	+38	−15	+14	− 29	+92	− 7	+0·00204	
March......	−48	− 2	−51	+ 9	− 27	−35	− 78	−0·00194	
Means......	+18	+ 2	−20	−16	− 12	+11	+ 18		

This Table shows that the assumption of a semiannual inequality, whose mean value

is $=0{\cdot}00025$, has led us into no sensible error, but that the corrections applied for the change of station between July 1864 and March 1868 should be somewhat modified.

Table III. gives as the probable error of a single observation, or monthly mean, $\pm 0{\cdot}00332$, and neglecting January 1868, $\pm 0{\cdot}00292$; and for the error of the mean $\pm 0{\cdot}00036$, or neglecting January 1868, $\pm 0{\cdot}00032$.

The Dip Observations may now be treated in a precisely similar manner as those of the Horizontal Force.

Table I.

Monthly mean values of the Dip.

April to September.	1863.	1864.	1865.	1866.	1867.	1868.	1869.	Mean.
April	69° 51′ 13″	46′ 29″	49′ 57″	48′ 52″	45′ 17″	40′ 45″	43′ 57″	69° 46′ 39″
May	48 37	46 40	50 42	47 26	44 16	41 9	37 34	45 12
June	47 50	49 15	49 21	47 33	43 22	39 57	38 45	45 9
July	51 27	47 44	50 29	48 32	47 24	42 11	38 9	46 34
August	49 51	47 29	51 8	46 19	46 15	36 58	40 49	45 33
September	48 15	50 35	50 0	46 11	45 16	40 45	35 15	45 11
Means	69 49 32	48 2	50 16	47 29	45 18	40 18	39 5	69 45 43
October to March.	1863–64.	1864–65.	1865–66.	1866–67.	1867–68.	1868–69.	1869–70.	
October	69 48 23	48 20	51 4	47 15	43 33	43 4	35 16	69 45 16
November	48 30	49 22	48 39	46 33	44 34	42 25	39 52	45 42
December	47 17	49 57	47 5	46 45	43 0	41 52	36 3	44 34
January	47 21	47 15	47 21	46 53	43 40	42 34	33 16	44 3
February	49 13	50 4	49 37	45 19	45 8	40 7	36 16	45 6
March	47 19	49 17	50 3	46 16	43 50	40 43	39 6	45 13
Means	69 48 1	49 3	48 58	46 30	43 58	41 47	36 38	44 59
Yearly means	69 48 47	48 32	49 37	47 0	44 38	41 2	37 52	69 45 21

The values for August and October 1863 are interpolated.

The mean dip for October 1st, 1866, is thus 69° 45′ 21″.

With a secular diminution$=1' \, 49''{\cdot}2$.

From these quantities we will now calculate the semiannual inequalities.

TABLE II.

Semiannual inequality of the Dip.

Date.	Correction for secular variation.	Mean ± secular variation.	Observed values.	Observed – Calculated.	
				April to September.	October to March.
July 1, 1863	+5′ 53″	69° 51′ 14″	69° 49′ 32″	−1′ 42″	
Jan. 1, 1864	+4 59	50 20	48 1	....	−2′ 19″
July 1, 1864	+4 4	49 25	48 2	−1 23	
Jan. 1, 1865	+3 10	48 31	49 3	....	+0 32
July 1, 1865	+2 15	47 36	50 16	+2 40	
Jan. 1, 1866	+1 21	46 42	48 58	....	+2 16
July 1, 1866	+0 27	45 48	47 29	+1 41	
Jan. 1, 1867	−0 27	44 54	46 30	....	+1 36
July 1, 1867	−1 21	44 0	45 18	+1 18	
Jan. 1, 1868	−2 15	43 6	43 58	....	+0 52
July 1, 1868	−3 10	42 11	40 18	−1 53	
Jan. 1, 1869	−4 4	41 17	41 47	....	+0 30
July 1, 1869	−4 59	40 22	39 5	−1 17	
Jan. 1, 1870	−5 53	39 28	36 38	....	−2 50
Mean differences in the semiannual periods				−0 5	+0 5

The resulting difference is small compared with the errors of observation, and therefore it can afford but a slight confirmation of the hypothesis of the dependence of this inequality on the position of the sun in the ecliptic.

TABLE III.

Residual errors in the monthly mean values of the Dip.

	1863–64.	1864–65.	1865–66.	1866–67.	1867–68.	1868–69.	1869–70.	Mean.	Semiannual mean.
April	−0′ 21″	−3′ 15″	+2′ 2″	+2′ 46″	+1′ 0″	−1′ 43″	+3′ 19″	+0′ 33″	
May	−2 48	−2 55	+2 56	+1 29	+0 8	−1 10	−2 55	−0 45	
June	−3 26	−0 11	+1 44	+1 45	−0 37	−2 13	−1 35	−0 39	
July	+0 10	−1 33	+3 1	+2 53	+3 34	+0 10	−2 2	+0 53	−0′ 1″
August	−1 7	−1 39	+3 49	+0 49	+2 34	−4 54	+0 47	+0 3	
September ...	−2 34	+1 36	+2 50	+0 50	+1 44	−0 58	−4 38	−0 10	
October	−2 26	−0 39	+3 54	+1 54	+0 1	+1 21	−4 37	−0 5	
November ...	−2 10	+0 32	+1 38	+1 21	+1 11	+0 51	+0 8	+0 30	
December ...	−3 14	+1 16	+0 13	+1 42	−0 14	+0 27	−3 32	−0 29	
January	−3 1	−1 17	+0 38	+1 59	+0 35	+1 19	−6 10	−0 51	+0 0·5
February ...	−1 0	+1 41	+3 3	+0 34	+2 12	−1 0	−3 1	+0 21	
March.........	−2 45	+1 3	+3 38	+1 40	+1 3	−0 15	−0 2	+0 37	
Means.........	−2 4	−0 27	+2 27	+1 39	+1 6	−0 40	−2 2		

In this Table the evidence is still stronger than before that the correction applied from July 1864 to March 1868 is in excess.

The probable errors deduced from the above Table are:

for any single monthly value $\pm 1'{\cdot}45$,

and for the deduced means $\pm 0'{\cdot}16$.

The Total Force.

This can be at once deduced from the mean values contained in Tables I.

For Epoch July 1st, 1866, H. F.$=3\cdot6028$, and Dip$=69^\circ\ 45'\ 43''$ from April to Sept.

For Epoch Jan. 1st, 1867, H. F.$=3\cdot6041$, and Dip$=69^\circ\ 44'\ 59''$ from Oct. to March

Applying the correction for the secular variation to reduce both these to a common epoch, we have for

Jan. 1st, 1867, from April to September, T. F.$=3\cdot6049\times$sec. $69^\circ\ 44'\ 48''=10\cdot4136$.

October to March, T. F.$=3\cdot6041\times$sec. $69^\circ\ 44'\ 59''=10\cdot4128$.

which would make the intensity greater when the sun is further from the earth, the difference being very small, viz. 0·0008.

This last result is opposed to that derived from the reductions of Sir EDWARD SABINE, but will have but little weight, both on account of the smallness of the amount, and still more from the uncertainty attached to the corrections applied for change of station.

The above mean value of the total force, 10·4136, for January 1st, 1867, compared with the observations in October 1858, gives an average annual rate of increase$=0\cdot0034$, which agrees closely with the amount calculated by General SABINE (*cf.* Proceedings of the Royal Society, February 1865).

Whilst drawing up this paper I was not aware that Dr. BALFOUR STEWART was engaged in a similar reduction of a second six years' series of the Kew observations, which has led to such a satisfactory confirmation of the results obtained by the discussion of the first six years' observations at the same observatory. When another six years have elapsed, we shall be able to determine to what extent the results obtained above are due to local influences.

V. *On the Theory of Resonance.* *By the Hon.* J. W. Strutt, *M.A., Fellow of Trinity College, Cambridge. Communicated by* W. Spottiswoode, *F.R.S.*

Received July 2,—Read November 24, 1870*.

Introduction.

Although the theory of aërial vibrations has been treated by more than one generation of mathematicians and experimenters, comparatively little has been done towards obtaining a clear view of what goes on in any but the more simple cases. The extreme difficulty of any thing like a general deductive investigation of the question is no doubt one reason. On the other hand, experimenters on this, as on other subjects, have too often observed and measured blindly without taking sufficient care to simplify the conditions of their experiments, so as to attack as few difficulties as possible at a time. The result has been vast accumulations of isolated facts and measurements which lie as a sort of dead weight on the scientific stomach, and which must remain undigested until theory supplies a more powerful solvent than any now at our command. The motion of the air in cylindical organ-pipes was successfully investigated by Bernoulli and Euler, at least in its main features; but their treatment of the question of the open pipe was incomplete, or even erroneous, on account of the assumption that at the open end the air remains of invariable density during the vibration. Although attacked by many others, this difficulty was not finally overcome until Helmholtz†, in a paper which I shall have repeated occasion to refer to, gave a solution of the problem under certain restrictions, free from any arbitrary assumptions as to what takes place at the open end. Poisson and Stokes‡ have solved the problem of the vibrations communicated to an infinite mass of air from the surface of a sphere or circular cylinder. The solution for the sphere is very instructive, because the vibrations outside any imaginary sphere enclosing vibrating bodies of any kind may be supposed to take their rise in the surface of the sphere itself.

More important in its relation to the subject of the present paper is an investigation by Helmholtz of the air-vibrations in cavernous spaces (*Hohlraüme*), whose three dimensions are very small compared to the wave-length, and which communicate with the external atmosphere by small holes in their surfaces. If the opening be circular of area σ, and if S denote the volume, n the number of vibrations per second in the fundamental

* Additions made since the paper was first sent to the Royal Society are inclosed in square brackets [ ].

† Theorie der Luftschwingungen in Röhren mit offenen Enden. Crelle, 1860.

‡ Phil. Trans. 1868, or Phil. Mag. Dec. 1868.

note, and a the velocity of sound,

$$n = \frac{a\sigma^{\frac{1}{4}}}{2^{\frac{1}{2}}\pi^{\frac{3}{4}}S^{\frac{1}{2}}}.$$

HELMHOLTZ's theory is also applicable when there are more openings than one in the side of the vessel.

In the present paper I have attempted to give the theory of vibrations of this sort in a more general form. The extension to the case where the communication with the external air is no longer by a mere hole in the side, but by a neck of greater or less length, is important, not only because resonators with necks are frequently used in practice, but also by reason of the fact that the theory itself is applicable within wider limits. The mathematical reasoning is very different from that of HELMHOLTZ, at least in form, and will I hope be found easier. In order to assist those who may wish only for clear general ideas on the subject, I have broken up the investigation as much as possible into distinct problems, the results of which may in many cases be taken for granted without the rest becoming unintelligible. In Part I. my object has been to put what may be called the dynamical part of the subject in a clear light, deferring as much as possible special mathematical calculations. In the first place, I have considered the general theory of resonance for air-spaces confined nearly all round by rigid walls, and communicating with the external air by any number of passages which may be of the nature of necks or merely holes, under the limitation that both the length of the necks and the dimensions of the vessel are very small compared to the wave-length. To prevent misapprehension, I ought to say that the theory applies only to the fundamental note of the resonators, for the vibrations corresponding to the overtones are of an altogether different character. There are, however, cases of multiple resonance to which our theory is applicable. These occur when two or more vessels communicate with each other and with the external air by necks or otherwise; and are easily treated by LAGRANGE's general dynamical method, subject to a restriction as to the relative magnitudes of the wave-lengths and the dimensions of the system corresponding to that stated above for a single vessel. I am not aware whether this kind of resonance has been investigated before, either mathematically or experimentally. Lastly, I have sketched a solution of the problem of the open organ-pipe on the same general plan, which may be acceptable to those who are not acquainted with HELMHOLTZ's most valuable paper. The method here adopted, though it leads to results essentially the same as his, is I think more calculated to give an insight into the real nature of the question, and at the same time presents fewer mathematical difficulties. For a discussion of the solution, however, I must refer to HELMHOLTZ.

In Part II. the calculation of a certain quantity depending on the form of the necks of common resonators, and involved in the results of Part I., is entered upon. This quantity, denoted by c, is of the nature of a length, and is identical with what would be called in the theory of electricity the *electric conductivity* of the passage, supposed to be occupied by uniformly conducting matter. The question is accordingly similar to that of determining the electrical resistance of variously shaped conductors—an analogy of

which I have not hesitated to avail myself freely both in investigation and statement. Much circumlocution is in this way avoided on account of the greater completeness of electrical phraseology. Passing over the case of mere holes, which has been already considered by HELMHOLTZ, and need not be dwelt upon here, we come to the value of the resistance for necks in the form of circular cylinders. For the sake of simplicity each end is supposed to be in an infinite plane. In this form the mathematical problem is definite, but has not been solved rigorously. Two limits, however (a higher and a lower), are investigated, between which it is proved that the true resistance must lie. The lower corresponds to a correction to the length of the tube equal to $\frac{\pi}{4}\times$(radius) for each end. It is a remarkable coincidence that HELMHOLTZ also finds the same quantity as an approximate correction to the length of an organ-pipe, although the two methods are entirely different and neither of them rigorous. His consists of an exact solution of the problem for an approximate cylinder, and mine of an approximate solution for a true cylinder; while both indicate on which side the truth must lie. The final result for a cylinder infinitely long is that the correction lies between ·785 R and ·828 R. When the cylinder is finite, the upper limit is rather smaller. In a somewhat similar manner I have investigated limits for the resistance of a tube of revolution, which is shown to lie between

$$\int\frac{dx}{\pi y^2}$$

and

$$\int\frac{dx}{\pi y^2}\left\{1+\frac{1}{2}\left(\frac{dy}{dx}\right)^2\right\},$$

where y denotes the radius of the tube at any point x along the axis. These formulæ apply whatever may be in other respects the form of the tube, but are especially valuable when it is so nearly cylindrical that $\frac{dy}{dx}$ is everywhere small. The two limits are then very near each other, and either of them gives very approximately the true value. The resistance of tubes, which are either not of revolution or are not nearly straight, is afterwards approximately determined. The only experimental results bearing on the subject of this paper, and available for comparison with theory, that I have met with are some arrived at by SONDHAUSS* and WERTHEIM†. Besides those quoted by HELMHOLTZ, I have only to mention a series of observations by SONDHAUSS‡ on the pitch of flasks with long necks which led him to the empirical formula

$$n=46705\,\frac{\sigma^{\frac{1}{4}}}{L^{\frac{1}{2}}S^{\frac{1}{2}}},$$

σ, L being the area and length of the neck, and S the volume of the flask. The corresponding equation derived from the theory of the present paper is

$$n=54470\,\frac{\sigma^{\frac{1}{2}}}{L^{\frac{1}{2}}S^{\frac{1}{2}}},$$

* POGG. Ann. vol. lxxxi. † Annales de Chimie, vol. xxxi. ‡ POGG. Ann. vol. lxxix.

which is only applicable, however, when the necks are so long that the corrections at the ends may be neglected—a condition not likely to be fulfilled. This consideration sufficiently explains the discordance. Being anxious to give the formulæ of Parts I. and II. a fair trial, I investigated experimentally the resonance of a considerable number of vessels which were of such a form that the theoretical pitch could be calculated with tolerable accuracy. The result of the comparison is detailed in Part III., and appears on the whole very satisfactory; but it is not necessary that I should describe it more minutely here. I will only mention, as perhaps a novelty, that the experimental determination of the pitch was not made by causing the resonators to speak by a stream of air blown over their mouths. The grounds of my dissatisfaction with this method are explained in the proper place.

[Since this paper was written there has appeared another memoir by Dr. SONDHAUSS* on the subject of resonance. An empirical formula is obtained bearing resemblance to the results of Parts I. and II., and agreeing fairly well with observation. No attempt is made to connect it with the fundamental principles of mechanics. In the Philosophical Magazine for September 1870, I have discussed the differences between Dr. SONDHAUSS's formula and my own from the experimental side, and shall not therefore go any further into the matter on the present occasion.]

PART I.

The class of resonators to which attention will chiefly be given in this paper are those where a mass of air confined almost all round by rigid walls communicates with the external atmosphere by one or more narrow passages. For the present it may be supposed that the boundary of the principal mass of air is part of an oval surface, nowhere contracted into any thing like a narrow neck, although some cases not coming under

this paper. Moreover the motion in the passage and its neighbourhood will not differ sensibly from that of an incompressible fluid, and its energy will depend only on the rate of total flow through the opening. A quarter of a period later this energy of motion will be completely converted into the potential energy of the compressed or rarefied air inside the reservoir. So soon as the mathematical expressions for the potential and kinetic energies are known, the determination of the period of vibration or resonant note of the air-space presents no difficulty.

The motion of an incompressible frictionless fluid which has been once at rest is subject to the same formal laws as those which regulate the flow of heat or electricity through uniform conductors, and depends on the properties of the potential function, to which so much attention has of late years been given. In consequence of this analogy many of the results obtained in this paper are of as much interest in the theory of electricity as in acoustics, while, on the other hand, known modes of expression in the former subject will save circumlocution in stating some of the results of the present problem.

Let h_0 be the density, and ϕ the velocity-potential of the fluid motion through an opening. The kinetic energy or *vis viva*

$$=\frac{1}{2}h_0\iiint\left[\left(\frac{d\phi}{dx}\right)^2+\left(\frac{d\phi}{dy}\right)^2+\left(\frac{d\phi}{dz}\right)^2\right]dxdydz,$$

the integration extending over the volume of the fluid considered

$$=\frac{1}{2}h_0\iint\phi\frac{d\phi}{dn}dS,$$

by GREEN's theorem.

Over the rigid boundary of the opening or passage, $\frac{d\phi}{dn}=0$, so that if the portion of fluid considered be bounded by two equipotential surfaces, ϕ_1 and ϕ_2, one on each side of the opening,

$$vis\ viva=\frac{1}{2}h_0(\phi_1-\phi_2)\iint\frac{d\phi}{dn}dS=\frac{1}{2}h_0(\phi_1-\phi_2)\dot{X},$$

if $\dot{X}$ denote the rate of total flow through the opening.

At a sufficient distance on either side ϕ becomes constant, and the rate of total flow is proportional to the difference of its values on the two sides. We may therefore put

$$\phi_1-\phi_2=\frac{1}{c}\iint\frac{d\phi}{dn}dS=\frac{\dot{X}}{c},$$

where c is a linear quantity depending on the size and shape of the opening, and representing in the electrical interpretation the reciprocal of the *resistance* to the passage of electricity through the space in question, the specific resistance of the conducting matter being taken for unity. The same thing may be otherwise expressed by saying that c is the side of a cube, whose resistance between opposite faces is the same as that of the opening.

The expression for the *vis viva* in terms of the rate of total flow is accordingly

$$vis\ viva=\frac{h_0}{2}\frac{\dot{X}^2}{c}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

If S be the capacity of the reservoir, the condensation at any time inside it is given by

$\frac{X}{S}$, of which the mechanical value is

$$\frac{1}{2}h_0a^2\frac{X^2}{S}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

a denoting, as throughout the paper, the velocity of sound.

The whole energy at any time, both actual and potential, is therefore

$$\frac{h_0}{2}\frac{\dot{X}^2}{c}+\frac{h_0}{2}a^2\frac{X^2}{S}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and is constant. Differentiating with respect to time, we arrive at

$$\ddot{X}+\frac{a^2c}{S}X=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

as the equation to the motion, which indicates simple oscillations performed in a time

$$2\pi\div\sqrt{\frac{a^2c}{S}}.$$

Hence if n denote the number of vibrations per second in the resonant note,

$$n=\frac{a}{2\pi}\sqrt{\frac{c}{S}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

The wave-length λ, which is the quantity most immediately connected with the dimensions of the resonant space, is given by

$$\lambda=\frac{a}{n}=2\pi\sqrt{\frac{S}{c}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

A law of SAVART, not nearly so well known as it ought to be, is in agreement with equations (5) and (6). It is an immediate consequence of the principle of dynamical similarity, of extreme generality, to the effect that *similar* vibrating bodies, whether they be gaseous, such as the air in organ-pipes or in the resonators here considered, or solid, such as tuning-forks, vibrate in a time which is directly as their linear dimensions. Of course the material must be the same in two cases that are to be compared, and the geometrical similarity must be complete, extending to the shape of the opening as well as to the other parts of the resonant vessel. Although the wave-length λ is a function of the size and shape of the resonator only, n or the position of the note in the musical scale depends on the nature of the gas with which the resonator is filled. And it is important to notice that it is on the nature of the gas in and near the opening that the note depends, and *not* on the gas in the interior of the reservoir, whose inertia does not come into play during vibrations corresponding to the fundamental note. In fact we

may say that the mass to be moved is the air in the neighbourhood of the opening, and that the air in the interior acts merely as a spring in virtue of its resistance to compression. Of course this is only true under the limitation specified, that the diameter of the reservoir is small compared to the quarter wave-length. Whether this condition is fulfilled in the case of any particular resonator is easily seen, *à posteriori*, by calculating the value of λ from (6), or by determining it experimentally.

Several Openings.

When there are two or more passages connecting the interior of the resonator with the external air, we may proceed in much the same way, except that the equation of energy by itself is no longer sufficient. For simplicity of expression the case of two passages will be convenient, but the same method is applicable to any number. Let X_1, X_2 be the total flow through the two necks, c_1, c_2 constants depending on the form of the necks corresponding to the constant c in formula (6); then T, the *vis viva*, is given by

$$T=\frac{h_0}{2}\left(\frac{\dot{X}_1^2}{c_1}+\frac{\dot{X}_2^2}{c_2}\right),$$

the necks being supposed to be sufficiently far removed from one another not to *interfere* (in a sense that will be obvious). Further,

$$V=\text{Potential Energy}=\frac{1}{2}h_0 a^2\frac{(X_1+X_2)^2}{S}.$$

Applying LAGRANGE's general dynamical equation, $\frac{d}{dt}\left(\frac{dT}{d\dot{\psi}}\right)-\frac{dT}{d\psi}=-\frac{dV}{d\psi}$,

we obtain

$$\left.\begin{aligned}\frac{\ddot{X}_1}{c_1}+\frac{a^2}{S}(X_1+X_2)=0,\\ \frac{\ddot{X}_2}{c_2}+\frac{a^2}{S}(X_1+X_2)=0\end{aligned}\right\} \quad \ldots \quad (7)$$

as the equations to the motion.

By subtraction,

$$\frac{\ddot{X}_1}{c_1}-\frac{\ddot{X}_2}{c_2}=0,$$

or, on integration,

$$\frac{X_1}{c_1}=\frac{X_2}{c_2} \quad \ldots \quad (8)$$

Equation (8) shows that the motions of the air in the two necks have the same period and are at any moment in the same phase of vibration. Indeed there is no essential distinction between the case of one neck and that of several, as the passage from one to the other may be made continuously without the failure of the investigation.

When, however, the separate passages are sufficiently far apart, the constant c for the system, considered as a single communication between the interior of the resonator and the external air, is the simple sum of the values belonging to them when taken separately, which would not otherwise be the case. This is a point to which we shall return later, but in the mean time, by addition of equations (7), we find

$$\ddot{X}_1+\ddot{X}_2+\frac{a^2}{S}(c_1+c_2)(X_1+X_2)=0,$$

so that

$$n=\frac{a}{2\pi}\sqrt{\frac{c_1+c_2}{S}}. \quad \ldots \ldots \ldots \ldots \quad (9)$$

If there be any number of necks for which the values of c are c_1, c_2, c_3, ..., and no two of which are near enough to interfere, the same method is applicable, and gives

$$n=\frac{a}{2\pi}\sqrt{\frac{c_1+c_2+c_3+\ldots}{S}}; \quad \ldots \ldots \ldots \quad (9')$$

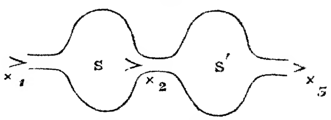
when there are two similar necks $c_2=c_1$, and

$$n=\sqrt{2}\times\frac{a}{2\pi}\sqrt{\frac{c}{S}}.$$

The note is accordingly higher than if there were only one neck in the ratio of $\sqrt{2}:1$, a fact observed by SONDHAUSS and proved theoretically by HELMHOLTZ for the case of openings which are mere holes in the sides of the reservoir.

Double Resonance.

Fig. 1.



Suppose that there are two reservoirs, S, S', communicating with each other and with the external air by narrow passages or necks. If we were to consider S S' as a single reservoir and to apply equation (9), we should be led to an erroneous result; for the reasoning on which (9) is founded proceeds on the assumption that, within the reservoir, the inertia of the air may be left out of account, whereas it is evident that the *vis viva* of the motion through the connecting passage may be as great as through the two others. However, an investigation on the same general plan as before meets the case perfectly. Denoting by X_1, X_2, X_3 the total flows through the three necks, we have for the *vis viva* the expression

$$T=\frac{1}{2}h_0\left\{\frac{\dot{X}_1^2}{c_1}+\frac{\dot{X}_2^2}{c_2}+\frac{\dot{X}_3^2}{c_3}\right\},$$

and for the potential energy

$$V=\frac{1}{2}h_0a^2\left\{\frac{(X_2-X_1)^2}{S}+\frac{(X_3-X_2)^2}{S'}\right\}.$$

An application of LAGRANGE's method gives as the differential equations to the motion,

$$\left.\begin{aligned}&\frac{\ddot{X}_1}{c_1}+a^2\frac{X_1-X_2}{S}=0,\\&\frac{\ddot{X}_2}{c_2}+a^2\left\{\frac{X_2-X_1}{S}+\frac{X_2-X_3}{S'}\right\}=0,\\&\frac{\ddot{X}_3}{c_3}+a^2\frac{X_3-X_2}{S'}=0.\end{aligned}\right\}\quad\quad(10)$$

By addition and integration

$$\frac{X_1}{c_1}+\frac{X_2}{c_2}+\frac{X_3}{c_3}=0.$$

Hence, on elimination of X_2,

$$\left.\begin{aligned}&\ddot{X}_1+\frac{a^2}{S}\left\{(c_1+c_2)X_1+\frac{c_1c_2}{c_3}X_3\right\}=0,\\&\ddot{X}_3+\frac{a^2}{S'}\left\{(c_3+c_2)X_3+\frac{c_3c_2}{c_1}X_1\right\}=0.\end{aligned}\right\}$$

Assuming $X_1=A\varepsilon^{pt}$, $X_3=B\varepsilon^{pt}$, we obtain, on substitution and elimination of $A:B$,

$$p^4+p^2a^2\left\{\frac{c_1+c_2}{S}+\frac{c_3+c_2}{S'}\right\}+\frac{a^4}{SS'}\left\{c_1c_3+c_2(c_1+c_3)\right\}=0\quad\quad(11)$$

as the equation to determine the resonant notes. If n be the number of vibrations per second, $n^2=-\frac{p^2}{4\pi^2}$, the values of p^2 given by (11) being of course both real and negative. The formula simplifies considerably if $c_3=c_1$, $S'=S$; but it will be more instructive to work this case from the beginning. Let $c_1=c_3=mc_2=mc$.

The differential equations take the form

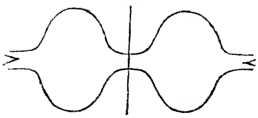
$$\left.\begin{aligned}&\ddot{X}_1+\frac{a^2c}{S}\left\{(1+m)X_1+X_3\right\}=0,\\&\ddot{X}_3+\frac{a^2c}{S}\left\{(1+m)X_3+X_1\right\}=0,\end{aligned}\right\}\text{while } X_2=-\frac{X_1+X_3}{m}.$$

Hence

$$\left.\begin{aligned}&(X_1+X_3)^{\cdot\cdot}+\frac{a^2c}{S}(m+2)(X_1+X_3)=0,\\&(X_1-X_3)^{\cdot\cdot}+\frac{a^2c}{S}m(X_1-X_3)=0.\end{aligned}\right\}$$

The whole motion may be regarded as made up of two parts, for the first of which $X_1+X_3=0$; which requires $X_2=0$. This motion is therefore the same as might take place were the communication between S and S′ cut off, and has its period given by

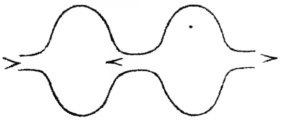
Fig. 2.



$$n^2=\frac{a^2c_1}{4\pi^2S}=\frac{a^2mc}{4\pi^2S}.$$

For the other component part, $X_1-X_3=0$, so that

Fig. 3.



$$X_2=-\frac{2X_1}{m}, \qquad n'^2=\frac{a^2(m+2)c}{4\pi^2 S} \quad . \quad . \quad . \quad (12)$$

Thus $\frac{n'^2}{n^2}=\frac{m+2}{m}$, which shows that the second note is the higher. It consists of vibrations in the two reservoirs opposed in phase and modified by the connecting passage, which acts in part as a second opening to both, and so raises the pitch. If the passage is small, so also is the difference of pitch between the two notes. A particular case worth notice is obtained by putting in the general equation $c_3=0$, which amounts to suppressing one of the communications with the external air. We thus obtain

$$p^4+a^2p^2\left(\frac{c_1+c_2}{S}+\frac{c_2}{S'}\right)+\frac{a^4}{SS'}c_1c_2=0\,;$$

or if $S=S'$, $c_1=mc_2=mc$,

$$p^4+a^2p^2\frac{c}{S}(m+2)+\frac{a^4c^2}{S^2}m=0,$$

$$n^2=\frac{a^2c}{8\pi^2S}\{m+2\pm\sqrt{m^2+4}\}.$$

If we further suppose $m=1$ or $c_2=c_1$,

$$n^2=\frac{a^2c}{8\pi^2S}(3\pm\sqrt{5}).$$

If N be the number of vibrations for a simple resonator (S, c),

$$N^2=\frac{a^2c}{4\pi^2S}\,;$$

$$\therefore\ n_1^2\div N^2=\frac{3+\sqrt{5}}{2}=2\cdot618,$$

$$N^2\div n_2^2=\frac{2}{3-\sqrt{5}}=2\cdot618.$$

It appears therefore that the interval from n_1 to N is the same as from N to n_2, namely, $\sqrt{2\cdot618}=1\cdot618$, or rather more than a fifth. It will be found that whatever the value of m may be, the interval between the resonant notes cannot be less than $2\cdot414$, which is about an octave and a minor third. The corresponding value of m is 2.

A similar method is applicable to any combination of reservoirs and connecting passages, no matter how complicated, under the single restriction as to the comparative magnitudes of the reservoirs and wave-lengths; but the example just given is sufficient to illustrate the theory of multiple resonance. In Part III. a resonator of this sort will be described, which was constructed for the sake of a comparison between the theory and experiment. In applying the formulæ (6) or (12) to an actual measurement, the question will arise whether the volume of the necks, especially when they are rather large, is to be included or not in S. At the moment of rest the air in the neck is com-

pressed or rarefied as well as that inside the reservoir, though not to the same degree; in fact the condensation must vary continuously between the interior of the resonator and the external air. This consideration shows that, at least in the case of necks which are tolerably symmetrical, about half the volume of the neck should be included in S.

[In consequence of a suggestion made by Mr. CLERK MAXWELL, who reported on this paper, I have been led to examine what kind of effect would be produced by a deficient rigidity in the envelope which contains the alternately compressed and rarefied air. Taking for simplicity the case of a sphere, let us suppose that the radius, instead of remaining constant at its normal value R, assumes the variable magnitude $R+\varrho$. We have

$$\text{kinetic energy}=\frac{h_0\dot{X}^2}{2c}+\frac{m}{2}\dot{\varrho}^2,$$

$$\text{potential energy}=\frac{h_0a^2}{2S}\{X+4\pi R^2\varrho\}^2+\tfrac{1}{2}\beta\varrho^2,$$

where m and β are constants expressing the inertia and rigidity of the spherical shell. Hence, by LAGRANGE's method,

$$\left.\begin{aligned}&\ddot{X}+\frac{ca^2}{S}(X+4\pi R^2\varrho)=0,\\&m\ddot{\varrho}+4\pi R^2\frac{h_0a^2}{S}(X+4\pi R^2\varrho)+\beta\varrho=0,\end{aligned}\right\}$$

equations determining the periods of the two vibrations of which the system is capable. It might be imagined at first sight that a yielding of the sides of the vessel would necessarily lower the pitch of the resonant note; but this depends on a tacit assumption that the capacity of the vessel is largest when the air inside is most compressed. But it may just as well happen that the opposite is true. Everything depends on the relative magnitudes of the periods of the two vibrations supposed for the moment independent of one another. If the note of the shell be very high compared to that of the air, the inertia of the shell may be neglected, and this part of the question treated statically. Putting in the equations $m=0$, we see that the phases of X and ϱ are opposed, and then X goes through its changes more slowly than before. On the other hand, if it be the note of the air-vibration, which is much the higher, we must put $\beta=0$, which leads to

$$4\pi R^2h_0\ddot{X}-cm\ddot{\varrho}=0,$$

showing that the phases of X and ϱ agree. Here the period of X is diminished by the yielding of the sides of the vessel, which indeed acts just in the same way as a second aperture would do. A determination of the actual note in any case of a spherical shell of given dimensions and material would probably be best obtained deductively.

But in order to see what probability there might be that the results of Part III. on glass flasks were sensibly modified by a want of rigidity, I thought it best to make a direct experiment. To the neck of a flask was fitted a glass tube of rather small bore, and the whole filled with water so as to make a kind of water-thermometer. On

removing by means of an air-pump the pressure of the atmosphere on the outside of the bulb, the liquid fell in the tube; but only to an extent which indicated an increase in the capacity of the flask of about a ten-thousandth part. This corresponds in the ordinary arrangement to a doubled density of the contained air. It is clear that so small a yielding could produce no sensible effect on the pitch of the air-vibration.]

Open Organ-pipes.

Although the problem of open organ-pipes, whose diameter is very small compared to their length and to the wave-length, has been fully considered by HELMHOLTZ, it may not be superfluous to show how the question may be attacked from the point of view of the present paper, more especially as some important results may be obtained by a comparatively simple analysis. The principal difficulty consists in finding the connexion between the spherical waves which diverge from the open end of the tube into free space, and the waves in the tube itself, which at a distance from the mouth, amounting to several diameters, are approximately plane. The transition occupies a space which is large compared to the diameter, and in order that the present treatment may be applicable must be small compared to the wave-length. This condition being fulfilled, the compressibility of the air in the space mentioned may be left out of account and the difficulty is turned. Imagine a piston (of infinitely small thickness) in the tube at the place where the waves cease to be plane. The motion of the air on the free side is entirely determined by the motion of the piston, and the *vis viva* within the space considered may be expressed by

$$\frac{1}{2}h_0\frac{\dot{X}^2}{c},$$

where $\dot{X}$ denotes the rate of total flow at the place of the piston, and c is, as before, a linear quantity depending on the form of the mouth. If Q is the section of the tube *and ψ the velocity potential,*

$$\dot{X}=Q\frac{d\psi}{dx}.$$

The most general expression for the velocity-potential of plane waves is

$$\psi=\left(\frac{A}{k}\sin kx+B\cos kx\right)\cos 2\pi nt+\beta\cos kx\sin 2\pi nt, \quad \ldots \ldots \quad (13)$$

$$\frac{d\psi}{dx}=(A\cos kx-Bk\sin kx)\cos 2\pi nt-\beta k\sin kx\sin 2\pi nt,$$

where

$$k=\frac{2\pi}{\lambda}=\frac{2\pi n}{a}.$$

When $x=0$,

$$\left.\begin{aligned}\psi&=B\cos 2\pi nt+\beta\sin 2\pi nt,\\ \frac{d\psi}{dx}&=A\cos 2\pi nt.\end{aligned}\right\}$$

The variable part of the pressure on the tube side of the piston

$$=-h_0\frac{d\psi}{dt}.$$

The equation to the motion of the air in the mouth is therefore

$$\frac{Q}{c}\frac{d}{dt}\frac{d\psi}{dx}+\frac{d\psi}{dt}=0,$$

or, on integration,

$$\frac{Q}{c}\frac{d\psi}{dx}+\psi=0. \qquad (14)$$

This is the condition to be satisfied when $x=0$.

Substituting the values of ψ and $\frac{d\psi}{dx}$, we obtain

$$\cos 2\pi nt\left(A\frac{Q}{c}+B\right)+\beta\sin 2\pi nt=0,$$

which requires

$$A\frac{Q}{c}+B=0, \qquad \beta=0.$$

If there is a node at $x=-l$

$$A\cos kl+Bk\sin kl=0;$$

$$\therefore k\tan kl=-\frac{A}{B}=-\frac{c}{Q}. \qquad (15)$$

This equation gives the fundamental note of the tube closed at $x=-l$; but it must be observed that l is not the length of the tube, because the origin $x=0$ is not in the mouth. There is, however, nothing indeterminate in the equation, although the origin is to a certain extent arbitrary, for the values of c and l will change together so as to make the result for k approximately constant. This will appear more clearly when we come, in Part II., to calculate the actual value of c for different kinds of mouths. In the formation of (14) the pressure of the air on the positive side at a distance from the origin small against λ has been taken absolutely constant. Across such a loop surface no energy could be transmitted. In reality, of course, the pressure is variable on account of the spherical waves, and energy continually escapes from the tube and its vicinity. Although the pitch of the resonant note is not affected, it may be worth while to see what correction this involves.

We must, as before, consider the space in which the transition from plane to spherical waves is effected as small compared with λ. The potential in free space may be taken

$$\psi=\frac{A'}{r}\cos(kr+g-2\pi nt), \qquad (16)$$

expressing spherical waves diverging from the mouth of the pipe, which is the origin of r. The origin of x is still supposed to lie in the region of plane waves.

$*4\pi r^2 \frac{d\psi}{dr}=$ rate of total flow across the surface of the sphere whose radius is r

$=-4\pi A'[\cos 2\pi nt\{\cos(kr+g)+kr\sin(kr+g)\}+\sin 2\pi nt\{\sin(kr+g)-kr\cos(kr+g)\}]$.

If the compression in the neighbourhood of the mouth is neglected, this must be the same as

$$Q\frac{d\psi}{dx=0}=QA\cos 2\pi nt.$$

Accordingly

$$AQ=-4\pi A'\{\cos(kr+g)+kr\sin(kr+g)\},$$
$$0=\sin(kr+g)-kr(\cos kr+g).$$

These equations express the connexion between the plane and spherical waves. From the second, $\tan(kr+g)=kr$, which shows that g is a small quantity of the order $(kr)^3$. From the first

$$A'=-\frac{AQ}{4\pi},$$

so that

$$\psi_r=-\frac{AQ}{4\pi r}\cos 2\pi nt-\frac{AQk}{4\pi}\sin 2\pi nt,$$

the terms of higher order being omitted.

Now within the space under consideration the air moves according to the same laws as electricity, and so

$$\frac{Q}{c}\frac{d\psi}{dx=0}=-\psi_{x=0}+\psi_r,$$

$$\frac{d\psi}{dx=0}=A\cos 2\pi nt,$$

$$\psi_{x=0}=B\cos 2\pi nt+\beta\sin 2\pi nt.$$

Therefore on substitution and equation of the coefficients of $\sin 2\pi nt$, $\cos 2\pi nt$, we obtain

$$\left.\begin{aligned} AQ\left(\frac{1}{c}+\frac{1}{4\pi r}\right)&=-B.\\ \beta&=-\frac{AQk}{4\pi}.\end{aligned}\right\}$$

When the mouth is not much contracted c is of the order of the radius of the mouth, and when there is contraction it is smaller still. In all cases therefore the term $\frac{1}{4\pi r}$ is very small compared to $\frac{1}{c}$; and we may put

$$\frac{AQ}{c}=-B,\quad \beta=-\frac{AQk}{4\pi}, \quad \ldots \ldots \ldots \quad (17)$$

* Throughout HELMHOLTZ's paper the mouth of the pipe is supposed to lie in an infinite plane, so that the diverging waves are hemispherical. The calculation of the value of c is thereby simplified. Except for this reason it seems better to consider the diverging waves completely spherical as a nearer approximation to the actual circumstances of organ-pipes, although the sphere could never be quite complete.

which agree nearly with the results of HELMHOLTZ. In his notation a quantity α is used defined by the equation

$$-\frac{A}{Bk}=\cot k\alpha,$$

so that

$$\cot k\alpha=\tan kl \text{ by (15)},$$

or

$$k(l+\alpha)=(2m+1)\frac{\pi}{2};$$

α may accordingly be considered as the correction to the length of the tube (measured, however, in our method only on the negative side of the origin), and will be given by

$$\cot k\alpha=-\frac{c}{kQ}.$$

The value of c will be investigated in Part II.

The original theory of open pipes makes the pressure absolutely constant at the mouth, which amounts to neglecting the inertia of the air outside. Thus, if the tube itself were full of air, and the external space of hydrogen, the correction to the length of the pipe might be neglected. The first investigation, in which no escape of energy is admitted, would apply if the pipe and a space round its mouth, large compared to the diameter, but small compared to the wave-length, were occupied by air in an atmosphere otherwise composed of incomparably lighter gas. These remarks are made by way of explanation, but for a complete discussion of the motion as determined by (13) and (17), I must refer to the paper of HELMHOLTZ.

Long Tube in connexion with a Reservoir.

It may sometimes happen that the length of a neck is too large compared to the quarter wave-length to allow the neglect of the compressibility of the air inside. A cylindrical neck may then be treated in the same way as the organ-pipe. The potential of plane waves inside the neck may, by what has been proved, be put into the form

$$\psi=A'\sin k(x-\alpha)\cos 2\pi nt;$$

if we neglect the escape of energy, which will not affect the pitch of the resonant note,

$$\frac{d\psi}{dt}=-2\pi nA'\sin k(x-\alpha)\sin 2\pi nt,$$

$$\frac{d\psi}{dx}=kA'\cos k(x-\alpha)\cos 2\pi nt,$$

where α is the correction for the outside end.

The rate of flow out of $S=Q\dfrac{d\psi}{dx}$.

Total flow $=Q\displaystyle\int\frac{d\psi}{dx}dt=kA'Q\cos kL\frac{\sin 2\pi nt}{2\pi n}$,

the reduced length of the tube, including the corrections for both ends, being denoted by L. Thus rarifaction in S

$$=k\frac{A'Q\cos kL}{S}\frac{\sin 2\pi nt}{2\pi n}=\frac{1}{a^2}\frac{d\psi}{dt}=\frac{2\pi nA'\sin kL}{a^2}\sin 2\pi nt.$$

This is the condition to be satisfied at the inner end. It gives

$$\tan kL=\frac{a^2}{4\pi^2n^2}\frac{kQ}{S}=\frac{Q}{kS}. \quad \quad (18)$$

When kL is small,

$$\tan kL=kL+\tfrac{1}{3}(kL)^3=\frac{Q}{kS};$$

$$\therefore\ k^2=\frac{Q}{LS}\left(1-\frac{1}{3}\frac{LQ}{S}\right),$$

$$n=\frac{a}{2\pi}\sqrt{\frac{Q}{LS}}\left(1-\frac{1}{6}\frac{LQ}{S}\right)=\frac{a}{2\pi}\sqrt{\frac{Q}{L(S+\frac{1}{3}LQ)}} \quad \quad (19)$$

In comparing this with (5), it is necessary to introduce the value of c, which is $\frac{Q}{L}$. (5) will accordingly give the same result as (19) if *one-third* of the contents of the neck be included in S. The first overtone, which is often produced by blowing in preference to the fundamental note, corresponds approximately to the length L of a tube open at both ends, modified to an extent which may be inferred from (18) by the finiteness of S.

The number of vibrations is given by

$$n=\frac{a}{2}\left(\frac{1}{L}+\frac{Q}{\pi^2S}\right). \quad \quad (20)$$

[The application of (20) is rather limited, because, in order that the condensation within S may be uniform as has been supposed, the linear dimension of S must be considerably less than the quarter wave-length; while, on the other hand, the method of approximation by which (20) is obtained from (18) requires that S should be large in comparison with QL.

A slight modification of (18) is useful in finding the pitch of pipes which are cylindrical through most of their length, but at the closed end expand into a bulb S of no great capacity. The only change required is to understand by L the length of the pipe down to the place where the enlargement begins with a correction for the *outer* end. Or if L denote the length of the tube simply, we have

$$\tan k(L+\alpha)=\frac{Q}{kS}, \quad \quad (20\,a)$$

and $\alpha=\frac{\pi}{4}R$ approximately.

If S be very small we may derive from (20 a)

$$n=\frac{a}{4\left(L+\alpha+\frac{S}{Q}\right)}. \quad \quad (20\,b)$$

In this form the interpretation is very simple, namely, that at the closed end the shape is of no consequence, and only the volume need be attended to. The air in this part of the pipe acts merely as a spring, its inertia not coming into play. A few measurements of this kind will be given in Part III.

The overtones of resonators which have not long necks are usually very high. Within the body of the reservoir a nodal surface must be formed, and the air on the further side vibrates as if it was contained in a completely closed vessel. We may form an idea of the character of these vibrations from the case of a sphere, which may be easily worked out from the equations given by Professor STOKES in his paper "On the Communication of Motion from a vibrating Sphere to a Gas"*. The most important vibration within a sphere is that which is expressed by the term of the first order in LAPLACE'S series, and consists of a swaying of the air from side to side like that which takes place in a doubly closed pipe. I find that for this vibration

$$\text{radius : wave-length} = \cdot 3313,$$

so that the note is higher than that belonging to a doubly closed (or open) pipe of the length of the diameter of the sphere by about a musical fourth. We might realize this vibration experimentally by attaching to the sphere a neck of such length that it would by itself, when closed at one end, have the same resonant note as the sphere.

Lateral Openings.

In most wind instruments the gradations of pitch are attained by means of lateral openings, which may be closed at pleasure by the fingers or otherwise. The common crude theory supposes that a hole in the side of, say, a flute establishes so complete a communication between the interior and the surrounding atmosphere, that a loop or point of no condensation is produced immediately under it. It has long been known that this theory is inadequate, for it stands on the same level as the first approximation to the motion in an open pipe in which the inertia of the air outside the mouth is virtually neglected. Without going at length into this question, I will merely indicate how an improvement in the treatment of it may be made.

Let ψ_1, ψ_2 denote the velocity-potentials of the systems of plane waves on the two sides of the aperture, which we may suppose to be situated at the point $x=0$. Then with our previous notation the conditions evidently are that when $x=0$,

$$\left.\begin{aligned} &\psi_1=\psi_2 \\ &\frac{Q}{c}\left(\frac{d\psi_1}{dx}-\frac{d\psi_2}{dx}\right)+\psi=0, \end{aligned}\right\} \quad \ldots \ldots \ldots \ldots \quad (20\,c)$$

the escape of energy from the tube being neglected. These equations determine the connexion between the two systems of waves in any case that may arise, and the working out is simple. The results are of no particular interest, unless it be for a comparison with experimental measurements, which, so far as I am aware, have not hitherto been made.]

* Professor STOKES informs me that he had himself done this at the request of the Astronomer Royal.

PART II.

In order to complete the theory of resonators, it is necessary to determine the value of c, which occurs in all the results of Part I., for different forms of mouths. This we now proceed to do. Frequent use will be made of a principle which might be called that of minimum *vis viva*, and which it may be well to state clearly at the outset.

Imagine a portion of incompressible fluid at rest within a closed surface to be suddenly set in motion by an arbitrary normal velocity impressed on the surface, then the actual motion assumed by the fluid will have less *vis viva* than any other motion consistent with continuity and with the boundary conditions*.

If u, v, w be the component velocities, and ρ the density at any point,

$$vis\ viva = \tfrac{1}{2}\iiint \rho(u^2+v^2+w^2)\,dx\,dy\,dz,$$

the integration extending over the volume considered. The minimum *vis viva* corresponding to prescribed boundary conditions depends of course on ρ; but if in any specified case we conceive the value of ρ in some places diminished and nowhere increased, we may assert that the minimum *vis viva* is *less* than before; for there will be a decrease if u, v, w remain unaltered, and therefore, *à fortiori*, when they have their actual values as determined by the minimum property. Conversely, an increase in ρ will necessarily raise the value of the minimum *vis viva*. The introduction of a rigid obstacle into a stream will always cause an increase of *vis viva*; for the new motion is one that might have existed before consistently with continuity, the fluid displaced by the obstacle remaining at rest. Any kind of obstruction in the air-passages of a musical instrument will therefore be accompanied by a fall of the note in the musical scale.

Long Tubes.

The simplest case that can be considered consists of an opening in the form of a cylindrical tube, so long in proportion to its diameter that the corrections for the ends may be neglected. If the length be L and area of section σ, the electrical resistance is $\frac{L}{\sigma}$, and

$$c=\frac{\sigma}{L}. \qquad (21)$$

For a circular cylinder of radius R

$$c=\frac{\pi R^2}{L}. \qquad (22)$$

Simple Apertures.

The next in order of simplicity is probably the case treated by HELMHOLTZ, where the opening consists of a simple hole in the side of the reservoir, considered as indefinitely thin and approximately plane in the neighbourhood of the opening. The motion of the

* THOMSON and TAIT's 'Natural Philosophy,' p. 230.

fluid in the plane of the opening is by the symmetry normal, and therefore the velocity-potential is constant over the opening itself. Over the remainder of the plane in which the opening lies the normal velocity is of course zero, so that ϕ may be regarded as the potential of matter distributed over the opening only. If the there constant value of the potential be called ϕ_1, the electrical resistance for *one side only* is

$$\phi_1 \div \iint \frac{d\phi}{dn} d\sigma,$$

the integration going over the area of the opening.

Now

$$\iint \frac{d\phi}{dn} d\sigma = 2\pi \times \text{ the whole quantity of matter;}$$

so that if we call M the quantity necessary to produce the unit potential,

$$\text{resistance for one side} = \frac{1}{2\pi M}.$$

Accordingly

$$c = \pi M. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

In electrical language M is the *capacity* of a conducting lamina of the shape of the hole when situated in an open space.

For a circular hole $M = \frac{2R}{\pi}$, and therefore

$$c = 2R. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

When the hole is an ellipse of eccentricity e and semimajor axis R,

$$c = \frac{\pi R}{F(e)}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

where F is the symbol of the complete elliptic function of the first order. Results equivalent to (23), (24), and (25) are given by HELMHOLTZ.

When the eccentricity is but small, the value of c depends sensibly on the area (σ) of the orifice only. As far as the square of e,

$$F(e) = \frac{\pi}{2}(1 + \tfrac{1}{4}e^2),$$

$$\sigma = \pi R^2 \sqrt{1 - e^2} = \pi R^2 (1 - \tfrac{1}{2}e^2),$$

$$R = \sqrt{\frac{\sigma}{\pi}}(1 + \tfrac{1}{4}e^2);$$

$$\therefore \; c = \pi \sqrt{\frac{\sigma}{\pi}} \div \frac{\pi}{2} = 2\sqrt{\frac{\sigma}{\pi}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

the fourth power of e being neglected—a formula which may be applied without sensible error to any orifice of an approximately circular form. In fact for a given area the circle is the figure which gives a minimum value to c, and in the neighbourhood of the minimum the variation is slow.

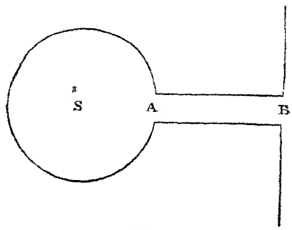
Next, consider the case of two circular orifices. If sufficiently far apart they act

independently of each other, and the value of c for the pair is the simple sum of the separate values, as may be seen either from the law of multiple arcs by considering c as the electric *conductivity* between the outside and inside of the reservoir, or from the interpretation of M in (23). The first method applies to any kind of openings with or without necks. As the two circles (which for precision of statement we may suppose equal) approach one another, the value of c diminishes steadily until they touch. The change in the character of the motion may be best followed by considering the plane of symmetry which bisects at right angles the line joining the two centres, and which may be regarded as a rigid plane precluding normal motion. Fixing our attention on half the motion only, we recognize the plane as an obstacle continually advancing, and at each step more and more obstructing the passage of fluid through the circular opening. After the circles come into contact this process cannot be carried further; but we may infer that, as they amalgamate and shape themselves into a single circle (the total area remaining all the while constant), the value of c still continues to diminish till it approaches its minimum value, which is less than at the commencement in the ratio of $\sqrt{2}:2$ or $1:\sqrt{2}$. There are very few forms of opening indeed for which the exact calculation of M or c can be effected. We must for the present be content with the formula (26) as applying to nearly circular openings, and with the knowledge that the more elongated or broken up the opening, the greater is c compared to σ. In the case of similar orifices or systems of orifices c varies as the linear dimension.

Cylindrical Necks.

Most resonators used in practice have necks of greater or less length, and even where there is nothing that would be called a neck, the thickness of the side of the reservoir could not always be neglected. For simplicity we shall take the case of circular cylinders whose inner ends lie on an approximately plane part of the side of the vessel, and whose outer ends are also supposed to lie in an infinite plane, or at least a plane whose dimensions are considerable compared to the diameter of the cylinder. Even under this form the problem does not seem capable of exact solution; but we shall be able to fix two slightly differing quantities between which the true value of c must lie, and which determine it with an accuracy more than sufficient for acoustical purposes. The object is to find the *vis viva* in terms of the rate of flow. Now, according to the principle stated at the beginning of Part II., we shall obtain too small a *vis viva* if at the ends A and B of the tube we imagine infinitely thin laminæ of fluid of infinitely small density. We may be led still more distinctly perhaps to the same result by supposing, in the electrical analogue, thin disks of perfectly conducting matter at the ends of the tube, whereby the effective resistance must plainly be lessened. The action of the disks is to produce uniform

Fig. 4.



potential over the ends, and the solution of the modified problem is obvious. Outside the tube the question is the same as for a simple circular hole in an infinite plane, and inside the tube the same as if the tube were indefinitely long.

Accordingly

$$\text{resistance}=\frac{L}{\pi R^2}+\frac{1}{2R}=\frac{1}{\pi R^2}\left(L+\frac{\pi}{2}R\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

The correction to the length is therefore $\frac{\pi}{2}R$, that is, $\frac{\pi}{4}R$ for each end,

$$c=\frac{\pi R^2}{L+\frac{\pi}{2}R}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

HELMHOLTZ, in considering the case of an organ-pipe, arrives at a similar conclusion,—that the correction to the length (α) is approximately $\frac{\pi}{4}R$. His method is very different from the above, and much less simple. He begins by investigating certain forms of mouths for which the exact solution is possible, and then, by assigning suitable values to arbitrary constants, identifies one of them with a true cylinder, the agreement being shown to be everywhere very close. Since the curve substituted for the generating line of the cylinder lies entirely outside it, HELMHOLTZ infers that the correction to the length thus obtained is too small.

If, at the ends of the tube, instead of layers of matter of no density, we imagine rigid pistons of no sensible thickness, we shall obtain a motion whose *vis viva* is necessarily *greater* than that of the real motion; for the motion with the pistons might take place without them consistently with continuity. Inside the tube the character of the motion is the same as before, but for the outside we require the solution of a fresh problem:—To determine the motion of an infinite fluid bounded by an infinite plane, the normal velocity over a circular area of the plane being a given constant, and over the rest of the plane zero. The potential may still be regarded as due to matter confined to the circle, but is no longer constant over its area; but the density of matter at any point, being proportional to $\frac{d\phi}{dn}$ or to the normal velocity, is constant.

The *vis viva* of the motion

$$=\tfrac{1}{2}\iint\phi\frac{d\phi}{dn}d\sigma=\tfrac{1}{2}\frac{d\phi}{dn}\iint\phi d\sigma,$$

the integration going over the area of the circle.

The rate of total flow through the plane

$$=\iint\frac{d\phi}{dn}d\sigma=\pi R^2\frac{d\phi}{dn};$$

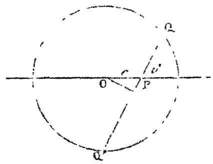
$$\therefore \frac{2 \text{ vis viva}}{(\text{rate of flow})^2}=\frac{\iint\phi d\sigma}{\pi^2R^4\frac{d\phi}{dn}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

We proceed to investigate the value of $\iint\phi d\sigma$, which is the *potential on itself* of a circular disk of unit density.

Potential on itself of a uniform circular disk.

r denoting the distance between any two points on the disk, the quantity to be evaluated is expressed by

Fig. 5.



$$\iint d\sigma \iint \frac{d\sigma'}{r}.$$

The first step is to find the potential at any point P, or $\iint\frac{d\sigma'}{r}$. Taking this point as an origin of polar coordinates, we have

$$\text{potential}=\iint\frac{d\sigma'}{r}=\iint\frac{rdrd\theta}{r}=\int rd\theta=\int(\mathrm{PQ}+\mathrm{PQ}')d\theta.$$

Now from the figure

$$\tfrac{1}{4}(\mathrm{QQ}')^2=\mathrm{R}^2-c^2\sin^2\theta,$$

where c is the distance of the point P from the centre of the circle whose radius is R. Thus potential at P

$$=2\mathrm{R}\int_0^\pi\sqrt{1-\frac{c^2}{\mathrm{R}^2}\sin^2\theta}\,d\theta=4\mathrm{R}\int_0^{\frac{\pi}{2}}\sqrt{1-\frac{c^2}{\mathrm{R}^2}\sin^2\theta}\,d\theta. \quad \ldots \ldots \quad (30)$$

Hence potential of disk on itself

$$=4\pi\mathrm{R}^3\int_0^1 dx\int_0^{\frac{\pi}{2}}\sqrt{1-x\sin^2\theta}\,d\theta,$$

if for the sake of brevity we put $\frac{c^2}{\mathrm{R}^2}=x$.

In performing first the integration with respect to θ we come upon elliptic functions, but they may be avoided by changing the order of integration.

$$\int_0^1 dx\sqrt{1-x\sin^2\theta}=\left\{-\frac{2}{3\sin^2\theta}(1-x\sin^2\theta)^{\frac{3}{2}}\right\}_0^1$$

$$=\frac{2}{3\sin^2\theta}(1-\cos^3\theta)=\frac{2}{3}\frac{1}{1+\cos\theta}+\frac{2}{3}\cos\theta;$$

$\therefore$ potential on itself

$$=\tfrac{8}{3}\pi\mathrm{R}^3\int_0^{\frac{\pi}{2}}d\theta\left\{\frac{1}{1+\cos\theta}+\cos\theta\right\}=\tfrac{8}{3}\pi\mathrm{R}^3\{1+1\}=\tfrac{16}{3}\pi\mathrm{R}^3. \quad \ldots \ldots \quad (31)$$

This, therefore, is the value of $\iint\phi d\sigma$ when the density is supposed equal to unity. The corresponding value of

$$\frac{d\phi}{dn}=2\pi.$$

and so from (29)

$$\frac{2 \textit{ vis viva}}{(\text{rate of flow})^2}=\frac{8}{3\pi^2 R}. \qquad (32)$$

This is for the space outside one end. For the whole tube and both ends

$$\frac{2 \textit{ vis viva}}{(\text{rate of flow})^2}=\frac{L}{\pi R^2}+\frac{16}{3\pi^2 R}. \qquad (33)$$

Whatever, then, may be the ratio of L : R, the electrical resistance to the passage in question or $\frac{1}{c}$ is limited by

$$\left.\begin{aligned}\frac{1}{c}&>\frac{L}{\pi R^2}+\frac{1}{2R}\\ &<\frac{L}{\pi R^2}+\frac{16}{3\pi^2 R}\end{aligned}\right\}. \qquad (34)$$

In practical application it is sometimes convenient to use the quantity α or correction to the length. In terms of α (34) becomes

$$\left.\begin{aligned}\alpha&>\frac{\pi}{2}R\\ &<\frac{16}{3\pi}R.\end{aligned}\right\}$$

or in decimals,

$$\left.\begin{aligned}\alpha&>(1\cdot571R=2\times\cdot785R)\\ \alpha&<(1\cdot697R=2\times\cdot849R)\end{aligned}\right\}. \qquad (34')$$

The corrections for *both* ends is the thing here denoted by α. Of course for one end it is only necessary to take the half *.

I do not suppose that any experiments hitherto made with organ-pipes could discriminate with certainty between the two values of α in (34′). If we adopt the mean provisionally, we may be sure that we are not wrong by so much as ·032 R for each end.

Our upper limit to the value of α expressed in (34′) was found by considering the hypothetical case of a uniform velocity over the section of the mouth, and we fully determined the non-rotational motion both for the inside and for the outside of the tube. Of course the velocity is not really uniform at the mouth; it is, indeed, infinite at the edge. If we could solve the problem for the inside and outside when the velocity (normal) at the mouth is of the form $a+br^2$, we should with a suitable value of $b:a$ get a much better approximation to the true *vis viva*. The problem for the outside may be solved, but for the inside it seems far from easy. It is possible, however, that we may

* Though not immediately connected with our present subject, it may be worth notice that if at the centre of the tube, or anywhere else, the velocity be constrained (by a piston) to be constant across the section, as it would approximately be if the tube were very long, without a piston, the limiting inequalities (34) still hold good. For large values of L the two cases do not sensibly differ, but for small values of L compared to R the true solution of the original problem tends to coincide with the lower limit, and of the modified (central piston) problem with the higher.

be able to find some motion for the inside satisfying the boundary conditions and the equation of continuity, which, though of a rotational character, shall yet make the whole *vis viva* for the inside and outside together less than that previously obtained. At the same time this *vis viva* is by THOMSON's law necessarily greater than the one we seek.

Motion in a finite cylindrical tube, the axial velocity at the plane ends ($x=0$ and $x=l$) being

$$u=u_0+\chi(r), \qquad (35)$$

where

$$\int_0^1 r\chi(r)dr=0, \qquad (36)$$

r being the transverse coordinate, and the radius of the cylinder being put equal to 1.

If u, v be the component velocities, the continuity equation is

$$\frac{du}{dx}+\frac{1}{r}\frac{d(rv)}{dr}=0, \qquad (37)$$

whence

$$\left.\begin{aligned} ru&=\frac{d\psi}{dr}, \\ rv&=-\frac{d\psi}{dx}, \end{aligned}\right\} \qquad (37')$$

where ψ is arbitrary so far as (37) is concerned.

Take

$$\psi=u_0\frac{r^2}{2}+\phi(x)\int_0^r r\chi(r)dr,$$

so that

$$\left.\begin{aligned} u&=u_0+\phi(x)\chi(r), \\ v&=-\phi'(x)\frac{1}{r}\int_0^r r\chi rdr. \end{aligned}\right\} \qquad (38)$$

It is clear from (38) that if

$$\phi(0)=\phi(l)=1. \qquad (39)$$

$$u_{x=0}=u_0+\chi r=u_{x=l}$$

$$v_{r=1}=0 \text{ for all values of } x.$$

Thus (38) satisfies the boundary conditions including (35), and ϕ is still arbitrary, except in so far as it is limited by (39).

In order to obtain an expression for the *vis viva*, we must integrate u^2+v^2 over the volume of the cylinder.

$$\left.\begin{aligned} \text{Twice } \textit{vis viva}&=u_0^2\pi l+2u_0\int_0^l \phi(x)dx\int_0^1 \chi(r)2\pi rdr \\ &+\int_0^l \overline{\phi x}^2dx\int_0^1 2\pi\overline{\chi(r)}^2rdr \\ &+\int_0^l \overline{\phi' x}^2dx\int_0^1 \frac{2\pi dr}{r}\left\{\int_0^r r\chi(r)dr\right\}^2. \end{aligned}\right\} \qquad (40)$$

The second term vanishes in virtue of (36), and we may write

$$\text{Twice } vis\ viva = u_0^2 \pi l + \int_0^l (\mathrm{A}y^2 + \mathrm{B}y'^2)dx, \quad \ldots \quad (40')$$

where A and B are known quantities depending on χ, and $y=\varphi(x)$ is so far an arbitrary function, which we shall determine so as to make the *vis viva* a minimum.

By the method of variations

$$y = \mathrm{C}\varepsilon^{-\sqrt{\frac{\mathrm{A}}{\mathrm{B}}}x} + \mathrm{C}'\varepsilon^{+\sqrt{\frac{\mathrm{A}}{\mathrm{B}}}x}; \quad \ldots \quad (41)$$

and in order to satisfy (39),

$$\left.\begin{aligned} 1 &= \mathrm{C} + \mathrm{C}', \\ 1 &= \mathrm{C}\varepsilon^{-\sqrt{\frac{\mathrm{A}}{\mathrm{B}}}l} + \mathrm{C}'\varepsilon^{+\sqrt{\frac{\mathrm{A}}{\mathrm{B}}}l} \end{aligned}\right\} \quad \ldots \quad (42)$$

(41) and (42) completely determine y as a function of x, and when this value of y is used in (40) the *vis viva* is less than with any other form of y. On substitution in (40′),

$$\text{Twice } vis\ viva = u_0^2 \pi l + 2\sqrt{\mathrm{AB}}\,\frac{1-\varepsilon^{-2\sqrt{\frac{\mathrm{A}}{\mathrm{B}}}l}}{1+\varepsilon^{-2\sqrt{\frac{\mathrm{A}}{\mathrm{B}}}l}}. \quad \ldots \quad (43)$$

The *vis viva* expressed in (43) is less than any other which can be derived from the equation (38); but it is not the least possible, as may be seen by substituting the value of ψ in the stream-line equation

$$\frac{d^2\psi}{dx^2} - \frac{1}{r}\frac{d\psi}{dr} + \frac{d^2\psi}{dr^2} = 0,$$

which will be found to be *not* satisfied.

The next step is to introduce special forms of χ. Thus let

$$u_{x=0} = 1 + \mu r^2.$$

Then
$$u_0 = 1 + \tfrac{1}{2}\mu,$$
$$\chi = \mu(-\tfrac{1}{2} + r^2).$$

Accordingly

$$\mathrm{A} = \frac{\pi\mu^2}{12}, \qquad \mathrm{B} = \frac{1}{16}\cdot\frac{\pi\mu^2}{12}, \qquad \sqrt{\mathrm{AB}} = \frac{\pi\mu^2}{48}; \qquad \sqrt{\frac{\mathrm{A}}{\mathrm{B}}} = 4,$$

and (43) becomes

$$2\ vis\ viva = \pi l(1+\tfrac{1}{2}\mu)^2 + \frac{\pi\mu^2}{24}\,\frac{1-\varepsilon^{-8l}}{1+\varepsilon^{-8l}}. \quad \ldots \quad (44)$$

We have in (44) the *vis viva* of a motion within a circular cylinder which satisfies the continuity equation, and which makes over the plane ends

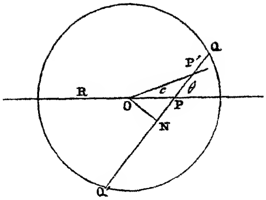
$$u = 1 + \mu r^2.$$

If $\mu=0$ we fall back on the simple case considered before; and this is the value of μ for which the *vis viva* in (44) is a minimum compared to the rate of flow $(1+\frac{1}{2}\mu)$. But for the part outside the cylinder the *vis viva* is, as we may anticipate, least when μ has

some finite value; so that when we consider the motion as a whole it will be a finite value of μ that gives the least *vis viva.*

The *vis viva* of the motion outside the ends is to be found by the same method as before, the first step being to determine the potential at any point of a circular disk whose density $=\mu r^2$;

Fig. 6.



$$\text{potential at P}=\iint\frac{\varrho d\varrho d\theta}{\varrho}\mu\text{OP}'^2,$$

where

$$\text{OP}'^2=c^2+\varrho^2+2c\varrho\cos\theta;$$

$$\therefore\ \text{potential at P}=\int\mu d\theta\left\{c^2\varrho+\frac{\varrho^3}{3}+c\varrho^2\cos\theta\right\};$$

or if previously to integration with respect to θ we add together the elements from Q to Q′,

$$=\mu\int_0^{\pi}d\theta(\text{PQ}+\text{PQ}')\left\{c^2+\frac{\text{PQ}^2+\text{PQ}'^2-\text{PQ}\,.\,\text{PQ}'}{3}+c\cos\theta(\text{PQ}-\text{PQ}')\right\}.$$

Now

$$\text{PQ}+\text{PQ}'=2\sqrt{\text{R}^2-c^2\sin^2\theta},$$

$$\text{PQ}-\text{PQ}'=-2\text{PN}=-2c\cos\theta,$$

$$\text{PQ}\,.\,\text{PQ}'=\text{R}^2-c^2.$$

Thus potential at $\text{P}=\frac{4\text{R}^3}{3}\mu\int_0^{\frac{\pi}{2}}d\theta\sqrt{1-c^2\sin^2\theta}(1+2c^2\sin^2\theta)$, c being written for $c\div\text{R}$.

To this must be added the potential for a uniform disk found previously, and the result must be multiplied by the compound density and integrated again over the area, the order of integration being changed as before so as to take first the integration with respect to c. In this way elliptic functions are avoided; but the process is too long to be given here, particularly as it presents no difficulty. The result is that the potential on itself of a disk whose density

$$=1+\mu\frac{r^2}{\text{R}^2}$$

is expressed by

$$\frac{16\pi\text{R}^3}{3}(1+\tfrac{14}{15}\mu+\tfrac{5}{21}\mu^2)^*.\quad\ldots\ldots\ldots\ldots\quad(45)$$

* [Mr. Clerk Maxwell has pointed out a process by which this result may be obtained much more simply. Begin by finding the potential at the *edge* of the disk whose density is $1+\mu r^2$. Taking polar coordinates (ρ,θ), the pole being at the edge, we have

$$r^2=\rho^2+a^2-2a\rho\cos\theta$$

and

$$\text{V}=\iint\{1+\mu(\rho^2+a^2-2a\rho\cos\theta)\}d\theta d\rho,$$

the limits of ρ being 0 and $2a\cos\theta$, and those of θ being $-\frac{\pi}{2}$ and $+\frac{\pi}{2}$. We get at once

$$\text{V}=4a+\tfrac{20}{9}\mu a^3.$$

Thus if for brevity we put $R=1$, we may express the *vis viva* of the whole motion (both extremities included) by

$$2\ vis\ viva=\pi l(1+\tfrac{1}{2}\mu)^2+\frac{\pi\mu^2}{24}\frac{1-\epsilon^{-8l}}{1+\epsilon^{-8l}}+\frac{16}{3}\left(1+\frac{14}{15}\mu+\frac{5}{21}\mu^2\right),$$

which corresponds to the rate of flow $\pi u_0=\pi(1+\frac{1}{2}\mu)$.

Thus

$$\frac{2\ vis\ viva}{(\text{rate of flow})^2}=\frac{l}{\pi}+\frac{1}{3\pi}\frac{\frac{\Lambda}{8}\mu^2+\frac{16}{\pi}\left(1+\frac{14}{15}\mu+\frac{5}{21}\mu^2\right)}{(1+\frac{1}{2}\mu)^2},\quad \ldots\ldots \quad (46)$$

where $\Lambda=\frac{1-\epsilon^{-8l}}{1+\epsilon^{-8l}}$.

The second fraction on the right of (46) is next to be made a minimum by variation of μ. Putting it equal to z and multiplying up, we get the following quadratic in μ:—

$$\mu^2\left\{\frac{\Lambda}{8}+\frac{5\cdot16}{21\pi}-\frac{z}{4}\right\}+2\mu\left\{\frac{16\cdot7}{15\pi}-\frac{z}{2}\right\}+\frac{16}{\pi}-z=0.$$

The smallest value of z consistent with a real value of μ is therefore given by

$$\left(\frac{16\cdot7}{15\pi}-\frac{z}{2}\right)^2-\left(\frac{16}{\pi}-z\right)\left(\frac{\Lambda}{8}+\frac{5\cdot16}{21\pi}-\frac{z}{4}\right)=0$$

$$z=\frac{2\Lambda+\frac{8192}{1575\pi}}{\frac{\Lambda\pi}{8}+\frac{12}{35}}=\frac{2\Lambda+1\cdot6556}{\cdot3927\Lambda+\cdot3429}=\frac{3\cdot6556-\cdot3444\epsilon^{-8l}}{\cdot7356-\cdot0498\epsilon^{-8l}}.$$

Thus

$$\frac{2\ vis\ viva}{(\text{rate of flow})^2}=\frac{l}{\pi}+\frac{1}{3\pi}\frac{3\cdot6556-\cdot3444\epsilon^{-8l}}{\cdot7356-\cdot0498\epsilon^{-8l}}.\quad \ldots\ldots \quad (47)$$

This gives an upper limit to $\frac{1}{c}$. In terms of α (including both ends)

$$\alpha<2\cdot305R\frac{10\cdot615-\epsilon^{-8\frac{l}{R}}}{14\cdot771-\epsilon^{-8\frac{l}{R}}}.\quad \ldots\ldots \quad (47')$$

From (47′) we see that the limit for α is smallest when $l=0$, and gradually increases with l.

Now let us cut off a strip of breadth da from the edge of the disk, whose mass is accordingly

$$2\pi a(1+\mu a^2)da.$$

The work done in carrying this strip off to infinity is

$$2\pi a da(1+\mu a^2)(4a+\tfrac{20}{9}\mu a^3).$$

If we gradually pare the disk down to nothing and carry all the parings to infinity, we find for the total work by integrating with respect to a from 0 to R,

$$\frac{8\pi R^3}{3}\left(1+\frac{14}{15}\mu+\frac{5}{21}\mu^2\right),$$

μ being written for μR^2. This is, as it should be, the half of the expression in the text.]

When $l=\infty$, it becomes

$$1{\cdot}6565\,\mathrm{R}=2\times{\cdot}8282\,\mathrm{R}.$$

Thus the correction for one end of an infinite tube is limited by

$$\left.\begin{array}{l}\alpha>{\cdot}785\,\mathrm{R}\\ \quad<{\cdot}8282\,\mathrm{R}\end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (48)$$

When l is not infinitely great the upper limit may be calculated from (47′), the lower limit remaining as before; but it is only for quite small values of l that the exponential terms in (47′) are sensible. It is to be remarked that the *real* value of α is least when $l=0$, and gradually increases to its limit when $l=\infty$. For consider the actual motion for any finite value of l. The *vis viva* of the motion going on in any middle piece of the tube is greater than corresponds merely to the length. If the piece therefore be removed and the ends brought together, the same motion may be supposed to continue without violation of continuity, and the *vis viva* will be more diminished than corresponds to the length of the piece cut out. *A fortiori* will this be true of the real motion which would exist in the shortened tube. Thus α steadily decreases as the tube is shortened until when $l=0$ it coincides with the lower limit $\frac{\pi}{4}\mathrm{R}$.

In practice the outer end of a rather long tube-like neck cannot be said generally to end in an infinite plane, as is supposed in the above calculation. On the contrary, there could ordinarily be a certain flow back round the edge of tube, the effect of which must be sensibly to diminish α. It would be interesting to know the exact value of α for an infinite tube projecting into unlimited space free from obstructing bodies, the thickness of the cylindrical tube being regarded as vanishingly small. HELMHOLTZ has solved what may be called the corresponding problem in two dimensions; but the difficulty in the two cases seems to be of quite a different kind. Fortunately our ignorance on this point is not of much consequence for acoustical purposes, because when the necks are short the hypothesis of the infinite plane agrees nearly with the fact, and when the necks are long the correction to the length is itself of subordinate importance.

Nearly Cylindrical Tubes of Revolution.

The non-rotational flow of a liquid in a tube of revolution or of electricity in a similar solid conductor can only in a few cases be exactly determined. It may therefore be of service to obtain formulæ fixing certain limits between which the *vis viva* or resistance must lie. First, considering the case of electricity (for greater simplicity of expression), let us conceive an indefinite number of infinitely thin but at the same time perfectly conducting planes to be introduced perpendicular to the axis. Along these the potential is necessarily constant, and it is clear that their presence must *lower* the resistance of the conductor in question. Now at the point x (axial coordinate) let the radius of the conductor be y, so that its section is πy^2. The resistance between two of the above-mentioned planes which are close to one another and to the point x will be in the limit

$dx \div \pi y^2$, if dx be the distance between the planes, the resistance of the unit cube being unity. Thus resistance

$$=\int \frac{dx}{\pi y^2} * \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (49)$$

Upper Limit.

Secondly, we know that in the case of a liquid the true *vis viva* is less than that of any other motion which satisfies the boundary conditions and the equation of continuity. Now u, v being the axial and transverse velocities, it will always be possible so to determine v as to satisfy the conditions if we assume $u=$ constant over the section, and therefore

$$u=\frac{u_0}{\pi y^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (50)$$

This may be seen by imagining rigid pistons introduced perpendicular to the axis. To determine v it is convenient to use the function ψ, which is related to u and v according to the equations (37),

$$ru=\frac{d\psi}{dr}, \qquad rv=-\frac{d\psi}{dx}.$$

These forms for u and v secure the fulfilment of the continuity equation. Since

$$u=\frac{u_0}{\pi y^2}, \qquad \frac{d\psi}{dr}=r\frac{u_0}{\pi y^2},$$

$$\psi=\psi_0(x)+\frac{u_0}{2\pi y^2}r^2,$$

and therefore

$$v=-\frac{\psi'_0(x)}{r}-\frac{u_0}{2\pi}r\frac{d}{dx}\left(\frac{1}{y^2}\right).$$

But since v cannot be infinite on the axis, but must, on the contrary, be zero,

$$\psi'_0(x)=0,$$

and we have

$$\left.\begin{aligned} u&=\frac{u_0}{\pi y^2}, \\ v&=-\frac{u_0}{2\pi}r\frac{d}{dx}\left(\frac{1}{y^2}\right) \end{aligned}\right\}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (51)$$

From the manner in which these were obtained, they must satisfy the condition of

* [It is easy to show formally that no error can arise from neglecting the effect of the curved rim. Imagine the planes at x and $x+dx$ extended, and the curves in which they cut the surface of the conductor projected by lines parallel to the axis. In this way a cylinder is formed which contains the whole surface between x and $x+dx$, and another cylinder which is entirely contained by the surface. The small cylinder may be obtained by supposing part of the matter not to conduct, and therefore gives too great a resistance. On the other hand, the real solid may be obtained from the large cylinder by the same process. The resistance of the slice lies accordingly between those of the two cylinders which are themselves equal in the limit. Hence, on the whole, the parts neglected vanish compared to those retained.]

giving no normal motion at the surface of the tube. That this is actually the case may be easily verified *à posteriori*, but it is scarcely necessary for our purpose to do so. To find the *vis viva*,

$$\int_0^y u^2 2\pi r dr = \frac{u_0^2}{\pi y^2},$$

$$\iint u^2 2\pi r dr dx = \frac{u_0^2}{\pi}\int \frac{1}{y^2}\, dx,$$

$$\int v^2 2\pi r dr = \frac{u_0^2 y^4}{8\pi}\left[\frac{d}{dx}\left(\frac{1}{y^2}\right)\right]^2,$$

$$\iint v^2 2\pi r dr dx = \frac{u_0^2}{8\pi}\int y^4\left[\frac{d}{dx}\left(\frac{1}{y^2}\right)\right]^2 dx = \frac{u_0^2}{2\pi}\int \frac{1}{y^2}\left(\frac{dy}{dx}\right)^2 dx.$$

Thus
$$\textit{vis viva} = \frac{u_0^2}{2\pi}\int \frac{1}{y^2}\left\{1+\tfrac{1}{2}\left(\frac{dy}{dx}\right)^2\right\}dx.$$

The total flow across any section is $\pi y^2 u = u_0$.

Therefore

$$\frac{2\ \textit{vis viva}}{(\text{rate of flow})^2} = \frac{1}{\pi}\int \frac{1}{y^2}\left\{1+\frac{1}{2}\left(\frac{dy}{dx}\right)^2\right\} dx. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (52)$$

This is the quantity which gives an upper limit to the resistance. The first term, which corresponds to the component u of the velocity, is the same as that previously obtained for the lower limit, as might have been foreseen. The difference between the two, which gives the utmost error involved in taking either of them as the true value, is

$$\frac{1}{2\pi}\int \frac{1}{y^2}\left(\frac{dy}{dx}\right)^2 dx.$$

In a nearly cylindrical tube $\frac{dy}{dx}$ is a small quantity, and so the result found by this method is closely approximate. It is not necessary that the section of the tube should be nearly constant, but only that it should vary slowly. The success of the approximation in this and similar cases depends in great measure on the fact that the quantity to be estimated is a minimum. Any reasonable approximation to the real motion will give a *vis viva* very near the minimum, according to the principles of the differential calculus.

Application to straight tube of revolution whose end lies on two infinite planes.

For the lower limit to the resistance we have

$$\frac{1}{\pi}\int \frac{dx}{y^2} + \frac{1}{4R_1} + \frac{1}{4R_2},$$

R_1, R_2 being the radii at the ends, and for the higher limit

$$\frac{1}{\pi}\int \frac{1}{y^2}\left\{1+\tfrac{1}{2}\left(\frac{dy}{dx}\right)^2\right\}dx + \frac{8}{3\pi^2}\left(\frac{1}{R_1}+\frac{1}{R_2}\right).$$

The first expression is obtained by supposing infinitely thin but perfectly conducting planes perpendicular to the axis to be introduced from the ends of the tube inwards, while in the second the conducting planes in the electical interpretation are replaced by pistons in the hydrodynamical analogue. For example, let the tube be part of a cone of semivertical angle θ.

The lower limit is

$$\frac{1}{\pi\tan\theta}\left(\frac{1}{R_1}\sim\frac{1}{R_2}\right)+\frac{1}{4}\left(\frac{1}{R_1}+\frac{1}{R_2}\right),$$

and the higher

$$\frac{1+\tan^2\theta}{\pi\tan\theta}\left(\frac{1}{R_1}\sim\frac{1}{R_2}\right)+\frac{8}{3\pi^2}\left(\frac{1}{R_1}+\frac{1}{R_2}\right).$$

Tubes nearly straight and cylindrical but not necessarily of revolution.

Taking the axis of x in the direction of the length, we readily obtain by the same process as before a *lower* limit to the resistance

$$\int\frac{dx}{\sigma}, \qquad (53)$$

where σ denotes the section of the tube by a plane perpendicular to the axis at the point x, an expression which has long been known and is sometimes given as rigorous. The conductor (for I am now referring to the electrical interpretation) is conceived to be divided into elementary slices by planes perpendicular to the axis, and the resistance of any slice is calculated as if its faces were at constant potentials, which is of course not the case. In fact it is meaningless to talk of the resistance of a limited solid at all, unless with the understanding that certain parts of its surface are at constant potentials, while other parts are bounded by non-conductors. Thus, when the resistance of a cube is spoken of, it is tacitly assumed that two of the opposite faces are at constant potentials, and that the other four faces permit no escape of electricity across them. In some cases of unlimited conductors, for instance one we have already contemplated—an infinite solid almost divided into two separate parts by an infinite insulating plane with a hole in it—it is allowable to speak of the resistance without specifying what particular surfaces are regarded as equipotential; for at a sufficient distance from the opening on either side the potential is constant, and any surface no part of which approaches the opening is approximately equipotential. After this explanation of the exact significance of (53), we may advantageously modify it into a form convenient for practical use.

The section of the tube at n different points of its length l is obtained by observing the length λ of a mercury thread which is caused to traverse the tube. Replacing the integration by a summation denoted by the symbol Σ, we arrive at the formula

$$\text{resistance}=\frac{l^2}{n^2V}\Sigma\lambda\,\Sigma\frac{1}{\lambda}, \qquad (54)$$

which was used by Dr. MATTHIESSEN in his investigation of the mercury unit of electrical

resistance, and was the subject of some controversy*. It is perfectly correct in the sense that when the number of observations is increased without limit it coincides with (53), *itself, however, only an approximation to the magnitude sought.* The extension of our second method (for the higher limit) to tube not of revolution would require the general solution of the potential problem in two dimensions. It may be inferred that the difference between the two limits is of the order of the square of the inclination of the tangent plane to the axis, and is therefore very small when the section of the tube alters but slowly.

Tubes not nearly straight.

In applying (53) to such cases, we are at liberty to take any straight line we please as axis; but if the tube is much bent, even though its cross section remain nearly constant, the approximation will cease to be good. This is evident, because the planes of constant potential must soon become very oblique, and the section σ used in the formula much greater than the really effective section of the tube. To meet this difficulty a modification in the formula is necessary. Instead of taking the artificial planes of equal potential all perpendicular to a straight line, we will now take them normal to a curve which may have double curvature, and which should run, as it were, along the middle of the tube. Consecutive planes intersect in a straight line passing through the centre of curvature of the "axis" and perpendicular to its plane.

The resistance between two neighbouring equipotential planes is in the limit

$$\delta\theta \div \iint \frac{d\sigma}{r},$$

where $\delta\theta$ is the angle between the planes, and r is the distance of any element $d\sigma$ of the

* See SABINE's 'Electric Telegraph,' p. 329. To prove (54), we have

$$\text{resistance} = \frac{l}{n}\Sigma\frac{1}{\sigma}, \text{ and } \sigma\lambda = \text{constant} = \kappa, \text{ say.}$$

$$\therefore \Sigma\frac{1}{\sigma} = \frac{1}{\kappa}\Sigma\lambda.$$

But $V = \text{volume} = \frac{l}{n}\Sigma\sigma = \frac{l}{n}\kappa\Sigma\frac{1}{\lambda}$.

$$\therefore \frac{1}{\kappa} = \frac{l}{nV}\Sigma\frac{1}{\lambda},$$

and

$$\Sigma\frac{1}{\sigma} = \frac{l}{nV}\Sigma\lambda\,\Sigma\frac{1}{\lambda}.$$

The correction for the ends of the tube employed by SIEMENS is erroneous, being calculated on the supposition that the divergence of the current takes place from the curved surface of a hemisphere of radius equal to that of the tube. This is tantamount to assuming a constant potential over the solid hemisphere conceived as of infinite conductivity, and gives of course a result too small—R for both ends together. The proper correction, which probably is not of much importance, would depend somewhat upon the mode of connexion of the tube with the terminal cups, but cannot differ much from $\frac{\pi}{2}$ R (for both ends), as we have seen. (I have since found that SIEMENS was aware of the small error in this correction.)

section from the line of intersection of the planes. Now $\delta\theta = ds \div \rho$, if ds be the intercept on the axis between the normal planes, and ρ the radius of curvature at the point in question. The lower limit to the resistance is thus expressed by

$$\int ds \frac{1}{\iint \frac{\rho d\sigma}{r}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (55)$$

In the particular case of a tube of revolution (such as an anchor-ring) $\iint \frac{\rho d\sigma}{r}$ is a constant, and the limit which now coincides with the true resistance varies as the length of the axis, and is evidently independent of its position. In general the value of the integral will depend on the axis used, but it is in every case less than the true value of the resistance. In choosing the axis, the object is to make the artificial planes of constant potential agree as nearly as possible with the true equipotential surfaces.

A still further generalization is possible by taking for the artificial equipotential surfaces those represented by the equation

$$F = \text{const.}$$

For all systems of surfaces, with one exception, the resistance found on this assumption will be too small. The exception is of course when the surfaces $F =$ const. coincide with the undisturbed equipotential surfaces. The element of resistance between the surfaces F and $F + dF$ is

$$\frac{1}{\iint \frac{1}{dn} d\sigma},$$

where dn is the distance between the surfaces at the element $d\sigma$, and the integration goes over the surface F as far as the edge of the tube. Now

$$dn = dF \div \sqrt{\left(\frac{dF}{dx}\right)^2 + \left(\frac{dF}{dy}\right)^2 + \left(\frac{dF}{dz}\right)^2};$$

$\therefore$ limit to resistance

$$= \int \frac{dF}{\iint d\sigma \sqrt{\left(\frac{dF}{dx}\right)^2 + \left(\frac{dF}{dy}\right)^2 + \left(\frac{dF}{dz}\right)^2}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (56)$$

an expression whose form remains unchanged when $f(F)$ is written for F. If $F = r$, so that the surfaces are spheres,

$$\Sigma \left(\frac{dF}{dx}\right)^2 = \left(\frac{dF}{dr}\right)^2 = 1;$$

$$\therefore \text{ limit } = \int \frac{dr}{\iint d\sigma} = \int \frac{dr}{\sigma}.$$

This form would be suitable for approximately conical tubes, the vertex of the cone being taken as origin of r.

The last formulæ, (55) and (56), are perhaps more elaborate than is required in the present state of acoustical science, and it is rather in the theory of electricity that their interest would lie; but they present themselves so readily as generalizations of previous results that I hope that they are not altogether out of place in the present paper. In all these cases we have the advantage that the quantity sought is determined by a minimum property, and is therefore subject to a much smaller error than exists in the conditions which determine it.

PART III.

Experimental.

The object of this Part is to detail some experiments on resonators instituted with a view of comparing some of the formulæ of Parts I. and II. with observation. HELMHOLTZ in his paper on organ-pipes has compared his own theory with the experiments of SONDHAUSS and WERTHEIM for the case of resonators whose communication with the external atmosphere is by simple holes in their sides. The theoretical result is embodied in (5) and (23), or for circular holes (24) and runs,

$$n=\frac{a}{2\pi}\sqrt{\frac{2R}{S}}=\frac{a}{\pi}\sqrt{\frac{R}{2S}}; \qquad (57)$$

or when the area of the opening is approximately circular and of magnitude σ,

$$n=\frac{a\sigma^{\frac{1}{4}}}{2^{\frac{1}{2}}\pi^{\frac{3}{4}}S^{\frac{1}{2}}}. \qquad (58)$$

On calculation HELMHOLTZ finds

$$n=56174\,\frac{\sigma^{\frac{1}{4}}}{S^{\frac{1}{2}}}*,$$

the unit of length being metrical.

The empirical formula found by SONDHAUSS is

$$n=52400\,\frac{\sigma^{\frac{1}{4}}}{S^{\frac{1}{2}}},$$

which agrees completely with theory as regards its form, but not so well in the value it assigns to the constant multiplier. The difference corresponds to more than a semitone, and is in the direction that the observed notes are all too low. I can only think of two explanations for the discordance, neither of which seem completely satisfactory. In the first place, SONDHAUSS determined his resonant notes by the pitch of the sound produced when he blew obliquely across the opening through a piece of pipe with a flattened end. It is possible that the proximity of the pipe to the opening was such as to cause an obstruction in the air-passage which might sensibly lower the pitch. Secondly, no account is taken of the thickness of the side of the vessel, the effect of which must be

* The velocity of sound is taken at the freezing-point; otherwise the discordance would be greater.

to make the calculated value of n too great. On the other hand, two sources of error must be mentioned which would act in the opposite direction. The air in the vicinity of the opening must have been sensibly warmer than the external atmosphere, and we saw in Part I. how sensitive resonators of this sort must be to small changes in the physical properties of the gas which occupy the air-passages. Indeed SAVART long ago remarked on the instability of the pitch of short pipes, comparing them with ordinary organ-pipes. The second source of disturbance is of a more recondite character, but not, I think, less real. It is proved in works on hydrodynamics that in the steady motion of fluids, whether compressible or not, an increased velocity is always accompanied by a diminished pressure. In the case of a gas the diminished pressure entails a diminished density. There seems therefore every reason to expect a diminution of density in the stream of air which plays over the orifice of the resonator, which must cause a rise in the resonant note. But independently of these difficulties, the theory of pipes or other resonators made to speak by a stream of air directed against a sharp edge is not sufficiently understood to make this method of investigation satisfactory. For this reason I have entirely abandoned the method of causing the resonators to speak in my experiments, and have relied on other indications to fix the pitch. The only other experiments that I have met with on the subject of the present paper are also by SONDHAUSS, who has been very successful in unravelling the complications of these phenomena without much help from theory*. For flasks with long necks he found the formula

$$n=46705\frac{\sigma^{\frac{1}{2}}}{L^{\frac{1}{2}}S^{\frac{1}{2}}}$$

as applicable when the necks are cylindrical and not too short, corresponding to the theoretical

$$n=\frac{a}{2\pi}\frac{\sigma^{\frac{1}{2}}}{L^{\frac{1}{2}}S^{\frac{1}{2}}}, \qquad (59)$$

obtained by combining (5) and (21), or, in numbers with metrical units,

$$n=54470\frac{\sigma^{\frac{1}{2}}}{L^{\frac{1}{2}}S^{\frac{1}{2}}}.$$

The discrepancy is no doubt to be attributed (at least in great measure) to the omission of the correction to the length of the neck.

In the experiments about to be described the pitch of the resonator was determined in various ways. Some of the larger ones had short tubes fitted to them which could be inserted in the ear. By trial on the piano or organ the note of maximum resonance could be fixed without difficulty, probably to a quarter of a semitone. In most of the experiments a grand piano was used, whose middle c was in almost exact unison with a fork of 256 vibrations per second. Whenever practicable the harmonic undertones were also used as a check on any slight difference which might be possible in the quality of consecutive notes. Indeed the determination was generally easier by means of the

* Pogg. Ann. t. lxxxi.

first undertone (the octave), or even the second (the twelfth), than when the actual note of the resonator was used. The explanation is, I believe, not so much that the overtones belonging to any note on the piano surpass in strength the fundamental tone, although that is quite possible*, as that the ear (or rather the attention) is more sensitive to an increase in the strength of an overtone than of the fundamental. However this may be, there is no doubt that a little practice greatly exalts the power of observation, many persons on the first trial being apparently incapable of noticing the loudest resonance. Another plan very convenient, though not to be used in measurements without caution, is to connect one end of a piece of india-rubber tubing† with the ear, while the other end is passed into the interior of the vessel. In this way the resonance of any wide-mouthed bottle, jar, lamp-globe, &c. may be approximately determined in a few seconds; but it must not be forgotten that the tube in passing through the air-passage acts as an obstruction, and so lowers the pitch. In many cases, however, the effect is insignificant, and can be roughly allowed for without difficulty. For large resonators this method is satisfactory, but in other cases is no longer available. I have, however, found it possible to determine with considerable precision the pitch of small flasks with long necks by simply holding them rather close to the wires of the piano while the chromatic scale is sounded. The resonant note announces itself by a quivering of the body of the flask, easily perceptible by the fingers. Since it is not so easy by this method to divide the interval between consecutive notes, I rejected those flasks whose pitch neither exactly agreed with any note on the piano nor exactly halved the interval. In some cases it is advantageous to sing into the mouth, taking care not to obstruct the passage; the resonant note is recognized partly by the tremor of the flask, and partly by a peculiar sensation in the throat or ear, hard to localize or describe.

The precision obtainable in any of these ways may seem inferior to that reached by several experimenters who have used the method of causing the resonators or pipes to speak by a stream of air. That the apparent precision in the last case is greater I of course fully admit; for any one by means of a monochord could estimate the pitch of a continuous sound within a smaller limit of error than a quarter of a semitone. But the question arises, what is it that is estimated? Is it the natural note of the resonator? I have already given my reasons for doubting the affirmative answer; and if the doubt is well grounded, the greater precision is only apparent and of no use theoretically. I may add, too, that many of the flasks that I used could not easily have been made to speak by blowing. If they sounded at all it was more likely to be the first overtone, which is the note rather of the neck than the flask: see equation (20). In carrying out the measurements of the quantities involved in the formula, the volume of the flask or reservoir was estimated by filling it with water halfway up the neck, which was then measured, or in some cases weighed. The measurements of the neck were made in two ways according to the length. Unless very short their capacity was measured by water,

* In this respect pianos, even by the same maker, differ greatly.

† The black French tubing, about $\frac{1}{2}$ inch in external diameter, is the pleasantest to use.

and the expression for the resistance (54) in a simplified form was used. The formula for n then runs,

$$n=\frac{a}{2\pi L}\sqrt{\frac{\text{vol. of neck}}{\text{vol. of flask}\times\left(1+\frac{\pi}{2}\frac{R}{L}\right)}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (60)$$

When, on the other hand, the necks were short, or simply holes of sensible thickness, the following formula was used,

$$n=\frac{a(r_1+r_2)}{4}\sqrt{\frac{1}{\pi LS\left(1+\frac{\pi(r_1+r_2)}{4L}\right)}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (61)$$

r_1, r_2 being the radii or halves of the diameters as measured at each end. It is scarcely necessary to say that the estimation of pitch was made in ignorance of the theoretical result; otherwise it is almost impossible to avoid a certain bias in dividing the interval between the consecutive notes.

TABLE I.

No. of observation.	S. in cub. centims.	V. in cub. centims.	L. in inches.	R, in inches.	n. by calculation.	n. by observation.	Difference, in mean semitones.
1	805	68	$4\frac{1}{2}$	$\frac{1}{2}$	127·7	126	+·23
2	1350	126·7	$5\frac{5}{8}$	$\frac{5}{8}$	107·7	108·7	−·16
3	7100	430	$\frac{13}{4}$	$\frac{23}{16}$, $\frac{15}{8}$	122·3	120	+·33
4	405	49·9	$3\frac{13}{16}$	$\frac{1}{2}$	179·7	180	−·03
5	180	21·26	$\frac{23}{8}$	$\frac{3}{8}$	233·7	228	+·42
6	785	36·84	$\frac{9}{4}$	$\frac{17}{32}$, $\frac{10}{16}$	174·3	176	−·16
7	210	32·5	$3\frac{7}{8}$	$\frac{13}{32}$	201·9	204	−·18
8	312	29·32	$3\frac{1}{4}$	$\frac{3}{8}$	186·3	182	+·41
10	6300	270	$3\frac{3}{8}$	$\frac{38}{32}$, $\frac{44}{32}$	104·2	102·4	+·29
12	54·69	11·6	$2\frac{5}{16}$	not recorded	391·6	384*	+·34

In Table I. the first column gives the number of the experiment, the second the volume of the reservoirs, including half the necks, the third the volume of the necks themselves, the fourth their lengths, and the fifth their radii measured, when necessary, at both ends. In the sixth column is given the number of vibrations per second calculated from (60), the velocity of sound being taken at 1123 feet per second, corresponding to 60° F., about the temperature of the room in which the pitch was determined.

* 12 was originally estimated an octave too low, so that the number in the Table is the double of what was put down as the result of observation.

Column 7 contains the values of n estimated by means of the pianoforte, while in 8 is given for convenience the discrepancy between the observed and calculated values expressed in parts of a mean semitone.

1, 2, 4, 5, 6, 7, 8, 12 were glass flasks with well-defined nearly cylindrical necks, the body of the flask being approximately spherical. Of these 1 and 2 had small tubes cemented into them, which were inserted in the ear; the pitch of the rest was estimated mainly by their quivering to the resonant note. 3 and 10 were globes intended for burning phosphorus in oxygen gas, and their pitch was fixed principally by the help of the india-rubber tube passed through the neck. A good ear would find no difficulty in identifying the note produced when the body of the globe is struck with the soft part of the hand. The agreement is I think very satisfactory, and is certainly better than I expected, having regard to the difficulties in the measurements of pitch and of the dimensions of the flasks. The average error in Table I. is about a quarter of a semitone, and the maximum error less than half a semitone. It should be remembered that there is no arbitrary constant to be fixed as best suits the observations, but that the calculated value of n is entirely determined by the dimensions of the resonator and the velocity of sound. If a lower value of the latter than 1123 were admissible, the agreement would be considerably improved.

TABLE II.

No. of experiment.	S, in cub. centims.	L, in inches.	d, in inches.	n. by calculation.	n. by observation.	Difference in mean semitones.
9	*1245*	$2\frac{11}{16}$	$\frac{14}{16}$, $\frac{15}{16}$	*107·3*	*108*	$-\cdot11$
11	216·6	$\frac{1}{6}$	1, $\frac{15}{16}$	526	538	$-\cdot39$
13	1245	1	1	163·2	170	$-\cdot71$
14	1245	$\frac{1}{6}$	1, $\frac{15}{16}$	219·4	213	$+\cdot51$
15	3090	$\frac{3}{32}$	$2\frac{1}{16}$	218·1	227·5	$-\cdot73$
16	3240	$1\frac{14}{16}$	$1\frac{3}{4}$	131·3	142	$-1\cdot36$
17	3240	$\frac{1}{8}$	$1\frac{1}{8}$, $1\frac{1}{16}$	149·1	153·5	$-\cdot50$
18	3240	$\frac{5}{16}$	$1\frac{5}{16}$	153·2	153·5	$-\cdot03$
19	3240	$\frac{5}{16}$	1	129·1	132	$-\cdot38$
20	3040	neglected	$\frac{32}{48}$	128·6	128	$+\cdot08$
21	3240	$1\frac{1}{8}$	$1\frac{1}{16}$, $1\frac{1}{32}$	101·5	103·5	$-\cdot34$
22	3240	$\frac{3}{16}$	$2\frac{7}{32}$	216	229	$-1\cdot01$

Table II. contains the results of the comparison between theory and observation for a number of resonators whose necks were too short for the convenient measurement of the volume. The length and diameter were measured with care and used in formula

(61). In 9, 13, 14 the reservoir consisted of the body of a flask whose neck had been cut off close, and which was fitted with a small tube for insertion in the ear. In 9 and 13 there was a short glass or tin tube fitted into the opening*, while in 14 the mouth was covered (air-tight) with a piece of sheet gutta percha pierced by a cork borer; 11 was a small globe treated in the same way. 15 to 22 were all experiments with a globe of a moderator-lamp, which also had a tube for the ear, one opening being closed by a piece of plate glass cemented over it. Sometimes a little water was poured in for greater convenience in determining the pitch, whence the slightly differing values of S. In 15 the opening was clear, and in 16 fitted with a brass tube; in 17 it was covered with a gutta-percha face, in 18, 19, 21 with a wooden face bored by a centre-bit, and in 20 with a piece of tin plate carrying a circular hole; 22 contains the result when the other opening of the globe was used clear.

On inspection of Table II. it appears that the discrepancy between theory and observation is decidedly greater than in Table I., in fact about double, whether we consider the maximum or the mean error. The cause of some of the large errors may, I think, be traced. 13 and 16 had necks of just the length for which the correction $\frac{\pi}{4}$ R may not be quite applicable. A decided flow back round the edge of the outer end must take place with the effect of diminishing the value of α. In order to test this explanation, a piece of millboard was placed over the outer end of the tube in 16 to represent the infinite plane. A new estimation of n, as honest as possible, gave $n=137$, which would considerably diminish the error. I fancied that I could detect a decided difference in the resonance according as the millboard was in position or not; but when the theoretical result is known, the difficulty is great of making an independent observation. In 15 and 22, where the apertures of the globe were used clear, the error is, I believe, due to an insufficient fulfilment of the condition laid down at the commencement of this paper. Thus in 15 the wave-length$=1123\div227=4\cdot9$ feet; or $\frac{1}{4}\lambda=1\cdot2$ feet, which is not large enough compared to the diameter of the globe (6 inches). The addition of a neck lowers the note, and then the theory becomes more certainly applicable.

It may perhaps be thought that the observations on resonance in Tables I. and II. do not extend over a sufficient range of pitch to give a satisfactory verification of a general formula. It is true that they are for the most part confined within the limits of an octave, but it must be remembered that if the theory is true for any resonant air-space, it may be extended to include all *similar* air-spaces in virtue of SAVART'S law alone—a law which has its foundations so deep that it hardly requires experimental confirmation. If this be admitted, the range of comparison will be seen to be really very wide, including all *proportions* of L and R. When the pitch is much higher or much lower than in the Tables, the experimental difficulties are increased. For much lower tones the ear is not sufficiently sensitive, while in the case of the higher tones some of the indications relied on to fix the pitch are no longer available.

* Gutta percha softened in hot water is very useful for temporary fittings of this sort.

Some experiments were next made with the moderator globe and two openings. The theoretical formulæ are

$$n=\frac{a}{2\pi}\sqrt{\frac{c_1+c_2}{S}},$$

$$c=\frac{\pi R^2}{L+\frac{\pi}{2}R}.$$

TABLE III.

No. of experiment.	S, in cub. centims.	c_1, in inches.	c_2, in inches.	c_1+c_2, in inches.	n, by calculation.	n, by observation.	Difference, in mean semitones.
23	3240	2·00	1·95	3·95	303·4	320	·92
24	3240	2·00	1·01	3·01	264·8	282	1·09
25	3240	·715	1·01	1·725	200·4	204	·31

In 23 both holes of the globe were clear, and in 24, 25 they were covered with wooden faces carrying holes of various diameters. The error in 23, 24 is to be ascribed to the same cause as in 15 and 22 above.

The last experiment that I shall describe was made in order to test the theory of double resonance, but is not quite satisfactory, for the same reason as 15, 22, 23, 24. Two moderator globes were cemented together so as to form two chambers communicating with each other and with the external air. The natural openings were used clear, and the resonance (which was not very good) was estimated by means of a tube connecting the ear with one of them. The observations gave for the values of n,

$$\left.\begin{array}{l}\text{High note}=384,\\ \text{Low note}\ =213.\end{array}\right\}$$

The result of calculation from the dimensions of the globes and openings by means of the formulæ of Parts I. and II. was

$$\left.\begin{array}{l}\text{High note}=360,\\ \text{Low note}\ =212.\end{array}\right\}$$

The error in the high note is about a semitone.

[The two moderator globes were fitted up again as a double resonator, only with bored wooden disks over the holes, so as to lower the note and render the theory more strictly applicable. The pitch was much better defined than before, and gave

$$\left.\begin{array}{l}\textit{Low note}\ =152\cdot5,\\ \textit{High note}=240.\end{array}\right\}$$

$$c_1=c_3=1\cdot008;\quad c_2=\cdot7152.$$
$$a=1133\ (70^\circ\ \text{F.})$$

Thus

$$n_1=\frac{1133\times 12}{6\cdot 28}\sqrt{\frac{16\cdot 39\times 1\cdot 008}{3200}}=155\cdot 6,$$

$$n_2=\frac{1133\times 12}{6\cdot 28}\sqrt{\frac{16\cdot 39\times 2\cdot 438}{3200}}=241\cdot 9.$$

The agreement is now very good.

One of the outer holes was stopped with a plate of glass. The resonance of the high note was feeble though well defined; that of the low was rather loud but badly defined.

$$\left.\begin{array}{l}\text{The high note was put at } 225\\ \text{,, low ,, ,, } 90\end{array}\right\}$$

$$S=3150,\quad S'=3250,$$

$$c_2=\cdot 7152,\quad c_1=1\cdot 008,\quad c_1+c_2=1\cdot 7232.$$

Calculating from these data, we get

$$\left.\begin{array}{l}n_1=225\cdot 2,\\ n_2=\ \ 90\cdot 5.\end{array}\right\}$$

The agreement is here much better than was expected, and must be in part fortuitous.

I will now detail two experiments made to verify the formula marked (20*a*). A moderator chimney was plugged at the lower end with gutta percha, through which passed a small tube for application to the ear. The bulb was here represented by the enlargement where the chimney fits on to the lamp. On measurement,

$$\frac{S}{\sigma}=4\cdot 16 \text{ inches},\quad L=5\cdot 367 \text{ inches}.\quad \alpha=\frac{\pi}{4}R=\cdot 471.$$

Thus

$$\tan k\times 9\cdot 611=\frac{1}{k\times 4\cdot 161};$$

from this the value of k was calculated by the trigonometrical tables. Finally,

$$n=251\cdot 4.$$

As the result of observation n had been estimated at 252.

In another case,

$$L=5\cdot 767,\quad \alpha=\cdot 537,\quad \frac{S}{\sigma}=3\cdot 737,\quad n \text{ by observation}=351.$$

The result of calculation is $n=350\cdot 3$. These are the only two instances in which I have tried the formula (20*a*). It is somewhat troublesome in use, but appears to represent the facts very closely; though I do not pretend that the above would be average samples of a large series. There is no necessity for the irregularity at the lower end taking the form of an enlargement. For example, the formula might be applied to a truly cylindrical pipe with a ball of solid material resting at the bottom.]

I had intended to have made these experiments more complete, particularly on multiple resonance, but have not hitherto had time. However, the results obtained seem

quite sufficient to establish a substantial agreement between theory and fact. It should be understood that those here presented are not favourable specimens selected out of a large number, but include, with one exception, all the measurements attempted. There are many kinds of bottles and jars, and among them some of the best resonators, which do not satisfy the fundamental condition on which our theory rests. The deductive treatment of the problem in such cases presents great difficulties of a different kind from any encountered in this paper. Until they are surmounted the class of resonators referred to are of no use for an exact comparison between theory and observation, though they may be of great service as aids to investigation in other directions.

VI. *On the Formation of some of the Subaxial Arches in Man.* By George W. Callender, *Assistant-Surgeon to and Lecturer on Anatomy at St. Bartholomew's Hospital. Communicated by* J. Paget, *F.R.S.*

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In a previous communication on the formation of the bones of the face*, I described the manner in which the nasal and incisor processes of the maxillary lobes united with the fronto-nasal process, and the way in which the intermaxillary bones were shut off from the front surface of the face. I also described the ossification of the membrane which surrounds the anterior portion of Meckel's cartilage on either side, and that of the anterior and mesial extremities of the cartilages themselves. I now propose to examine the formation of the arches below these structures, commencing with that commonly known as the hyoid.

To simplify the grouping of these several arches I propose to term them subaxial. The first of the subaxial arches is the fronto-nasal, the basis of which is formed by the trabeculæ cranii. I take the notochord as the central line around which the development of the cerebro-spinal axis progresses. Now the trabeculæ, at first two separate cartilage-rods, grow out beneath (in man in front of) the extremity of the notochord. After the fusion of the trabeculæ their double origin is indicated by the median septum growing from their under surface. They inclose, by being prolonged forwards and downwards, the nasal passages, and they end in the intermaxillary bones. They are therefore subaxial in relation to the notochord, and conform generally to the plan on which the other subaxial arches are developed, the modifications which distinguish them being due to their terminal position, to the tubular shape of the long nasal passages which they have to form, and to the support which they give to those outgrowths from the cerebral vesicles, the olfactory bulbs and the front portions of the cerebral hemispheres.

The second subaxial arch grows from the lower and outer part of the alæ of the sphenoid, and is the one usually described as the maxillary; it is said not to contain cartilage. The third subaxial arch is the mandibular; it is strengthened by rods of cartilage (Meckel's) growing from that in which the internal ear is developed. Of other points in the formation of this arch, and of the connexions of the fourth arch with the cartilage of the internal ear, I hope to give an account when I have examined the various steps in the formation and growth of the cartilaginous base of the skull. At present it was convenient to pass them by until I had dissected the structures in the cervical region.

* Philosophical Transactions, 1869, p. 163.

The subaxial arches below the mandibular are the fourth, or lingual; the fifth, or hyoid; the sixth, or laryngeal; and the seventh, which, differing from the preceding in its relation to the notochord, may be named the exoccipital or shoulder-girdle. There is also an imperfect arch connected with the laryngeal which will be described with it.

The changes which will be described as occurring in the formation of these structures take place between the fifth and twelfth weeks of fœtal life (30th to 84th day)*.

The Fourth Subaxial Arch—the Lingual.

In a fœtus less than ·4 of an inch in length this arch is recognized as a small bud-like process below the mandibular†; in a fœtus measuring ·45 this bud has grown out to the middle line, and has joined its fellow below the mandible (Plate I. fig. 1, *l*). The mandible, rather less than ·18 from side to side and ·03 deep, is yet membranous, and is closely connected by its lower border with the lingual arch; but in the midst of the connecting membrane two distinct lines of dense tissue are distinctly traced, meeting in the middle line. On either side of this they swell up into a pointed mass, and here the tongue is rapidly formed; so rapidly, indeed, that whilst in a fœtus of ·45 its position is indicated only by an ill-defined thickening of the tissue above the centre of the arch, in another fœtus of ·55 it is seen completely formed. From the eighth to the ninth week (fœtus ·9) the lingual arch, distinct from the hyoid, grows into a rounded cord which begins to contain cartilage. It is attached by a globular head to the cartilage for the internal ear below and behind the root of MECKEL's cartilage; its anterior extremity descends to the next lower arch, becomes slightly swollen, and loses its distinct character in the structures passing into the base of the tongue; so that in a fœtus of 1·2 this extremity is no longer recognized, although the rising of its tissue into the root of the tongue is still traceable (Plate I. fig. 2, *l*, fig. 3, *l*). The lower border is now, by the anterior extremity, in close relationship with the upper surface of the hyoid and contains cartilage, the future lesser horn of the hyoid bone.

At this stage, ninth to tenth week, the connexion of this arch with the cartilage at the base of the skull is as follows: the globular head is still distinct, and seems partially cut off from the basal cartilage (Plate I. fig. 3, *l l*). From this point, for ·05 of an inch forward, the rod is formed of cartilage; the remainder (·07 long), except the extreme anterior portion already spoken of, is a dense membrane. In the opposite direction (Plate I. fig. 3, *l l* to *l l l*) there is continued from the rod a ridge of membrane, lying on and becoming identified with the cartilage of the base; and this, after ascending for ·05 of an inch, turns forward and is continuous with membrane, a descending spur from which forms a distinct ring, which extends to that in which the squamous and zygomatic bones are ossified, and which appears to be continued into the membrane in front of MECKEL's cartilage (Plate I. fig. 3, *m c*), now ossified as the inferior maxilla (Plate I.

* The specimens on which the dissections were made are in the Museum of St. Bartholomew's Hospital. The figures indicating the size of each fœtus give its length in inches and tenths of an inch.

† Here and after the relations are given as for human anatomy.

fig. 3, *m*) or mandible: MECKEL's cartilage is seen growing out beneath the membrane and then passing under cover of the lower jaw.

In a fœtus 1·8 long, tenth to eleventh week, the lingual arch consists of five portions,—(1) cartilage from the base of the skull, (2) a short piece of membrane, (3) a second very small and short rod of cartilage, these three structures having a length of ·09 of an inch (Plate I. fig 5, *l* to *l**), (4) a long strip of membrane, and (5) a nodule of cartilage at the anterior extremity of the arch, which is connected with the hyoid below. The entire length of these structures is ·19 of an inch.

The Fifth Subaxial Arch—The Hyoid.

On a deeper plane and under cover of the parts in which the lingual arch is formed, a layer of membrane, comparatively broad, is seen growing out from the base of the skull in connexion with that part in which the basioccipital ossification takes place. In a fœtus ·45 long it has the form shown in Plate I. fig. 1. This mass widens as it descends into the cervical region and towards the middle line, and divides into two chief parts; the higher of these is the hyoid arch.

This arch is at first represented by the upper portion of the layer of membrane, and by a bud of this membrane from the upper and mesial extremity of either side piece (Plate I. fig. 1, *h*). These buds meet in the middle line and there unite, so that in a fœtus ·9 long they form a distinct rod, now cartilaginous as far back as the point *h**, fig. 2, rather less than ·1 of an inch long, where it is connected with the mass of membrane continued down to the larynx. The band of tissue connecting this arch with the thyroid cartilage at the point *h**, fig. 2, is now just discernible; in the next dissection, that of a fœtus 1·2, it is distinctly formed. The higher portion of this arch, towards the base of the skull, forms the middle constrictor muscle, which in a fœtus 1·8 can be traced from the base into the posterior portion of the cartilage of the arch, the greater or posterior horn of the hyoid bone.

The posterior portion of this arch, therefore, is membranous, and developes into the middle constrictor muscle; the cartilage in the anterior two-thirds (·06 long in a fœtus of ·9, and 1·1 in a fœtus of 1·8) consists of two portions, the one the greater horn, the other the body of the hyoid bone.

The Sixth Subaxial Arch—The Laryngeal.

Whilst the upper portion of the layer of membrane from the basioccipital cartilage grows forward to form the hyoid arch, the lower division, first distinctly separated from the upper by anterior bud-like outgrowths, forms a mass in the cervical region which in a fœtus of ·45 is about ·05 of an inch in length (Plate I. fig. 1, *l g*). This mass, uniting with a corresponding growth from the opposite side, forms an irregular line in the middle of the neck, all trace of which line rapidly disappears. At first it lies flat upon the vertebræ, but quickly thickens, and in a fœtus ·9 forms a thick block advanced to the level of the anterior extremity of the lingual arch (Plate I. fig. 2, *th*, *cr*). As it thickens

and rises to the front, a passage is left behind it leading from the mouth to the œsophagus, and common to the pharynx and to the larynx.

At this period the membrane consists of three layers. The inner of these folds in and forms a septum, which divides the larynx from the common pharyngeal tube, and growing up below the root of the tongue forms the epiglottis. These growths are completed in a fœtus ·9 of an inch long. The outer layer is that in which the constrictor muscles originate, whilst in the anterior portion of the arch there is an intermediate or third layer of dense granular and subsequently cartilaginous tissue, in which are formed the several cartilages of the larynx, excepting the epiglottis.

The upper portion of this arch, therefore, forms the inferior constrictor muscles, and at this early period of development blends with that tissue in which the middle constrictor of either side is developed (Plate I. figs. 3, 4, 5). The remainder of the arch becomes, like the hyoid, cartilaginous in the greater part of its extent. In a fœtus ·9 the outline of the thyroid cartilage is distinct, and also that of the front and sides of the cricoid, but posteriorly the horns of the thyroid are imperfectly recognized, the upper one not being as yet cartilaginous, whilst the arytenoid and cricoid cartilages are not yet completely separated from the thyroid behind (Plate I. fig. 2, *th*, *cr*, *ar*).

Thus far these structures whilst bent over towards the middle line are on either side perfectly flat, and altogether form a mass ·06 long and ·05 wide. Their continuity with the pharyngeal wall is easily traced.

In a fœtus 1·2 the separation of the principal cartilages is complete and the cornua of the thyroid are formed (Plate I. fig. 3, *th*, *cr*, *ar*). Although the first granular masses of cartilage appear in the thyroid and in the fore part of the cricoid, yet the chondrification of the membrane extends so rapidly to the posterior parts of the cricoid and to the arytenoid, that no practical distinction can be made between the times of these quickly consecutive changes.

be detached, and grows downwards and forwards to meet the corresponding structure from the opposite side in the middle line (Plate I. fig. 2, fig. 3, *th. b*). Although it is difficult to assign to it any particular function, it may be noticed that it is in relation with the respiratory tract in its development, and is virtually a rudimentary branchial arch *; and further, that it forms a girdle beneath which the air-tube is continued, and perhaps guided, as the trachea into the thorax.

The Trachea.

Whilst the cartilages of the larynx are acquiring their permanent characters, the trachea is formed by a continuous growth from the membrane below the cricoid cartilage. I have not as yet definitely traced out the formation of the bronchi; there is, I believe, no doubt but that they result from the division of the trachea.

The Seventh or Exoccipital Arch—The Shoulder-girdle.

At first it occurred to me that this should be termed a subaxial arch, and it still seems desirable to number it as following the sixth or laryngeal outgrowth of membrane from the base of the skull. But exception might be taken to the term subaxial, as defined to refer to the position of an arch with reference to the notochord, when applied to this structure; for the membrane from which the clavicle and scapula take origin grows out from the cartilage of the base of the skull in a line from the extremity of the membrane for the hyoid and laryngeal arches, below the lingual arch, and all along the side and hind plate in which the exoccipital and supraoccipital ossifications commence. From these the membrane of this arch is separated with some difficulty, leaving a rough edge. Thus whilst the membrane for the hyoid and laryngeal arches grows from the basioccipital region and is subaxial, that for the shoulder-girdle extends from the lateral and posterior portions of the occipital cartilage, having also for a short distance relations with the cartilage for the structures connected with the internal ear; it therefore is best described as an exoccipital arch, growing from the cartilage of the base of the skull, but at the side of and behind the axis line represented by the notochord.

The growth of this membrane is extremely simple. Dissecting at about the thirty-fifth day (fœtus ·45), it is seen as a fold of membrane extending from the root of the mandible on either side downwards, its connexion with the mandible being formed by

* With reference to this point I would refer to the remarks which I made when examining the relations of this body. "One cannot but be attracted by this connexion with the trachea, on which tube the thyroid (even if it be not developed from the membranous air-tube) buds and attains some little size, a formation reminding one of that of the lungs coming out from the front wall of the œsophagus, that is, from the trachea, and of the view of Mr. Simon, as afterwards expressed by the editors of Cuvier respecting the thyroid, 'C'est la fausse branchie, branchiole des poissons.' Indeed, from its relation to the air-tube during the early period of life, or in fish to the vertebral or hyoidal extremity of the gill, from its curious alternation with the supplementary gill of Broussonet, and from its structure (Kölliker), it may be not inaptly referred to as a pseudo-lung rather than as an associate with the thymus and the so-called ductless glands."—Proceedings of the Royal Society, vol. xvi. p. 185.

the integumental covering. If this fold is everted, a band of tissue can be traced to an angle above, where it comes into contact with the well-defined tissue in which the larynx is to be formed (Plate I. fig. 6). On the inner surface of the fold two ridges (Plate I. fig. 6, *c*, *s*) are imperfectly marked, but at the lower end of the membrane they are more distinct,—one ending in a rounded border (and here the clavicle is developed), the other, which lies behind the first, descending a little lower and being slightly bent with the convexity of the curve backwards (in this the scapula is formed).

In a fœtus more advanced (·55), after removal of the integument and of the upper thoracic extremity, the plate of membrane is more clearly defined. It grows from the occipital region, and above is in close relationship with the membrane for the laryngeal arch (Plate I. fig. 7). Traced downwards, it sends forward a comparatively delicate layer of tissue (Plate I. fig. 7, *c*), which extends to the superior extremity of that (*st*) in which the sternum and the costal cartilages are being formed. At about the middle of the lowest border of the arch a rounded mass, which is granular, and in which cartilage is appearing, forms a considerable elevation (*s*), pushed forward as it is by the ribs which have now grown up beneath it (*r*). This rod is curved, as shown in fig. 7; its superior and anterior extremity is the acromion, its inferior extremity is the coracoid process; and nearly midway between the two a bud-like outgrowth (which seems to originate in a folding over of the integument), the thoracic extremity, has been removed in the specimen from which the drawing (fig. 7) was made.

The changes thus far related consist in the division of the membrane growing out from the occipital region into two portions, the anterior of which developes into the hyoid and laryngeal arches, whilst the posterior division descends and forms a broad layer in the neck (covered by integument), which eventually forms the sterno-mastoid and trapezius muscles of either side, and behind joins above in the middle line the tissue from the opposite side at the ligamentum nuchæ: below it sends out two processes; one of these grows forward to form the clavicle and to join the sternum, the other curves downwards and forms the scapula, and from this there grows out the thoracic extremity. It will be noticed that these outgrowths, including the lingual, correspond not inexactly with the course of the cerebral nerves distributed below the mandible,—(1) the hypoglossal, (2) the glossopharyngeal, (3) the pneumogastric, and (4) the spinal accessory.

In a fœtus ·9 long the clavicle has ossified in its middle third, the remainder is cartilaginous; it is ·09 of an inch long. When the ossification of this bone is described as taking place in membrane, it is to be understood that only the earliest formed portion of the bone is so ossified; the greater part of the clavicle, about two-thirds of its length, ossifies in cartilage, which rapidly takes the place of the original membrane, and of which only small plates are left at either end of the bone in a fœtus 1·2 (in Plate I. figs. 8 and 9, the earliest membrane ossification and the extension of the ossification in cartilage are shown).

By the growth of the sternum to the middle line and by the attachment to it of the

clavicle, which is also connected with the first rib, the anterior portion of the seventh arch is brought from an oblique into a line nearly horizontal (in Man), and at this period the portion of membrane above it, which at first lies close along the upper border of the clavicle, becomes detached from the outer half of this border, or is united only by a thin layer of tissue. The tissue which remains connected with the inner half, or thereabouts, of the clavicle forms the sterno-mastoid muscle.

The inner extremity (Plate I. fig. 7, *a*) of the rod of cartilage (which is the scapula) being connected with the outer extremity of the clavicle is brought into the same horizontal line with that bone and forms the acromion. The coracoid extremity of the scapula (Plate I. fig. 7 and following, *co*) becomes more curved upon the acromial, to the level of which it eventually reaches (chiefly by the straightening in a downward direction of the acromion), and thus the entire scapula forms a slightly spiral curve, and the root of the coracoid portion becomes attached to the clavicle by a strong fibrous band. In this spiral curving of the rod of cartilage which forms the scapula the upper surface of the coracoid corresponds with the upper surface, its internal surface with the anterior, and its external with the posterior border of the acromion process. By the same curving of this portion of the arch the lower portion of the posterior border of the membrane ascending to the occipital region is bent or folded inwards; and this seems to explain the connexions acquired by the levator anguli scapulæ muscle in the cervical region, and its insertion into the end of the plate of the scapula which grows from the root of the coracoid process.

The growth of the plates of the scapula takes place from the posterior border of the rod of cartilage, the form of the bone being determined by the curve of the primitive rod. Following the posterior border the acromion plate is seen (fig. 8, *a*) growing backwards, whilst from the remainder of the rod to the root of the coracoid the larger plate bounded by the glenoid border is developed. Scarcely any growth has yet taken place from the coracoid (fig. 8, *co*), but in a fœtus 1·2 (fig. 9, *co*) it is beginning to throw out its plate, and in a fœtus 1·8 (fig. 10, *co. p*) this plate is being rapidly completed. The scapula is thus built up in the three-sided prismatic form referred to by Professor Flower *.

Later, comparatively, in the formation of the scapula (fœtus 1·8, fig. 10) the glenoid cavity grows out from the rest of the cartilage, and thus leaves a considerable notch between it and the acromion, and the scapula now acquires its permanent characters.

Up to this point no ossification is observable, nor have I recognized any segmentation of the primitive rod. When ossification commences it begins, as shown by Mr. Parker, in the central portion of the rod, as is the case with the clavicle, and, as with the clavicle, it leaves two cartilage ends, which are the acromion and the coracoid process.

* Osteology of the Mammalia, p. 334.

Explanation of the Plate.

PLATE I.

Fig. 1. Subaxial and exoccipital arches in a fœtus (·45 of an inch long). × 6.

Fig. 2. Lingual, hyoid, and laryngeal arches, with the thyroid body, in a fœtus ·9 of an inch long. ×6.

Fig. 3. The same arches as in fig. 2, with the addition of the mandibular. The divisions of the hyoid and laryngeal arches are shown, and also the connexion of the lingual arch with cartilage and membrane at the base and side of the skull. From a fœtus 1·2 of an inch long. ×6.

Fig. 4. The same arches at a more advanced stage, from a fœtus 1·4 of an inch long. ×6. The anterior portion of the lingual arch has been removed.

Fig. 5. The same structures, including the lingual arch, in a fœtus 1·8 of an inch long. ×6.

Fig. 6. Ridges seen on the inner surface of the exoccipital arch in a fœtus ·45 of an inch long. ×6.

Fig. 7. Outer surface of the exoccipital arch, showing the division of its lower border into the clavicular and scapular rods, and the growth beneath these of the sternum and ribs. From a fœtus ·55 of an inch long. ×6.

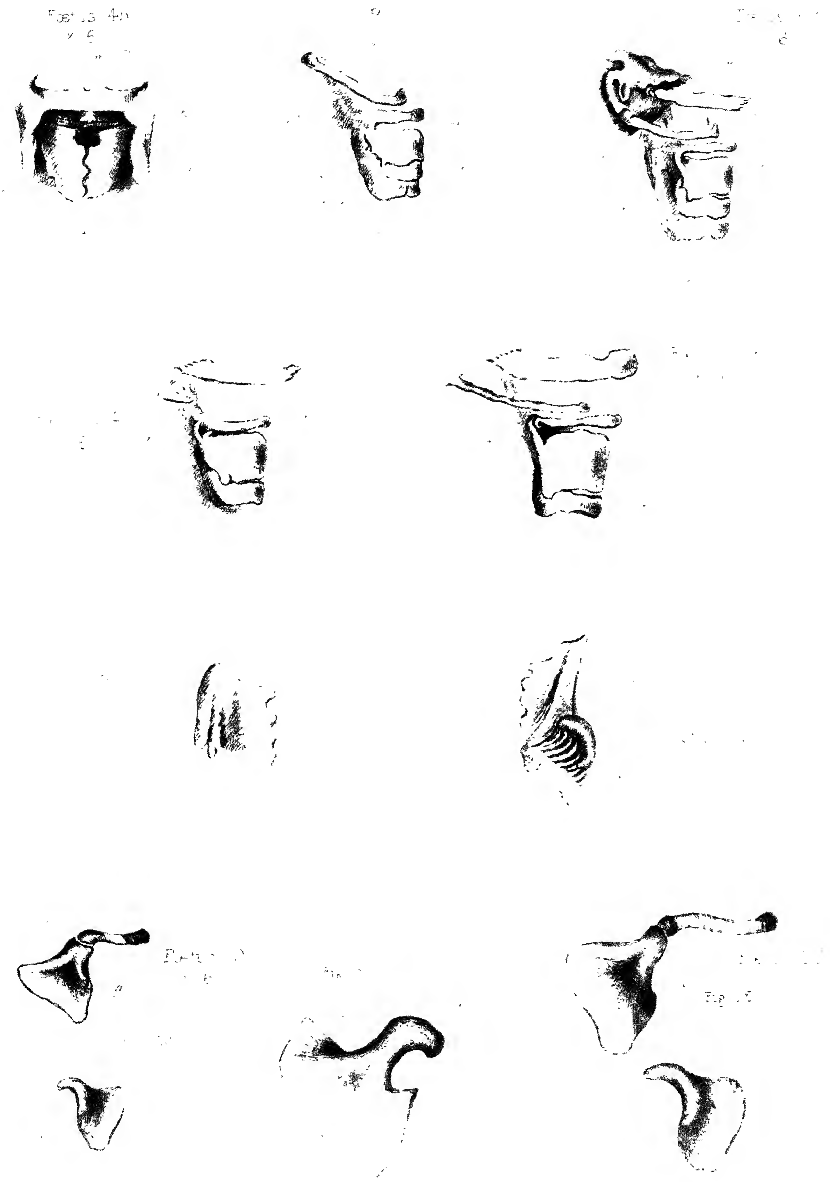
Fig. 8. Dorsal and ventral surfaces of the scapula of a fœtus ·9 of an inch long. ×6.

Fig. 9. The same surfaces of the scapula of a fœtus 1·2 of an inch long. In this and in the preceding figure the anterior surface of the clavicle is also represented. ×6.

Fig. 10. Dorsal surface of the scapula of a fœtus 1·8 of an inch long, showing the outgrowth of the glenoid cavity and of the coracoid plate. *×6.*

Explanation of Abbreviations.

a. Acromion.
ar. Arytenoid cartilage.
c. Clavicle.
c m. Constrictor muscles of pharynx.
co. Coracoid process.
co. p. Coracoid plate of scapula.
cr. Cricoid cartilage.
g. Glenoid cavity and border.
h. Hyoid arch.
h.* Posterior extremity of the cartilage of the hyoid arch.


Foetus 4m
x 6

l. Lingual arch.
l l. Basal extremity, or head, of lingual arch.
l l l. Membrane connected with basal extremity of lingual arch.
l.* Second cartilage of lingual arch.
lg. Laryngeal arch.
m. Mandible.
m c. MECKEL's cartilage.
r. Ribs.
s. Scapula.
st. Sternum.
th. Thyroid cartilage.
th. b. Thyroid body.

Fig 1.

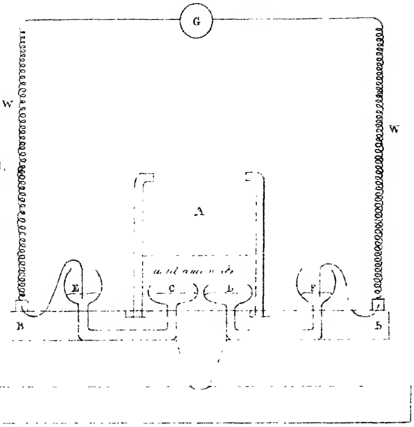


Fig. 2.

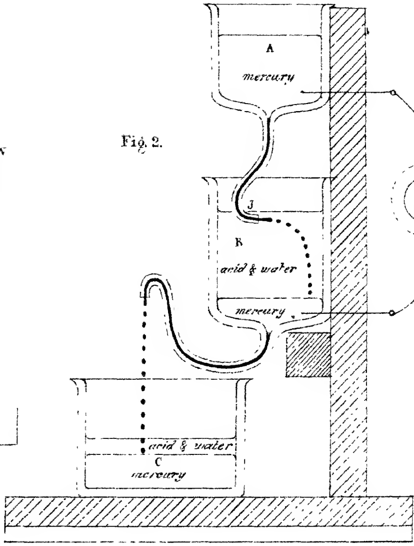


Fig 3

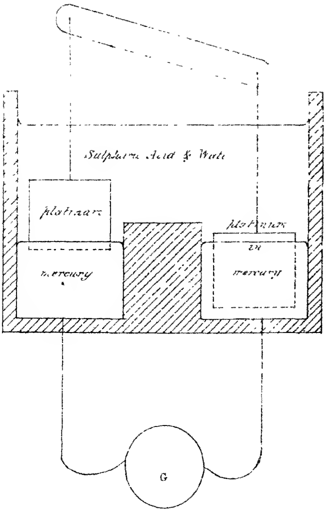
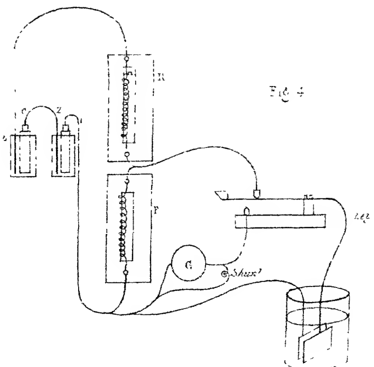


Fig 4



VII. *Polarization of Metallic Surfaces in Aqueous Solutions. On a new Method of obtaining Electricity from Mechanical Force, and certain relations between Electrostatic Induction and the Decomposition of Water.* By CROMWELL FLEETWOOD VARLEY. *Communicated by Sir* W. THOMSON, *F.R.S.*

Received October 5, 1870,—Read January 12, 1871.

IN 1860, having need of condensers of enormous capacity, the author found that platinum plates immersed in a solution of sulphuric acid and water had enormous capacity, and could, under certain conditions, be used as condensers with potentials below that necessary for decomposing water.

When one of the platinum plates was replaced by mercury, and a powerful battery was applied so as to make the mercury negative, the latter flattened out and increased its surface.

When a pasty amalgam was employed of the proper consistency on a flat surface, this flattening out was sometimes increased to more than double the original surface. The reversion of the current immediately brought the amalgam to its original dimensions.

This experiment suggested a means of obtaining dynamic electricity by reversing this process.

Plate II. fig. 1 represents a large glass vessel (A) cemented into a groove in the board B. The funnel-shaped glasses C and D, which are inside A, are connected with E and F outside of A. Under the board B is fixed a transverse bar of wood, so that the whole can be rocked from side to side. When tilted to the right the mercury in C runs into E, while that in F runs into D. The reverse operation takes place when the board is tilted to the left, consequently when the surface of mercury in C is diminished, that in D is augmented, and *vice versâ*. The galvanometer G connects the two by the spiral wires W.

The glass vessel (A) is filled half full with diluted sulphuric acid. If the two poles of a voltaic battery be immersed in A, the positive pole in connexion with the acid and water, the negative in connexion with the mercury in C, the latter becomes polarized with "nascent hydrogen."

The mercurial surface in C will retain its polarity for a long time after the removal of the polarizing-battery.

If the two cups E F be now connected by the galvanometer G, a current of electricity will be seen to flow between them, the polarization will be equally shared by the two mercurial surfaces, and then the current will cease. If, now, the board be rocked from right to left, and *vice versâ*, the mercury surface in the one vessel will diminish, while

that in the other increases. During this change of surface-dimensions currents of electricity will be found to pass from one to the other, the diminishing surface acting as the zinc plate, and the increasing surface as the copper plate of a voltaic couple.

All attempts to polarize the mercury with oxygen have entirely failed. This experiment enables one to obtain a mercurial surface neutral to the fluid in which it is placed, *i. e.* free from polarization, or very nearly so indeed.

All the specimens of mercury which were tested showed traces of this hydrogen polarization, but they can be almost absolutely depolarized (if the mercury be pure) by connecting it with the positive pole of a very feeble battery through a large resistance, and the aqueous solution with the negative pole, until the rocking to and fro of the vessel ceases to generate a current, or shows hardly any trace of one. The mercury-surfaces are then neutral, or very nearly so, to the fluid; if, however, they have been previously charged highly with hydrogen, this depolarizing process must be repeated over and over again until, after resting, the mercury is found not to yield a current by rocking. This "neutral" surface is likely to be of use in investigating the source of force in batteries.

The following arrangements will be found to give a continuous electric current. The three vessels A, B, and C are arranged as shown in figure 2 (Plate II). The vessel A receives mercury which issues from the jet J into the acid and water in the middle vessel B. The mercury at the bottom of B runs through a siphon into the vessel C. The mercury in A and B is connected through a galvanometer (G) by means of platinum wires.

If the mercury be pure and not polarized, the running of the mercury from A into B gives rise to no current. If, now, the larger surface of mercury in B be polarized as before, it will share its polarization with each drop of mercury that falls from J, and thus produce a current from A through the galvanometer to B; each drop as fast as it is polarized falls by gravitation into B, and so gives back the polarization it had received. By carrying the mercury from C back to A, this current is continued for a very long time.

The following method (fig. 4) is a convenient one for showing the experiment on a larger scale; and on this principle, several years ago, the author constructed an apparatus of twelve cells (which has never been published), which was worked by clock-work and gave a current of a potential rather greater than one cell of a DANIELL's battery.

When clean platinum plates are polarized with hydrogen, and dipped into polarized mercury, they instantly amalgamate all over.

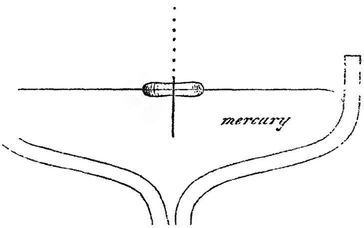
If a gutta-percha trough of the form shown in fig. 3 in section be constructed, so as to contain two separate cells of mercury covered and united together by the supernatant aqueous solution, and a bunch of platinum plates be inserted in each parcel of mercury and amalgamated, the mercurial surface exposed to the fluid will be augmented when the amalgamated plates are partially withdrawn from the mercury, and will diminish when they are inserted again; in this way, in a small space, a large extent of mercurial surface is easily obtained. If the platinum plates be well polarized, it will be found that, on raising those in the one cell and simultaneously immersing those in the other,

a current of considerable volume will flow from one to the other cell through the galvanometer. If a grain of zinc be added to the mercury this effect will be very persistent, and is more powerful, the zinc maintaining the polarization. Only one platinum plate in each vessel is shown in fig. 3.

When the funnel-shaped vessel C in fig. 1 was 5 inches in diameter, and the tube connecting it with E the $\frac{1}{30}$ of an inch in diameter, and the galvanometer connexion between E and F was severed, it was found, after having polarized the mercury surface C by the power of rather less than one DANIELL'S cell (a power too small to evolve hydrogen gas), that on tilting the vessel so as to let the mercury run from C to E, a few small bubbles of hydrogen gas were given off just as the last drop of mercury ran out of C; *thus the contraction of the surface concentrated the polarization* until it had power enough to evolve the hydrogen as gas. The other funnel might have been removed, the surface D having no influence on the result.

This evolution of gas is better shown by floating a minute piece of fine platinum wire on the mercury, which gives off the gas as the surface of mercury becomes reduced.

In this experiment the piece of platinum wire was about 0·002 inch in diameter and 0·5 in length. It was floated on the mercury by a small lump of shellac, thus:



The exposed portion of the wire was as short as possible. The contracting polarized mercurial surface acted like zinc to the platinum, which evolved hydrogen as it would have done when in contact with a piece of zinc.

The author had many times endeavoured to ascertain the electrostatic capacity of metallic surfaces exposed to an aqueous solution, but has only recently, and by the following means, been able to get sufficiently reliable results.

A reflecting galvanometer was constructed with copper wire, No. 18 gauge, and having 20 Ohms resistance; this was again reduced by a shunt to 4 Ohms; the mirror and magnet are hung in water to destroy the oscillations rapidly.

Two sets of platinum surfaces were used:—1st. Two platinum spheres about $\frac{3}{4}$ of an inch in diameter, and having each about 1·6 square inch of surface exposed to the acid and water, the two bulbs being placed in a glass of diluted sulphuric acid and water. 2nd. To reduce the resistance as much as possible two platinum plates, each of 1 square inch surface, were coated on one side with bees'-wax and paraffin, so as to leave only the one surface exposed to the fluid; these two plates were made to face each other at a very

short distance by placing one on each side of a double thickness of blotting-paper, and immersing the whole in diluted sulphuric acid, one volume of pure acid to four volumes of distilled water. By these means the resistance of the *fluid* between the plates was reduced to about $\frac{1}{25}$ part of an "Ohm," according to *calculation* from BECQUEREL's data. These plates were connected as shown in fig. 4 (Plate II.).

The current from two cells of DANIELL's battery was made to pass through the adjustable resistance-coils R and R′. The united resistance of these coils, when the platinum bulbs were used, was made 1000 Ohms; when the platinum plates were used, it was made 100 Ohms.

The two cells of DANIELL's battery had a resistance of about 12 Ohms, and by experiment their potential was found to be reduced from 344 to 317, or 8 per cent., by the resistance-coils R+R′ connecting the poles of the battery when R+R′=100 Ohms, and therefore the potential of them was reduced to about two volts.

By varying the resistance in R and R′, it is easy to get any desired subdivisions of the potential of the two volts, as Sir WILLIAM THOMSON and Sir C. WHEATSTONE have shown.

The apparatus was sufficiently rapid and sensitive to read the discharge from the platinum bulbs and from condensers of 311 microfarads capacity for comparison therewith, when they were charged with a potential not greater than 0·02 volt.

Table II. shows the discharge from the platinum plates when charged by potentials varying from 0·2 of a volt up to 1·6 volt, also the discharge obtained from the condensers of 311 microfarads capacity for comparison therewith.

During each of the experiments in Table I. the sensitiveness of the galvanometer was maintained constant, and the results are directly comparable with those obtained from the condensers; the same is the case in Table II.; but Table I. is not comparable with Table II., excepting by means of the last column. They were made at different dates, when the sensitiveness of the galvanometer was not the same. Thus it will *be seen by reference to column 6 that, while the electrostatic capacity of the ordinary condensers remains constant (i. e. the discharge varies directly as the "potential"),* with the fluid plates this regularity is only observed while the charge is very low, not more than 0·08 volt (see Table I.). As the potential increases from $\frac{1}{10}$ of a volt upwards, the discharge from the platinum plates increases in a greater ratio, as will be seen by column 6, which shows the deflection of the galvanometer divided by the "potential" to give the ratio; and it will be seen in Table I. that with potentials from ·02 to ·08 the capacity was 1 as against 3·5 with a potential of about 1·6.

An inference which the author thinks these experiments suggest is, that the water does not actually touch the platinum surface, and as the potential increases the water is attracted nearer to the electrified plate, thereby augmenting its electrostatic capacity as the distance between the platinum and the water is diminished by the electric attraction.

When, however, the repulsion between the platinum plate and the water is overcome by the electric attraction, then conduction of the current would seem to take place,

accompanied by decomposition of the water. If there be such a film between the water and platinum, the following considerations will enable us to get an idea of its depth. The electrostatic capacity of a layer of pure gas is sensibly the same as that of a vacuum. Gutta percha has a capacity of between three and four times this amount.

If we assume that the film between the platinum and the water is a pure gas, the comparison of the capacity of the 1 inch of platinum in water with the surface of air necessary to produce at a distance of 1 inch a similar capacity will give an idea of the thickness, or rather the extreme thinness, of this hypothetical film separating the water from the platinum.

The capacity of the French Atlantic Telegraph-Cable, Brest to St. Pierre, is 0·4 microfarads per nautical mile; each nautical mile has 400 lbs. of gutta percha, specific gravity 0·98, and 400 lbs. of seven-strand conductor filled with gutta-percha compound, giving it a specific gravity of about 8·8.

Calculating from this how large a sheet of gutta percha, 1 inch in thickness, and coated on each side with metal, is necessary to give an inductive capacity of one microfarad, we shall find that a sheet built up out of 1,040,000 cubic inches of gutta percha will have the capacity sought.

The condensers used for comparison with the two sheets of platinum, offering each 1 square inch of surface, had a capacity of 311 microfarads—that is to say, they were equal to a sheet of gutta percha 1 inch in thickness, and having a surface of 323 millions of square inches; and as gutta percha has about $3\frac{1}{2}$ times the capacity of pure gas, 1131 million square inches of metal separated from another such surface by 1 inch of pure gas would about give the capacity sought, viz. 311 microfarads.

At the commencement of the experiment with the platinum plates in fluid, when the potential was very small, the capacity was about 175 microfarads per square inch of (double) platinum surface; and as a sheet of air one inch thick and having a surface of $1{,}040{,}000 \times 3\frac{1}{2} = 3{,}640{,}000$ square inches has a capacity of 1 microfarad, $3{,}640{,}000 \times 175 = 637{,}000{,}000$ square inches will have the same capacity as the two platinum plates 1 inch square.

If the two hypothetical films have the same thickness on each platinum plate, the film on each will, if a pure gas, be $\frac{1}{1274{,}000{,}000}$ of an inch, with small potentials, and decrease to $\frac{2}{7}$ of this amount with a potential of 1·6 volt.

A useful inference can be drawn from these experiments by the telegraph engineer.

It has been repeatedly proposed to telegraph by means of a naked wire laid in the Ocean.

The speed of a long telegraph-cable varies inversely as the square of its length for similar sectional dimensions.

If L be its length, I its inductive capacity, R its resistance, and A a constant, its speed will be $\frac{A}{L^2} \times \frac{1}{I \times R}$.

Thus, then, if the product of $I \times R$ remain constant, their ratio to one another may vary without varying the speed of the cable.

The French Atlantic conductor consists of 400 lbs. to the nautical mile, and if made into a *solid* wire would in a length of 8·1 inches have the surface of $3\frac{1}{3}$ square inches.

The cable in round numbers is 2500 nautical miles long, and this length has a capacity of 1000 microfarads, the same as $3\frac{1}{3}$ square inches of platinum surface in water at a potential of ·6 or ·7 volt.

$I=3\frac{1}{2}$ inches, $R=2500$ miles; by varying these until $I=R$ in length (their product remaining constant), we get the length in round numbers of rather less than 1100 yards (or, say, half a mile) as that at which the bare wire if coated with platinum would equal in speed the cable now working from France to St. Pierre, the longest cable in the world.

As the surface of the wire increases with its diameter, and its conducting-power with the square of the diameter, the speed of transmission through a bare wire varies only as its diameter and not as its mass; therefore a bare solid conductor capable of working ten words a minute through 2500 miles of ocean must be more than 250,000 feet in diameter to have the same speed as the present French Atlantic Cable.

This mode of telegraphing is only practically available within distances of less than a mile, and this explains why Lord DUDLEY's uninsulated cable between Dover and Calais would not work. The distance being 20 miles ($40 \times \frac{1}{2}$ miles) and the conductor about half the diameter of the French Atlantic (which gives on an average ten words a minute in actual practice), the speed would be $\frac{10}{(40)^2} \times \frac{1}{2} = 1$ word of five letters in 320 minutes $=1$ letter of three signals in 53 minutes $=1$ simple signal in 18 minutes.

The measures in the Tables of the capacity of the platinum surfaces in the water are only approximately true.

There is considerable difficulty in these experiments, owing to the discharge not being sensibly instantaneous, and the absorption being very large.

The general truth that the capacity increases when the potential increases is, however, beyond all manner of doubt established.

This is the very reverse of what was expected by the author when he commenced the investigation.

Table I.

Two platinum bulbs about 0·75 inch in diameter in diluted sulphuric acid.

Owing to the large resistance (1000 Ohms) used in R and R′ the actual potential is uncertain in this experiment, because the conduction across the fluid reduces it.

1. Potential in terms of a cell of Daniell's battery.	2. Duration o electrification in seconds.	3. Swing of reflecting galvanometer by the discharge of the bulbs on raising the key.	4. Current after magnet came to rest.	5. Mean *minus* the remaining current.	6. Mean divided by potential and 100 to give relative capacity for various potentials.	7. Approximate capacity in microfarads.
0·02	10	2½ 2½ 2½ 2½	½	2	1	348
"	20	2½ 2½ 2½ 2½	"			
0·04	10	4½ 4½ 4½ 4½	"	4	1	"
0·06	10	6½ 6½ 6½ 6½	"	6	1	"
0·1	10	11½ 11½ 11½	1	10½	1·05	365
"	20	11 12 11½	"			
0·16	10	18 17½ 18	1¼	16¾	1·09	379
"	20	18 18 18	"			
0·2	10	24 24 24	1¾			
"	20	24 24 24	1½	22½	1·12	390
"	30	24 24 24	"			
0·4	10	59 58 58	2¼			
"	20	58 57 58	2½	55½	1·39	484
"	30	58 58 57	"			
0·6	10	105 105½ 104	"			
"	20	105 104 105	"	104½	1·74	606
"	30	104 105 103	"			
0·8	10	164 163 162 162	3			
"	20	162 162 161	"	159	1·99	693
"	30	162 162 161	"			
1·0	10	230 235 230 230	5			
"	20	232 231 230 231	"	226	2·26	786
"	30	231 231	"			
1·2*	30	318 320 314	14	303	2·53	880
1·4*	30	440 446 451	23	426	3·04	1057
1·6*	30	about 603	30 41 52	562	3·5	1218

Table I. (continued).

Condensers of 311 microfarads.

Potential in terms of a cell of Daniell's battery.	Throw of image by discharge of condensers.			Mean.	Ratio of capacity with different potentials as observed.	Value in microfarads.
0·02	1½	1½	1½	1½	1·03	311
0·04	3	3	3	3	"	"
0·06	4½	4½	4½	4½	"	"
0·08	6	6	6	6	"	"
0·10	7½	7½	7½	7½	"	"
0·20	14½	14½	14½	14½	1	"
0·40	29	29	28½	29	"	"
0·60	43	43½	44	43½	"	"
0·80	58	58	58	58	"	"
1·00	72½	73	73	73	"	"
1·60	116	116	116	116	"	"
2·00	143	143	143	143	"	"

* Last three readings doubtful, the current remaining after the discharge being considerable. The true reading would be greater than indicated.

The condensers of 311 microfarads capacity consisted of 24300 square feet of metal surface, insulated by thin paper and paraffin-wax.

TABLE II.

Two platinum plates in acid and water, each exposing 1 square inch surface. The resistance of $R+R'=100$ Ohms in this Table; by experiment the potential of the two cells was found to be reduced 8 per cent., and was therefore very nearly 2 volts instead of two cells DANIELL'S.

1. Approximate potential in volts.	2. Time of electrification.	3. Throw of image by discharge of plates on raising the key.	4. Current remaining after discharge.	5. Mean *minus* the current.	6. Ratio of capacity with different potentials.	7. Value in microfarads.
	seconds.					
0·2	10	19 20 19	1	18	1	175
,,	20	18 19 19	,,			
0·4	10	45 46 46	3	43	1·2	210
,,	20	46 46	,,			
0·8	10	175 170 170 165	11	159	2·2	385
1·0	,,	230 228 226	18	210	2·33	408
*1·2	,,	310 308 311	22	288	2·67	467
*1·4	,,	373 380 382	30	350	2·77	484
*1·6	,,	460 460 467 475	33	428	3·10	542
Condensers of 311 microfarads.						
0·2	....	32 32 32	0	32	1	311
0·4	....	63 64 63½	,,	63½	,,	,,
0·8	....	127 127	,,	127	,,	,,
1·0	....	159 159	,,	159	,,	,,
1·2	....	188 187 189	,,	188	,,	,,
1·4	....	220 220 221	,,	220	,,	,,
1·6	....	252 254 252 254	,,	253	,,	,,
1·8	....	284 283 284	,,	284	,,	,,
2·0	....	316 317 317	,,	317	,,	,,

* The author considers these readings uncertain, having been obliged to guess how much current remained after the image had swung out and back, the momentum of the galvanometer lasting longer than with smaller deflections; the true reading would be greater than those observed.

VIII. *On the Structure and Development of the Skull of the Common Frog* (Rana temporaria, *L.*). *By* William Kitchen Parker, *F.R.S.*

Received October 10, 1870,—Read January 19, 1871.

Introductory Remarks.

Since the sending in of my last communication, that on the Skull of the Fowl, our knowledge of the morphology of the facial arches has been very greatly extended by Professor Huxley's invaluable paper "On the Representatives of the Malleus and the Incus of the Mammalia in the other Vertebrata" (see Proc. Zool. Soc. May 1869, pp. 391–407).

After comparing the components of the mandibular and hyoid arches in an extended series of vertebrate types, the author concludes his paper by saying (p. 406), "in the higher Amphibia changes of a most remarkable kind take place, of which I do not now propose to speak, as my friend Mr. Parker is engaged in working out that part of the subject."

The most important result of Professor Huxley's more recent researches into the subject is the rectification of a very obstructive error—namely, the supposition that the incus of the Mammal was the pier of the mandibular arch, thus making it the counterpart of the os quadratum of the Sauropsida.

The type which has been most instructive in this matter is that remarkable New-Zealand Lizard (*Hatteria*) the structure of which has been so well worked out and described by Dr. Günther (Phil. Trans. 1867, pp. 595–629), in which the stapes is continuous with the hyoid arch.

Taking this form as a practical stand-point, Professor Huxley has, after comparing its facial structures with those of the Crocodile and the Bird, proposed a nomenclature for the parts of the middle ear (largely formed by metamorphosis of the top of the hyoid arch), which should have the term "stapes" as a foundation.

We thus get the terms "suprastapedial," "infrastapedial," and the like, all very useful terms in the description of these modified parts of the facial arches in the Sauropsida.

Whilst my friend was working out this subject, it occurred to me to reexamine the condition of these parts in the tailless Amphibia; I soon saw enough to allure me on to an extended observation of these structures in the Frog; and the longer I worked the more I saw the necessity for doing what Professor Huxley strongly advised me, namely, extending my observations *backwards* and *downwards* into the condition of these parts in very minute embryos.

One thing soon appeared certain; and that was the absolute morphological distinction between the "stapes" or *ear-plug*, and the other bones or cartilages related to it physiologically as part of the middle ear. Nevertheless I see no reason for the non-adoption

of Professor HUXLEY's nomenclature of these parts, if it be held in mind that the *prefixes* signify physiological relation and not morphological representation.

I make these remarks now because, when once fairly started, it will be necessary to keep clear of all other Vertebrate types: a comparison of the Frog with itself, in its marvellously varying morphology as its structure is traced from the egg to the adult form, will give us sufficient employment for the time.

My object in this paper being *special*, I shall describe what has been seen without reference either to the works of other anatomists, or to what I am familiar with in the structure of the skull in other types.

But as the Frog has received large attention from the best writers, it may be truly supposed that I am greatly indebted to their writings, the most important of which are the following, namely:—

I. ANT. DUGÈS. "Recherches sur l'Ostéologie et la Myologie des Batraciens à leurs différens âges." Paris, 1834.

II. K. B. REICHERT. "Vergleichende Entwickelungsgeschichte des Kopfes der nackten Amphibien, nebst den Bildungsgesetzen des Wirbelthierkopfes im Allgemeinen und seinen hauptsächlichsten Variationen durch die einzelnen Wirbelthierklassen." Königsb. 1838.

But in endeavouring to form a clear conception of the morphology of the skull in its simple condition, I am most of all indebted to JOH. MÜLLER's magnificent work on the lower types of Fish, entitled "Vergleichende Anatomie der Myxinoiden, der Cyclostomen mit durchbohrtem Gaumen." Berlin, 1835.

This work, Professor HUXLEY's Croonian Lecture (Proc. Roy. Soc. Nov. 1858), and his 'Elements of Comparative Anatomy,' 1864, have been always open before me whilst engaged in this piece of research.

My observations on the structure of the skull in the Common Frog have, in several stages, been corroborated by what I have seen in *Rana boans* vel *pipiens*, *Pseudis paradoxa*, and *Bufo vulgaris*; these, however, will be described at some future time*.

Before commencing a description of the stages of the Frog, it is necessary to speak of the terms which will be made use of, both histological and morphological.

In my memoir on the Shoulder-girdle (Ray Soc. 1868, p. 4), the *varieties* of ossification are spoken of as "parostoses," "ectostoses," and "endostoses;" to these another variety might have been added, namely, "dermostoses."

But the endosteal mode of ossification is further divisible into three kinds, namely, superficial, subcentral, and central endostosis.

These distinctions hold good in many cases, whilst in others they completely break down, and therefore they have only a limited and *varietal* value. For instance, the three divisions of the parasphenoid in the Bird, as I showed in my last paper, are true

* I have been greatly helped as to *materials* by Dr. MURIE, Professor W. H. FLOWER, F.R.S., Dr. GUNTHER, F.R.S., and Dr. MICHAEL FOSTER; but the best help of all has been frequent discussion of the matter, in its gradual unveiling, with Professor HUXLEY.

"parostoses" for some days, being formed in a thick web of fibrous blastema; they soon, however, apply themselves to the overlying cartilage of the basis cranii, and become ectosteal in relation to it.

Again, there can be no better instance of an ectosteal sheath than the shaft-bone of a young Sea-turtle's rib ("Shoulder-girdle and Sternum," pl. 12. fig. 5); but in a short time the bony matter spreads into the surrounding fibrous tissue and into the overlying derm, affecting that tissue *outside* much faster than the cartilage within (ibid. fig. 6). RATHKE had shown this long before, in his work 'Ueber die Entwickelung der Schildkröten,' 1848.

In this case that which is primarily a true ectostosis becomes parosteal during development, and then spreads into the substance of the skin, forming a dermosteal layer.

That which is remarkable in the "Anura" is the paucity of bony plates as compared with Osseous Fishes and Reptiles, and also the long time that the ossification of the fibrous bony layer keeps independent of the *calcification* which takes place somewhat later in the superficial cells of the hyaline cartilage within; this latter is "superficial endostosis," as in the Sharks and Rays*.

Here, in the Amphibia, both tailed and tailless, there is no original distinction to be seen between a parostosis and an ectostosis, and therefore the question as to whether the fibrous bones are ectoskeletal or endoskeletal has to be determined arbitrarily, by comparison with their counterparts in Osseous Fishes and in the abranchiate Vertebrata.

That this distinction is merely arbitrary, as far as fibrous bones are concerned, is evident from what I shall have to describe in this paper; for in the Frog the *dentary*, which is nearly always a reliable parostosis in the other great groups, forms here an ectosteal sheath to MECKEL's cartilage; whilst the *articulare*, which everywhere else bears an ectosteal relation to the upper part of that rod, is in the Frog a mere splint applied to its surface. In the early condition of the *larvæ* of the "Amphibia Urodela," the bony plates which appear in the palatal region are all similar films of bony deposit in delicate tracts of fibrous blastema; and so far there is no distinction as to exoskeleton and endoskeleton between the parasphenoid and vomers on the one hand, and the palato-pterygoid plates on the other.

In my mode of illustration, however, I shall continue to colour those bony plates *yellow* which, as a rule, are the immediate setters-up of ossification in the cartilaginous endoskeleton; and the plates which normally keep free from the cartilage will be left uncoloured, as though they were truly exoskeletal; they are, indeed, the connecting bond between the two systems.

Structure of the Frog's Skull, First Stage.—Embryos from 2 to 3 lines long; extending in time from 2 days before to 2 days after hatching.

It will be necessary to describe both external and internal characters in their early stage;

* See ROLLETT's paper (p. 114) in STRICKER's 'Human and Comparative Histology,' translated for the New Sydenham Society, by H. POWER, 1870.

for the cutaneous system, now complete over both embryo and yelk-sac, is just ready to undergo very remarkable changes, becoming highly modified over the sense-organs, and undergoing dehiscence in the facial walls.

At present, in embryos two lines long, still in the jelly-ball, the cutaneous system has only one opening, namely, the oral (Plate III. fig. 1, *m.*).

Elevations and hollows there are, however, in abundance, which will show their own meaning afterwards, when dehiscence takes place, or even now, when they indicate the form of already developed organs or parts within. A front view of the skull and face of the unhatched frog-embryo (two lines long) is very instructive (see Plate III. fig. 1, $\times 20$).

In the centre is the small lozenge-shaped oral opening (*m.*), above this the "fronto-nasal process" (*f.n.*), ending below in a right and left horn; this median, bilobate region forms the inner half of the boundary of each of the rudimentary olfactory sacs (*ol.*).

Below the oral opening we have the right and left cheeks or facial walls, *swollen* in this early stage; for the fore part of the "yelk-chasm" is permanent, and enlarges to form the pharyngo-stomal cavity, which is already lined by a layer of cells, derived, according to Reichert, from the lower part of the "cumulus germinativus;" this layer becomes the mucous membrane of this region, and passes, some way within the oral opening, into the cutaneous system. The two sides of the face are separated anteriorly by a deep fossa; below they terminate in those thick reduplications of the cutaneous system which are called the "claspers" (*cp.*).

On each side of the fronto-nasal region (*f.n.*) there is a reniform depression covered by a thinner tract of the cutaneous system, and having as its inner rim the thickened edge of the fronto-nasal process; above and on the outside the rim is still more strongly marked; *this is the rudiment of the olfactory sac* (*ol.*).

At present this is merely a mass of cells formed by modification (differentiation) of this part of the cutaneous system; it is at present *blind* within and without; but the *depression* will become a *tube*, and the *rim* will afterwards enclose the alinasal cartilage.

Above, behind, and external to these rudiments there is another pair, having a like relation to the cutaneous system; these become the eyeballs.

The thick rim of these somewhat triangular depressions is a fold of the skin which is open in front, the structure of the eyeball being mostly horseshoe-shaped at first and circular afterwards.

Surmounting the whole we have the frontal part of the cranial region, which contains at this part the "2nd cerebral vesicle" above, and the "1st cerebral vesicle" below, the "mesocephalic flexure" having turned the latter downwards and given the 2nd vesicle (fig. 4) a *frontal* position.

This front view (fig. 1) shows a very sharp distinction between the *frontal* and *nasal* regions; for the depression on the eye-rudiment is continued across the top of the face, the two *seams* meeting in an arched manner at the mid line.

Towards this seam or "raphe" the cutaneous system is thickened both above and

below, so that where the skin of the forehead meets that of the top of the face there is a very evident selvedge; whilst the furrow between the "raphe" is of a paler colour, and is but little differentiated from the gelatinous blastema which separates the membranous cranium from the oral mucous membrane*.

The *lips*, which are so highly developed afterwards, are at present merely represented by the somewhat inturned edges of the oral opening (*m.*), the upper being formed by the emarginate lower edge of the fronto-nasal process, and the lower on each side by the bevelled supero-internal angle of the descending facial plate. These facial plates are greatly inturned at the mid line, where they form an obliquely descending raphe which is but imperfectly finished above; the oral opening being continuous with a slit which descends for some distance, the right and left facial walls being imperfectly soldered together above.

If a vertical section of the head of an embryo at this stage be made, we have the appearance seen in fig. 4.

The thick dermal layer (*d.*) is seen to follow the inflections of the membranous cranium (*m.c.*), which has already a considerable consistence, and the inflections of which relate to the form of the enclosed cerebral vesicles (fig. 4, C 1, C 2, C 3) and the rudimentary pituitary body (*py.*). The flexure of the brain upon itself (mesocephalic flexure) is shown in this figure, although its straightness is in some degree recovered. Underlying the medulla oblongata (*m.ob.*) and medulla spinalis (*m.s.*) there is a thick rod of gelatinous tissue enclosed in its own sheath; it is blunt-pointed and decurved, and it terminates a little distance behind the pituitary vesicle (*py.*); this is the notochord (*n.c.*).

The tissue forming this rod is very similar to that which everywhere fills up the spaces between the rudimentary organs in the embryo at this stage; it is a very watery kind of blastema, interspersed in all directions with delicate membranous bands, a structure well displayed in MÜLLER's figure of a transverse section of the notochord of the Hag-fish (*Myxine glutinosa*) (see 'Myxinoids,' pl. 9, fig. 1).

Already, patches of cartilage have appeared in the outer part of the notochordal tube (sheath); the foremost of these become the postpituitary part of the basis cranii, and the following patches form vertebral arches. In this section the yelk-mass (*y.*) is seen to persist up to nearly the fore end of the notochord above, but lower down it is deficient further back; this is caused by the bulging behind of the mouth-chasm or stomo-pharyngeal cavity.

Still further down the substance of the yelk itself has undergone transformation, being applied to the formation of the rudimentary heart and its sac (better seen in next stage, fig. 12, *p.c.d.*). Beneath the sac the cutaneous system has joined from the right and left side and has produced the thick claspers (*cp.*).

In this section the oral mucous membrane is shown lining the cheek, except ante-

* Professor HUXLEY, at first sight of this figure, pointed out to me that the middle part of this "raphe" occupies the place of the azygous nasal opening of the young Lamprey (see MÜLLER's 'Myxinoids,' pl. 4, figs. 9, 10, *f*).

riorly, where the naso-frontal process has been cut through a little to the right of the mid line, so as to expose its cavity, which contains gelatinous tissue surrounding an upper and a lower cartilage; these cartilages underprop the 1st cerebral vesicle (C 1). The form of the cheek, as seen from the outside at this stage, is simply swollen; but it becomes *ribbed* soon afterwards (fig. 2, side view); this ribbing, however, is largely developed already on the inner side (fig. 4); the subvertical thickenings contain the solidifying "visceral arches," and the *furrows* are the commencing visceral clefts: at present this dehiscence of the facial wall has not affected the cutaneous investment; but the mucous membrane has been thrown into sharp folds, as is well shown in transverse sections (figs. 5–7). The side view of a recently hatched embryo shows in its further advancement much that is instructive (fig. 2): in this figure the skin has been removed from over the auditory sac (*au.*).

Here it is seen that the three sense-capsules occupy what may be called the "lateral cranio-facial line;" they follow the curve of the cranial cavity, and lie immediately above the visceral arches, the first of which is beneath the nose but *above* and in *front* of the mouth; the postoral arches reach to the yelk-mass (*y.*) which fills the thoracico-abdominal cavity; and here the skin is already cleft on the outside; this early dehiscence is the first appearance of the opening between the last branchial arch and the anterior margin of the thoracic wall. The outer wall of the face is now rather flatter; and in lines corresponding with the internal ribbings (fig. 4) shallow sulci are seen, over which the skin is becoming attenuated. On about the middle of the 3rd and 4th post-oral arches there is a small *bud*; this is the first appearance of the branchial papillæ on the 1st and 2nd branchial arches, which are thus seen to be developments of the cutaneous system. If the skin be now peeled away from the rest of the cheek (fig. 3), we shall see what caused the ribbings in the mouth-cavity (fig. 4); here are displayed a series of parallel subvertical rods of rapidly hardening cartilage, which occupy the whole space from the middle of the fronto-nasal process to the thoracic yelk-mass. Here we see that the swollen condition of the cheeks seen in the front view (fig. 1) is not so much due to the size of the oral "chasm," as to the very solid walls of the face; nearly all the tissue intervening between the oral lining and the outer skin has been expended in the formation of these bars, which are very closely packed together externally. It will be seen, both from the inner and outer view (figs. 4 & 3), that the shape of these bars is gently sigmoid, that they bulge outwards in their middle part, that they turn inwards both above and below, and that the foremost and the last two or three do not reach so far downwards as those in the middle. This arises from the fact that the foremost pair lie above and in front of the mouth, and the hinder arches lie above the pericardium. They thus follow the shape of the "stomo-pharyngeal cavity," of which they are the skeleton; and they are by no means to be confounded with the arches that grow from the segmented vertebral axis, which, when finished afterwards, enclose the *heart* and all the other viscera. The point, *first of all in importance*, which is to be noted here is that these rods are all *absolutely distinct* (fig. 3); their upper termination

is by blunt points, which are curved inwards and then a little backwards; the arches then gently curve forwards to the middle, from which part they turn backwards and inwards again. Thus, notwithstanding all the melting together of facial bars, sense-capsules, and cranial walls, in more advanced stages, nothing of the kind now exists; the after-blending is due to *secondary connectives* that fuse together the elementary organs. The first pair of these arches correspond to the "palatal bands" (MÜLLER) of the larval Lamprey (*Ammocœtes*), but they indicate a much earlier stage; for MÜLLER's figures ('Myxinoids,' pl. 4, figs. 7–10, D) show a continuity of these bands both with each other in front, and with the "investing mass" of the notochord behind; this will be illustrated in my "third stage." These bars, the "trabeculæ cranii" of RATHKE, are the shortest and the thickest of the series; they retain their parallelism with the next pair as far downwards as the middle, and then turn forward and inward, lying like *little beams* beneath the *totally distinct* membranous cranium (see fig. 8, a transversely vertical view). The second of the series (first poststomal) is the rudiment of the mandibular arch; but at present there is no segment freed from its lower end, answering to "MECKEL's cartilage." The third, or second poststomal, is the hyomandibular and hyoid cornu in one undivided piece; the fourth is the first branchial, and has a branchial "bud" upon it; so also has the fifth, but not the sixth; the seventh arch (sixth poststomal, fourth branchial) is not yet clearly differentiated. The horizontal views are very instructive: figs. 5–7 are of an embryo younger than that dissected from the outside (fig. 3); and fig. 9 is a corresponding section to the lowermost of the earlier stage (fig. 7), in one somewhat older than that displayed in fig. 3.

In fig. 5 the rods are cut through, by a somewhat oblique section, immediately below the cranial capsule; the "investing mass," the rudimentary first vertebra (*v*. 1), and the enclosed notochord are displayed in this section, and also the shape of the mouth-chasm. Fig. 6 is a little lower down, and shows what fig. 5 does not, namely, the top of the sixth arch, or third branchial; here the inner walls of the cheeks are seen to some extent. Lower down still (fig. 7) we see the floor of the mouth, with the rudimentary tongue (*tg.*); and behind the arches, only five of which could be clearly seen, we have the large yelk-mass (*y.*) with its emarginate front outline. Expecting to find the earliest condition of the "trabeculæ" (first pair of rods) in the *ammocœtine* condition (that is, continuous with the "investing mass"), these sections cost me much thought, and these observations were repeated again and again. The fact is that the *last* of the arches comes nearest to the investing mass, and not the *first*, and it is wholly distinct from the *axial* structures.

Thus we see that "the branchial arches have the same morphological value as the hyoid, and the latter as the mandibular arc" (HUXLEY, "Croonian Lecture," p. 53); and not only so, but the "trabeculæ cranii" are merely the foremost of these arches, as both of us have for some time felt satisfied.

In the earliest condition of my "first stage" (fig. 7), although these thick rods gradually decrease in size from before backwards, yet the first are twice as thick as the

second, and their section is most perfectly ovoidal. In the rest the shape in section is round within and flattish on the outside; this arises from the fact that dehiscence commences on the inner side several days before the clefts appear in the cutaneous system; yet thus early the cheeks are becoming forrowed (figs. 6 & 7) between the arches.

The incurved lower end of the arches stops short some distance from the mid line, leaving a space for the azygous pieces, which, however, do not appear for several days to come. The oblong tract of tissue between the arches below (fig. 7), which has a rounded free margin anteriorly, and which has been partly cut away in the section behind, is the basihyal and basibranchial region; the free-ending anterior part becomes the tongue (*tg.*). The tissue of which the rods are composed is cellular, very solid, and is rapidly passing into hyaline cartilage; a very evident concentric line indicates the differentiation of the perichondrium; they are imbedded between the skin and mucous membrane in a very thin layer of delicate gelatinous tissue. A transversely vertical section (fig. 8) displays the manner in which the first pair, or "trabeculæ," underprop the membranous cranium; the section is through the "first cerebral vesicle" and the rudimentary eyeball (*e.*). Here these rods seem, as seen from behind, to come into contact more nearly than they do in reality; this is caused by their being seen through the thick mucous membrane; the other sections (figs. 5–7) correct this. Immediately below the clubbed ends of these rods (fig. 8) is the upper lip; this contains a trace of solidifying cartilage of the upper labial, seen better in figs. 3 & 4 (*u.l.*). Below the mouth (*m.*) there is another and more vertical patch of young cartilage (figs. 3, 4, and 8, *l.l.*); this becomes the lower pair of labials; the first poststomals are seen from behind, and partly cut away in fig. 8: the great density and thickness of the cutaneous investment, and the tracts filled by gelatinous tissue, are shown in this figure, as also the manner in which the membranous cranium rests upon the palatal portion of the oral mucous membrane.

In the horizontal section of a hatched embryo 3 lines long (fig. 9) we have a transition towards the second stage.

The decreased "mesocephalic flexure" has carried the trabecular rods into a more horizontal position, so that the section of them is very oblique and not directly across as in fig. 7.

The next pair have freed themselves below from the better-formed rudiment of the tongue (*tg.*), and the free clubbed ends of these rudiments of the mandibular arch are now preparing for transverse fission; thus "MECKEL's cartilage" will soon be differentiated. The next or hyoid arch is now seen to belong to the rudimentary tongue; the *gill-buds* are seen enlarging outside the first and second branchials; the fourth branchial is not yet distinct. In both the lateral views (figs. 2 & 3) the skin has been removed from over the auditory sac (*au.*): it was perfect, however; but the sac itself, already with very solid walls, is *open* on the outer side; this opening is large, oval, and turned downwards and forwards; it looks at this stage as though the whole structure had been formed as an involution of the cutaneous system. Along the "lateral cranio-facial line," in the

hinder part of which the ear-sac is planted, there is a considerable space beneath the derm, which is filled now, and for some weeks to come, with gelatinous stroma.

Second Stage.—Frog-tadpoles 4 lines long.

If figs. 2 & 10 be compared, it will be seen what great advances have been made in the development of the larval Frog, although the size is only one line longer, and the yelk-mass still fills the greater part of the thoracico-abdominal cavity.

The fore part of an embryo at this stage presents the appearance shown in fig. 10 when seen laterally, and in fig. 11 when viewed from below.

The lips (*lp.*) and mouth (*m.*) are now well developed, and the head is less bent upon itself; but the change which is of intensest interest to the morphologist is the completion of the visceral clefts (1, 2, 3, 4, 5, 6).

This dehiscence into free bands of the cephalic visceral walls characterizes the Vertebrate animal, and wholly distinguishes the stomo-pharyngeal series from all other arches.

In one sense the mouth itself is a cleft in the lowest part of the visceral wall between the preoral and first postoral arches; but I would rather consider it comparable to a neural "fontanelle," and therefore as an imperfect closure of the "membrana reuniens inferior," than to secondary openings (by dehiscence), such as we have seen the true clefts to be*. The opening between the first and second postoral arches (figs. 10 & 11, 1) only appears on the outside, in the lower third, and that imperfectly, for the skin is rather greatly attenuated than completely cleft; this is the rudiment of the "tympano-eustachian" passage. A perfect cleft, however, does appear between the preoral and 1st postoral arch; at present it can only be seen from the inner side (fig. 13, *i.n.*): this is the first appearance of the "inner nostril;" and the cleft is only completed when the nasal tube is perfect. This takes place when the *central depressed* skin over the nasal sac (*ol.*) has grown inwards, and coalesced with the mucous membrane lining the inner edges of the cleft (fig. 13, *i.n.*).

Only for a short time, and at this stage, can the cleft between the first and second postoral arches be seen externally (figs. 10 & 11, 1); it is lozenge-shaped, and has the same direction as its more perfect successors; and although the skin becomes greatly attenuated at this part, I could not discover a perfect passage. It soon closes up, afterwards to reappear as the "membrana tympani," with its overlying thin dermal layer.

The gill-buds have now become compound papillæ, of a palmate form; and the first (fig. 10, *br.* 1) has eight secondary papillæ upon it. A retral fold of the skin over the second postoral arch covers the cleft between itself and the first branchial on the outside; this is the "operculum" (figs. 10 & 11, *op.*).

The branchial clefts (2–6) are about half as long as the depth of the embryo at the same part, and are equidistant from the dorsal and ventral line; they are rounded above and more acute below; the last is the most patulous; and behind it the thoracic wall is incurved. The mouth (fig. 11, *m.*) is beginning to take on the rounded form proper to

* The lozenge-shaped opening (Plate III. fig. 1, *m.*) may be due to the absorption of a layer of cutaneous cells

the Tadpole and like that of the Myxinoid, which it represents when its larval condition is perfect. The fronto-nasal process is now transformed into the large upper lip, with its unequal halves separated by a small sulcus.

If the fore part of an embryo at this stage be cloven vertically, we shall see what is depicted in Plate III. fig. 12.

The simple anterior cerebral vesicle (fig. 12, C 1) is now evidently composed of three lobes, namely, a small one in front, the "rhinencephalon" (fig. 12, 1^a), then a large lobe, more than hemispherical, the prosencephalon (C 1^b), and above and behind this a lobe which is compressed below; this is the "deutencephalon" (C 1^c).

Behind this third division of the anterior vesicle is the middle, and behind it the third. These are now arranged in a less curved line; for the mesocephalic flexure is becoming less day by day. This section shows the heart (*h.*) and pericardium (*pc.d.*), and also that the yelk-mass (*y.*) is retreating, the pharyngeal cavity having grown much larger and the thoracic organs more differentiated. Anteriorly, we see more clearly defined labials (*u.l.*, *l.l.*); and above the lower labial a bud of cartilage is seen ascending from the lower end of the first preoral arch; this is the first definite appearance of MECKEL'S cartilage (*m.k.*). A side view, with the skin dissected away, shows how far the facial arches have advanced since the last stage (compare Plate III. fig. 3 with Plate IV. fig. 1). The whole of the facial wall up to the yelk-mass is occupied by the moieties of the closely packed visceral arches.

This series now shows a seventh, the last, branchial (*br.* 4); it is the smallest; and the trabecula, which was the most massive, begins now to yield to the second and third. These bars have the same general form, and are much more elegant than in the first stage; they all curve *inward* both above and below, and are bowed *forward* both above and below; at their middle they are gently arcuate in the same direction (Plate IV. fig. 1).

The first pair, or preoral, have begun to adapt themselves more perfectly to the floor of the brain-sac, but they still largely retain their parallelism with the next, or first postoral pair. Below, the first postoral has not only gained a Meckelian *bud*, but it lies also at a further distance from the one behind it; this is caused principally by a commencing bend backwards of the lower half of the second postoral, preparatory to the subdivision of this pair of rods into the hyo-mandibular and cerato-hyal pieces. This arch also has now grown to be much larger than its successors (compare Plate III. fig. 3 with Plate IV. fig. 1); it also projects further outward, the skin covering it becoming the gill-cover (Plate III. fig. 10, *op.*, and Plate IV. fig. 5, *op.*). The top of this third arch is immediately below the fore end of the auditory capsule (*au.*), which is still an open sac. The remaining four arches form a series regularly decreasing in size, and their outer surface becomes more and more buried in the neck of the embryo (see also horizontal section, Plate IV. fig. 5, *br.* 1–4).

All these seven bars are still totally free, both from each other and the surrounding organs: the free, incurved upper end of the first is postorbital in position; the next

does not touch the auditory sac by a distinct space; and the third, like the four branchials, roots inward *beneath* the sac; it will afterwards coalesce with it and be carried outward to be again segmented off in a later stage. Of the many horizontal sections made in my study of this stage I have figured five (Plate IV. figs. 2–6): they dip more or less; and indeed the head is as yet bent upon itself, so as to require the section to incline forwards. In fig. 2 the three pairs of sense-capsules have been cut through (*ol.*, *e.*, *au.*), and the first cerebral vesicle, now divided into prosencephalon and deutencephalon ($C\ 1^b$, $C\ 1^c$), has been so laid open as to expose part of the pituitary body (*py.*); the upper part of the notochord (*n.c.*) is also laid bare. All that can be seen of the facial arches here, is the free incurved tops of the first and second—the third or hyoid lying in a lower plane: this view is of a solid preparation, seen from *above.* In fig. 3 the section was *arched*; the razor passing through the upper part of the yelk-mass (*y.*) below the notochord, and then severing the pituitary body from the "deutencephalon," passed through the latter and the prosencephalon ($C\,1^b$, $C\,1^c$); this object was seen from *below.*

The nasal sacs are left with the lower half of the head; but the eyeballs (*e.*) are exactly halved; here we have the foremost facial arches severed lower down than in the last figure. The next section (fig. 4) is exactly horizontal in relation to the axis of the head; it is made through the notochord (*n.c.*) with its investing mass (*i.v.*), the pituitary body (*py.*), the base of the "deutencephalon" ($C\ 1^c$), and of the "prosencephalon" ($C\ 1^b$), between the eyes and nasal sacs, neither of which are displayed. The ear-sacs (*au.*) hide the tops of the third arch; but the second and first are shown in this section, which is seen from above.

In another section (Plate IV. fig. 5), which dipped considerably forwards and cleared the floor of the cranium, the whole circle of facial arches is shown with part of the notochord (*n.c.*), not its actual termination, however, but obliquely, near the first vertebra; this is shown as a *thin* slice, and demonstrates the perfect independence of the bars, besides showing the completion of the branchial clefts. Compared with the first stage (Plate III. fig. 6), we see a great change in the trabeculæ or first pair; they are growing towards each other in front, each showing a broad, squared end; posteriorly the "commissure" will be formed; anteriorly the free rounded angle of each bar will grow into a "trabecular horn." The groove on the inside, between the first and second bar, is less deep than the next; behind the third (hyoid) the clefts are perfect. The third arch (second postoral) shows its opercular fold (*op.*); and the two next have attached to them the roots of the palmate *free gills*; the branchial arches (*br.* 1–4) are seen to be compressed in sectional form. This second stage is illustrated by another still lower section; this is a solid object seen from below (Plate III. fig. 13). Here the upper lip is cut through, and *the azygous* rudiment of the two upper labials (*u.l.*) is displayed. The mandibular and hyoid arches (first and second postorals) are cut through, and a bird's-eye view is given of the fore part of the palate. Here the mucous membrane is seen folded round the *first true cleft*, that between the preoral and first postoral arches: it is not complete at present on the outer side; but being the rudiment of the

inner nostril, it *will extend to* the outer side as the external nostril; like the cleft between the first and second postorals (Plate III. fig. 10, 1), it is of small *vertical* extent.

Third Stage.—Frog-tadpoles 5 *lines long.*

This stage illustrates the period when the *free gills* are at their fullest development before the operculum has enclosed them; they have *tertiary* papillæ at this time. In Plate IV. fig. 7, a Tadpole at this stage is shown with the skin removed from one side so as to expose the skull and face; the eye has been dissected away so as to expose the first arch more perfectly; but the nasal sac (*ol.*) and the now *finished* ear-sac are left *in situ.* All the chondrified parts, except the "investing mass" and the azygous basal elements of the face, are shown at once in this view, which is extremely instructive, and the meaning of which will be evident to the ichthyotomist at once. The facial arches have not changed their relative size so much as their *shape* and *position*; transverse segmentation is now complete in the first and second postorals; and *secondary growths,* which may be called "connectives," have now begun to bind together these primordially free rods. The cranial cavity has now nearly lost its bend upon the spinal tube (figs. 7 and 8); and the first three arches have followed in its ascent, so that they now form an *acute* instead of a *right* angle with the general axis of the Tadpole; the gill-arches are still almost vertical. The first or preoral arch has now begun to hide itself beneath the membranous cranium, with which it will soon coalesce, and outside which it will set up the chondrifying process. Even externally it can be seen *that the first and second rods are coalescing with each other above, where they lie close to each other;* below this part they diverge—not, however, to lose connexion; for at the lower *third a transverse band appears, a "connective" uniting the trabecular arch with the mandibular.* This band, which had no existence in the last stage (Plate IV. fig. 1), is the first rudiment of the pterygo-palatine bar, so largely developed in the adult Frog. This "connective" lies, of necessity, *inside* the temporal (crotaphite) muscle. But the mandibular pier sends upwards and forwards another leafy growth of cartilage, which embraces the temporal muscle from its outside; this is the "orbitar process" (*or.p.*), a very characteristic larval batrachian structure. We can now see what becomes of the imperfectly open cleft between the first and second primordial arches: the space between the upper coalesced part and the pterygo-palatine connective becomes the "subocular space," whilst the space below and in front of the pterygo-palatine becomes open, first *within* and then on the *outside,* as the canal of the nasal sac, the openings being the internal and external nares. The upper part of the first postoral arch may now be called the "mandibular pier," because the true mandible is now well developed; this is MECKEL's cartilage (*mk.*); it is a short stout club, with a fossa for the condyle on the fore part of the lower end of the pier. It is not functional at present to a great degree, and turns upwards and inwards in thickness of the lower lip (see also fig. 8, *mk.*). The upper part of the mandibular pier, as far down as to the pterygo-palatine connective, may now take the name of the "metapterygoid region," whilst the rest of the bar is the

"quadrate." There is a part, however, which is not represented by the metapterygoid of the teleostean fish; this is the connective band, which has now attached it to the "investing mass," and which I shall describe anon. In the last stage we saw (Plate III. fig. 1) that the first and second postorals were diverging below; that divergence is now very great (fig. 7), and its meaning is self-evident. With the most perfect fidelity to the ichthyic type, the auditory capsule is now overshadowing the apex of the second poststomal, ready to coalesce with it; and this bar has now begun a series of metamorphic changes that yield in interest to nothing that the morphologist encounters. It has become divided into two nearly equal parts; and the lower piece diverges backwards so far as to leave a large space between it and the "quadrate region." The upper piece is the "hyo-mandibular," the lower the "stylo-cerato-hyal;" the space is preparing to become the "tympano-eustachian cavity;" and the small projection of cartilage below the new condyloid hinge is a rudiment of the "symplectic" in a *plagiostomous* condition. The hyo-mandibular piece and the symplectic *bud* cleave close to the preceding bar, save at the top: all this will be seen to be full of meaning in subsequent stages. The four next, diminishing, faucial arches, the branchials, have not altered much; but they are more perfect in form, and are still quite distinct from each other and from surrounding organs. Two other cartilages are seen in this side view, namely the upper and lower labials (Plate IV. fig. 7, *u.l.*, *l.l.*); these are to be seen in the other figures. Much as a lateral view displays, there is also much to be learned from vertical and horizontal views. In fig. 9 the membranous cranium and its contents have been removed, all but the diverticulum, which contains the pituitary body (*p.y.*); the cephalic part of the notochord and the "investing mass" (*n.c.*, *i.v.*) are shown as horizontally cut through; all the rest of this view *is facial*. In this view, a solid object seen from above, the upper labial cartilage (*u.l.*), is seen to be divided into a right and a left piece; these pieces only half fill the thickness of the upper lip, which is filled with gelatinous tissue anteriorly. The middle of the fore part of the palate has been cut away, exposing the oral cavity (*m.*).

Within three or four days, whilst the embryo has grown a line more in length, the facial arches have become largely confluent. The trabeculæ have not only grown beneath the skull more perfectly, and become fused to the mandibular pier above and tied to it below by the pterygo-palatine bar, but they have also formed a large *commissure* in front of the pituitary body*.

This palato-facial balk is the common support and foundation, indeed, of all that is termed ethmoidal in the higher types; yet at the mid line it is *a secondary* structure. If we compare this horizontal view of the trabeculæ with that of the previous stage (fig. 5), we shall see how rapid have been the changes, not the least remarkable of which is this *closing-in* of the "palatal bands" (Gaumenleisten) round the pituitary body, which in the earlier conditions (see the position of the pituitary body, *py.*, in relation to them in fig. 5)

* This is termed by MÜLLER "vordere Commissur der Gaumenleisten unter der Nasencapsel" in the Ammocœtine stage of the Lamprey; and in the adult it forms his "knöcherner Gaumen, or hard palate" ('Myxinoids.' pl. 4. fig. 3, H, and fig. 7^b, d').

were so totally unrelated. The parallel faces of the ends of the trabeculæ in fig. 5 cannot have coalesced wholly, but only behind—the free anterior rounded angle growing into the leaf-like trabecular cornua (fig. 9, *tr.c.*).

That I have not overrated the rapidity nor the kind and degree of the changes which have taken place in three or four days will be seen if fig. 2 be compared with fig. 9, as to the apex of both the first and second arch (trabecular and mandibular).

In fig. 2 we see that these tops are apiculated, inturned, free, and a long distance from the pituitary body (*py.*); in fig. 9 they have become wholly fused at the top, have nearly gained the mid line *behind* the pituitary body, and although distinct *above* from the investing mass (*i.v.*), embracing it, and forming a retral lobe, yet, seen from below (fig. 10, *tr.*, *m.n.*, *i.v.*) we have clear proof that these facial bars have now grafted themselves upon the investing mass, *skull and face* being now no longer distinct.

In fig. 9 the eyeball (*e*) and its surrounding stroma hide the pterygo-palatine bar; but in fig. 10 this is seen.

The top of the second postoral, or hyo-mandibular (*h.m.*), is now seen to be just *inside* the fore part of the pyriform auditory capsule (*au.*); it is close to it and is creeping outwards, ready to coalesce and be carried to the outside of the face by that part of the capsule which contains the ampulla of the horizontal semicircular canal. Behind the hyo-mandibular the section (fig. 9, *br.* 1) has displayed the top of the fourth arch, or first branchial; it is the pharyngo-branchial portion; and its counterpart is well known in Fishes, after attaching itself, quite normally, below the prootic region of the cranium. We can now see the structures on which the membranous cranium rests (figs. 8 and 9)—namely, the cephalic part of the notochord (*n.c.*), its investing mass (*i.v.*). and the auditory capsules (*au.*); the division between the cephalic and vertebral portions of the *axial skeleton now more clearly displays the occipito-atlantal* articulation, the ends of the investing mass forming the paired *occipital condyles. The nature of the vertebrate skull, as, morphologically, highly compound, is well shown in these horizontal views.* Even the floor of the pituitary space, the fundus of the "sella turcica," is *cranio-facial* in its nature. In fig. 10 it is seen that cartilage is creeping beneath the pituitary body *into the substance of* that diverticulum of the membranous cranium which contains this part of the brain. Now this centripetally growing floor is derived, behind, from the "investing mass," laterally from the trabeculæ, and in front from their secondary commissure. In this figure, which is *solid*, and seen from below, the cranio-facial base is exposed up to the front of the "commissure;" but between this part and the severed upper lip with its cartilages, the fore part of the palatal roof is shown with the *first pair of visceral clefts*, namely those which form the nasal passages. The meaning of these parts will be better shown in the *perfect Tadpole*, the next stage; but here the swelling of the mucous membrane, mesiad of the opening, is seen to be due to the overlying trabecular horn which is *inside* the nasal passage; the pterygo-palatine connective (*pg.p.*) bounds the passage *behind*; and the quadrate condyle (*q.*), here cut away, lies on its *outside*. The connective which binds the metapterygoid part of the mandibular pier to the investing mass turns suddenly

backwards, and then as suddenly forwards: these curves are clearly seen in the large Tadpole, the next stage. In this figure (10) it must be held in mind that these roots lie on a much higher plane than the amputated quadrates (*q.*). The figure must be compared with the lateral view (fig. 7); in this last the outer part of the metapterygoid is really the *elbow* shown in fig. 10; out of sight in the lateral view, the secondary connective bar grows suddenly *forwards* as well as inwards. In this lower view the upper part of the *cleft* between the prestomal and first poststomal arches is shown; it is a sigmoid tract of membrane (see also fig. 7, *s.o.f.*), the "subocular fenestra." The direction of the pterygo-palatine bar (*ppg.*) is downwards, outwards, and backwards; its development is a measure of the distance travelled by the second from the first primordial arch—the lower end of the mandibular pier (the quadrate hinge) ultimately lying directly below the exoccipital, the pterygo-palatine bar becoming nearly as long as the very elongated mandibular ramus. In the vertical section (fig. 8) many parts are brought into view and shown in their relation, which cannot be so well seen in other aspects. In the large lips we see the upper and lower left labials (*u.l.*, *l.l.*), the end of the left "trabecular cornu" under the prosencephalic region of the membranous cranium, the trabecular commissure (*tr.c.*) in section lying obliquely in a fold of palatal mucous membrane; the edge of the left moiety of the investing mass (*i.v.*) seen beneath the notochord (*n.c.*) and in the floor of the mouth cartilages are seen either as ends or sections; these will be better understood by reference to horizontal views.

The intestinal canal has now become ready for its functions, and the pharynx is narrowing where it is passing into the opening *cardia*. The heart is well formed in an ichthyic manner. In a horizontal section through the upper lip and labials (*u.l*) and at the lower third of the facial arches (fig. 11) we get a bird's-eye view of the lower lip bulged into two rounded ridges by Meckel's cartilage (*mk.*), and of the tongue (*t.g.*) bulged by the massive stylo-cerato-hyals (*hy.*); the broad ramus of each side is seen in section further outwards. In front of this ramus is an oblique view, in section, of the quadrate region of the first poststomal arch; and behind the hyoid is a deep fossa severing it from the first branchial (*br.* 1); all the branchials (*br.* 1–4) are seen to be flattened from before backwards obliquely; and the termination of their intervening clefts, a good distance from the mid line, is seen. In a section made lower down and more perfectly dissected (fig. 12) the lower part of the quadrate is followed by the massive hyoid, and, the soft parts of the tongue being dissected away, there is seen a pisiform keystone of soft cartilage, soft for weeks to come, being *late* in becoming hyaline; this does not occupy the whole mesial space, but is only behind; this is the "basihyal" (*b.hy.*), and is seen in section in fig. 8.

Behind the basihyal (figs. 8 & 12, *b.hy.*) there is a large elegantly pyriform plate of cartilage forming a keystone to the first and second branchials; it is the basibranchial bar (*b.br.*), and contains two *potential* segments, the first and second of ichthyotomy. In the figure (12) the first branchial is seen to be becoming angular in section; this will be understood afterwards; the last two, seen from their lower extremity in fig. 8, are curved upwards towards the mid line, in conformity with the much narrowed pharynx. These

horizontal sections are from embryos somewhat in advance of those which are illustrated in figs. 7–10.

Fourth Stage.—Tadpoles 1 *inch long.*

After the digestive canal is complete the growth of the larval Frog is very rapid.

The *floating gills* become covered with the opercular fold, and at the same time lose their individuality amongst copious tufts of the same character which grow both from the outer or dermal and inner or mucous surface of the gill-arches. On the *right* side the fold of skin from outside the second poststomal (hyoid) arch coalesces with the skin of the thorax; but on the *left* side (Plate V. fig. 1, *op.*) there is a vertical oval opening for the currents of water. The coalescence of primordially distinct organs had begun in earnest in the last stage; but now, having suffered some two or three weeks to elapse, we shall find a most remarkable development of this process of mutual engrafting by the direct fusion of closely applied parts, by connective bands in lateral bars, and by azygous commissures at the mid line. The *sense-capsules* are not in the least uniform in their relation to the cranio-facial structures; for the auditory sacs are primarily distinct and then coalesce, the cartilaginous "sclerotics" are permanently distinct, whilst the nasal labyrinth, at first entirely membranous, has its floors, roofs, and walls entirely developed as "outgrowths" from the first visceral arch ("trabeculæ") and its prepituitary "commissure."

Rightly to understand the skull and face of this *perfect Tadpole* (the next stages lead to the *Frog*), it will be well to compare the corresponding figures. Beginning with the lateral view, let the eye refer back to Plate IV. fig. 7 (third stage), to Plate IV. fig. 1 (second stage), and to Plate III. fig. 3 (first stage).

Beginning with the primordial skeleton of the mouth, we see the labials at their fullest state of development; and as they are very important in relation to the various ichthyic types, especially the lower, they may now be described.

When the skin is removed from the side of the Tadpole's head (Plate V. fig. 1), or when the head is completely bisected (fig. 2), then the upper and lower labials (*u.l.*, *l.l.*) are seen to form the thickness of the wrinkled dentigerous lips (*l.p.*); between these is the nearly vertical, very small, suctorial mouth.

The upper pair (Plate V. figs. 1–6, *u.l.*) are relatively large falcate flaps of solid hyaline cartilage, the cells of which proliferate rapidly (fig. 6^a). They are almost vertically placed (figs. 1, 2), are thick and heart-shaped in section (fig. 2), the upper thick edge being grooved and embracing the decurved end of the trabecular horn (fig. 2, *u.l.*, *tr.*). They are not the germs of the præmaxillæ, nor the premaxillary axis (endoskeleton)*,

* In the "Croonian Lecture" (p. 33, fig. 9), Professor HUXLEY named the upper labial "premaxillary," and the lower "mandible;" the error was discovered too late for correction in print, as he now informs me. That is almost the only flaw I can detect in his description of the larval amphibian skull: it only extends over four or five pages (p. 31 to 35); but that and the small woodcuts illustrating it have been of inestimable value to me in working out the present paper; I should have repeatedly lost myself during my research but for this ever-present guide.

but by metamorphosis into fibrous tissue do yield *some* of the matrix in which the parosteal præmaxillæ are formed. The first possible slice that can be taken from the fore part of the Tadpole's face (Plate V. fig. 6, *u.d.g.*) when seen from its posterior or cut surface, shows that the upper horny dentigerous (*Chimæroid*) plate is formed between the upper labials and the outer skin.

The lower labials (Plate V. figs. 1–5, and Plate VI. figs. 1–3, *l.l.*) are drumstick-shaped rods of cartilage, having the lower end the stoutest; the upper end is attached to the anterior face of MECKEL'S cartilage near the symphysis. Their direction is almost vertical; they nearly meet below; and the lower dentigerous plate (*l.d.g*) is rather below than behind these rods (Plate VI. fig. 2, *l.l*, *l.d.g*). The creases of the lips (Plate VI. figs. 1 & 2, *l.p.*) are covered with small hooked teeth, in addition to the two principal plates. The angle at which the lower labial is attached to the Meckelian rod (*m.k.*) is suggestive of a very different origin for the two cartilages thus unconformably related.

It would appear, from the imbedded condition of the Meckelian rods at this stage, that the lower labials are really the effective lower jaws of the Tadpole; we have in the Vertebrata a procession of three orders of mandibles—namely, the lower labials, the Meckelian rods, and the 'dentary' "parostoses."

The "trabecular horns" (Plate V. fig. 1, *tr.c.*), which sprang from the outer angle of the commissure, have grown into long, gently diverging bands, which are strongly decurved where they articulate with the upper labials (*u.l.*).

These thickish bands are rounded at the edge; their breadth is best seen in the horizontal views (Plate V. figs. 3–5).

They are at present distinct up to the "commissure" (*eth.*, *t.c.*) (the rudimentary ethmoid), which now rises in front of the cranium as a low, rounded, transverse wall. This wall lies immediately above the pterygo-palatine bar (fig. 1, *pg.*); in this view the rest of the trabecula is largely hidden by the eyeball (*e*); that part, however, which has coalesced with the next bar is seen behind.

The sectional view (fig. 2) best shows the manner in which the ethmoidal wall has been formed, and how the rhinencephalon (C 1^{a}) lies behind its concave face.

The subcranial part of the trabeculæ has nearly the same thickness as the free "horns" (see edge view of right bar in Plate V. fig. 2); nor is their breadth altered in any degree (figs. 3 & 4); and in this stage it seems difficult to suppose that they do not with these horns form one continuous bar with the investing mass; nothing but a study of their development could prevent such a view being taken of them. The trabeculæ help to form the skull in the same manner as the investing mass; both lie beneath the membranous cranium, and both, by upward continuous growth of cartilaginous laminæ, enclose the fibrous sac. In the postpituitary region this is a very exact repetition of the manner in which the neural laminæ of the vertebral moieties enclose the "theca vertebralis;" and the notochordal pith is enveloped by cartilage in the same way as in a vertebra.

But the hinder part of the skull is merely a modification of the vertebral axis, whilst from the posterior boundary of the pituitary space the apparent continuity of structure has only come about by a most remarkable metamorphosis of primordial parts. The transversely oval space caused by the closing-in of the trabeculæ and the formation of the "commissure" (Plate IV. fig. 9, *py.*), has now changed into an oblong form, rounded in front and pointed behind (Plate V. fig. 3, *py.*).

We saw in the last stage (Plate IV. fig. 10, *py.*) that a floor of cartilage was forming to the pituitary space; this floor, derived both from the "investing mass" and the trabeculæ, is now complete. It no longer, however, answers merely to the floor of the "sella turcica," but is in reality the middle and anterior part of the basisphenoidal region, and also the whole of the presphenoidal. The outer edges of the trabeculæ, also, are growing upwards so as to form a cartilaginous wall outside the fibrous brain-sac; this, however, will be described when I come to the sectional views.

Referring again to the side-views (Plate III. fig. 3, Plate IV. figs. 1 & 7, and Plate V. fig. 1), we see what morphological changes the first and second poststomals have undergone. The first of these, the mandibular arch, is seen to have its once free "metapterygoid" apex connected in front to the trabecula and behind to the investing mass. As in the last stage, this connective band grows *backwards* as well as outwards and downwards (Plate V. figs. 1, 3 & 4, and Plate VI. fig. 8, *m.pg.*); above, it is separated from the ear-sac by the "foramen ovale" (5), and the remnant of the first poststomal cleft is continued as far as to the outer edge of the now dilated ear-sac. From this point, to below the cup for the styloid head of the hyoid cornu, the two arches, which were cleaving to each other in the last stage (Plate IV. fig. 7), have now entirely coalesced. Opposite the part where this hyoid portion of the large bar ceases, the mandibular pier gives off two processes and one ray.

The first of these processes is still, as in the last stage, a very short connective band, tying together the trabecula and its successor (compare Plate IV. fig. 7, *ppg.*, with Plate V. figs. 1–4, *pg.*); the pterygo-palatine bar shows as yet no signs of what it will become in the Frog. The other process (*or.p.*) is *free*, and has now grown very large, strongly bending upon the temporal muscle from without, and reaching up to the fore edge of the eye (*e*). The remaining square end of the bar (Plate V. figs. 1–5, *qu.*) is the quadrate region, with its condyle for the free ray, or Meckel's cartilage (*mk.*). This stout, short, clubbed rod is shown in many aspects in Plate V.; its proximal end is deeply and roundly notched to hinge upon the condyle of the quadrate; it grows upwards, inwards, and forwards, and when the mouth is closed its upper end nearly reaches the trabecula; it is joined to its fellow by fibrous tissue.

The hyoid arch has made its second great morphological change; it has coalesced with the mandibular pier in front and with the auditory capsule above (Plate V. figs. 1–4, and Plate VI. fig. 8, *s.h.m.*, *i.h.m.*). The upper part, or supra-hyomandibular (*s.h.m.*), is attached to the auditory sac much lower down and more outward than the top of the arch in front.

In the second stage (Plate IV. fig. 1) the second poststomal was seen to be lower than the first; and now, attaching itself to the prootic region, it is carried both outwards and downwards.

This upper distinct part is small; it answers to only the upper part of the Teleostean hyo-mandibular; there is a broad sub-bifid upper head answering to the two ichthyic condyles, then a narrow neck, and then behind and below an "opercular process" (*op.p.*). Below this the two arches are fused together; but the hyoid part is demonstrated just above the commencement of the lower third, by the lunate fossa for the "styloid condyle" (Plate V. figs. 2 & 4, *st.h.*).

No further light will be thrown upon the amount of "symplectic" growth below the hinge (*sy.*) until we come to the next stage. The free terminal portion of the second poststomal bar has now become a very broad and massive plate of cartilage, the "stylo-cerato-hyal" or "hyoid cornu" (Plate V. figs. 1, 2 & 5, *hy.*); it is roughly 4-sided or lozenge-shaped, smoothly convex without, scooped within into an antero-inferior and a postero-superior fossa, divided by a ridge which passes down from the semilunar condyle. Only the posterior part of the basal line of this plate articulates with the small pisiform "basihyal," as we saw in the last stage (Plate IV. fig. 12, and Plate V. figs. 2 & 5, *b.h.*).

The four remaining arches (branchials) have arrived at their full development, and are greatly modified since the last stage (compare Plate IV. figs. 7, 11 & 12, and Plate V. figs. 1 & 5, *br.*). In the first place they have all coalesced together both above and below (Plate V. figs. 1, 1^a & 5). Three of the apices can still be seen (fig. 1^a, *p.br.*), but there is only one finger-shaped pharyngo-branchial above the first and second arches. In meeting above they form a miniature gothic arch, and the secondary bonds are thin and fenestrate; the two regions when they are joined are the "pharyngo- and epi-branchial." Below, they all unite to form a continuous "hypobranchial" region (Plate V. fig. 5, *h.br.*), which articulates on each side with the round first basibranchial and its rudimentary second segment. This hypobranchial plate, behind the basal element, overlies the pericardium (fig. 7, *h.br.*, *pc.d.*), and lies on nearly the same plane as the summits of the arches, close beneath the mucous membrane of the throat; so that the arches, which were once immediately beneath the skin and parallel with the sides of the face (Plate III. fig. 3, *br.*), now hang like hammocks obliquely across the throat (Plate V. figs. 5 & 7, and Plate VI. fig. 5). The whole mass on each side looks like a fruit, the carpels of which are in a state of dehiscence.

This resemblance is increased by the form of the first and fourth arches; in the last stage these were acquiring an angular form as seen in section (Plate IV. fig. 12, *br.* 1 & *br.* 4); but now they are baggy and crumpled, and are extremely thin. The toothings seen on their edges, which more or less alternate with each other, are covered with papillæ; but these are not calcified into teeth as in the Cod-fish and other "Teleostei," but form a rich series of transversely placed tufts—branchial tufts (Plate VI. fig. 5); in the figure the outer tufts are not shown, but they are indicated in Plate V. fig. 7. In a note

below* I give a description of the branchial tufts in a larger species, as they have been little understood hitherto.

At present little need be said of the cranial cavity; it is, as yet, largely membranous; and the cartilage related to it is, from the notochord forwards, entirely facial in its origin; and yet we have seen that the facial and the axial regions behave to the primordial membranous sac in a similar manner.

The first bone related to the skull has appeared below the elongated intertrabecular (pituitary) space; it has the same shape, and is but little larger; this is the "parasphenoid" (Plate V. figs. 2, 4, 7, *pa.s.*). The ear-sacs (*au.*) being implanted in the side-walls of the now long straight cranium, at its posterior third, the cartilage, which might have sprung up along the whole upper edge of the "investing mass," is aborted, and only appears *above*, behind, and to some degree in front of the large periotic mass (Plate V. fig. 1). That which grows up *directly from the auditory capsule* in front is the rudimentary alisphenoid (*al.s.*); that behind is the occipital arch (*so.*, *eo.*); and that above is the pterotic (*pt.o.*): so we see that all these coalesced parts, first facial arch, periotic capsules, and alæ of investing mass, behave in a like manner, and not otherwise than the symmetrical rudiments of a vertebra as they grow upwards to enclose the "myelon" and its fibrous sheath. Yet, morphologically, how diverse are these elements! In the last stage (Plate IV. fig. 9) the periotic capsule was obliquely pyriform; but now its contents have distended it in various directions, thus altering very much its original elliptical form. The semicircular canals and their ampullæ have done this (Plate V. fig. 3); and here it is shown that even on the outside their form can be seen; and the capsule is hollowed between them, above. The horizontal canal (*h.sc.*) has not only carried the capsule outwards in front, but it has also formed a ledge which projects outwards; this is the rudiment of the "tegmen tympani" (Plate V. fig. 4, *t.ty.*). We saw that the auditory sac was *open* in the first and second, and closed

* *On the Branchiæ of* Rana pipiens, *Linn.*

The whole series of arches is a deep suboval shell of cartilage open at the top, and placed so as to look outwards and backwards.

There are three clefts, two cochleate bars, and between these two widish rods with rather sinuous posterior edges. On the edge of the first cochleate bar there is a rich row of tufts (the Tadpole is "Lophobranchiate"), which are principally *within*; they are well supplied with *pigment-coated* vessels, and are divided into short transverse groups.

On the second and third narrower band-like arches the vessels, which are richly coated with pigment, are covered on the outer (lower) side, behind, with short transverse rows of rich tufts; whilst from the anterior edge of the *inner* face of the bars shorter tufts, in transverse rows, grow upwards into the cavity of the shell-like branchial case; these are not pigmented, and they are large in front.

On the fourth arch the tufts are large and pigmented at the edge, and grow *less* and *less* in *longer* transverse rows on the inside of the spoon-shaped bar, and are not coloured black *within*, but only where they hang downwards at the free anterior edge.

The result is a most rich and almost crowded crop of free depending tufts, and a less rich development *within* and *above*.

The posterior edge of the third ray is sinuous; the *inner* tufts are set on conical elevations of *cellular* tissue.

in the third stage (Plate III. fig. 3, Plate IV. figs. 1 & 7, *au.*); it is now open again, for a long-oval segment has been as it were cut out, and yet left in, like a miniature bung; this is the "stapes" (Plate V. figs. 1, 2, 4, *st.*), and the opening is the "fenestra ovalis;" its situation is infero-lateral, and is nearly midway between the ends of the capsule.

The "glosso-pharyngeal nerve" escapes near the end of the capsule below (fig. 4, 8^a), and the "vagus" (8^b) between it and the occipital cartilage; the ganglion of the fifth nerve (5) lies close to the *prootic* region, opposite the end of the notochord, and sends its branches *over* the metapterygoid connective (*m.pg.*).

The portio mollis (7^b) enters the capsule below the junction of the anterior and posterior canals (fig. 3, *a.sc.*, *ps.c.*, 7^b). Posteriorly, the epiotic-region (*ep.*) framed upon the posterior canal (*p.sc.*) projects nearly as far backwards as the now well-formed occipital condyles (*o.c.*).

The notochord (*n.c.*) has now become very much diminished (figs. 3–5), and is bridged over by the investing mass (*i.v.*) behind. It has *relatively* retreated in front, leaving an open fissure, the "posterior basicranial fontanelle."

Having described the face and skull from lateral and horizontal views, and from a longitudinally vertical section, I must describe the transversely vertical sections. This will involve some repetition, but a much more perfect idea will be obtained of the relation and thickness of the parts; the reader must keep the other views before him if he would understand the numerous sections.

Two things will be noticed at once in these views, namely the thickness of the "derm," and the large quantity of subcutaneous *gelatinous* stroma in which the more solid organs are enveloped.

The first transverse slice of the face (Plate V. fig. 6) has already been described; the second (Plate VI. fig. 1) shows the form of the oral opening (*m*), and exposes the nasal tube (*ol.*) both near the skin and also near the trabeculæ, which are shown in their relation to the fore part of the palatal region. A pair of papillæ are seen lower down, projecting into the mouth, the floor of which is a deep angular fossa.

The clubbed ends of the Meckelian rods (*mk.*) are cut through on each side; and a vertical section has been made through the lower labials (*l.l.*), the form and relations of which are well shown; they are invested below with the lower dentigerous plate (*l.d.g.*); and on each side the smaller teeth of the labial rugæ (*lp.*) are also shown.

This is the back view of a solid slice of the face; and the next (Plate VI. fig. 2) is the *front* view of another slice made further backwards. This section is still in front of the "trabecular commissure," and lays open a reduplication of the nasal sac (*ol.*).

Here the Meckelian rods (*mk.*) are completely displayed up to their condyloid fossa, and the lower labials (*l.l.*) are seen, as also the lower dentigerous plate and labial dentigerous rugæ (*l.dg.*, *lp.*).

The *posterior* face of the next slice (fig. 3) displays the trabeculæ in the ethmoidal region (*eth.*) immediately behind the commissure; the rhinencephala (C 1^a) are cut through.

We now see MECKEL'S cartilages (*mk.*) and the lower labials (*l.l.*) from behind; the quadrate, however, is shown in section. Here we get a perfect idea of the relation of the first two visceral arches, for the section has been made through the secondary band or pterygo-palatine bar (*pa.*, *pg.*).

The remarkable "orbitar" process is shown embracing the temporal muscle (*or.p.*, *t.m.*); and this muscle is almost entirely surrounded by cartilage; for there is a "pterygo-palatine process" (well seen from above in Plate V. fig. 3, *p.p.p.*), the meaning of which is not very evident, although from its relation to the lateral parts of the ethmoidal cartilage it is very probably a rudimentary "pars plana." As the section is close behind the "internal nares," a fold is seen in the mucous membrane on each side of the palate.

The next section (Plate VI. fig. 4) has been made through the hemispheres ($C1^b$), the eyes (*e.*), and the middle of the temporal muscle (*t.m.*). The manner in which the barge-shaped skull of the Frog is formed, anteriorly, is well seen in this section; for here the elongated pituitary space is bridged across; a concavo-convex plate of cartilage, representing the presphenoid (*p.s.*), and underlain by the "parasphenoid" (*pa.s.*), occupies the mid line.

Laterally, also, the trabeculæ (*tr.*) are growing gently upwards around the membranous brain-sac, and will soon form the cartilage of the orbito-sphenoidal region.

The suspensorium has been cut through close in front of the fossa for the stylo-hyal condyle, it is therefore the quadrate (*qu.*); the hinder part of the orbitar process is also seen (*or.p.*). This section has been made through the hyoid cornu (*hy.*), close in front of the condyle; the manner in which these secondary (segmented) bars imitate those immediately in front of them is well shown (compare fig. 3, *mk.*, and fig. 4, *hy.*). Here, however, we encounter an additional element not seen in the mandibular arch, namely a basal azygous piece; this is the basihyal (*b.h.*); it is pear-shaped in section, the thick part below, and is composed of soft cartilage. This is a *back* view, and would seem to show that the hyoid cornua (*hy.*) were extremely solid.

A front view (fig. 4^a, *hy.*, *b.h.*) corrects this, however, and the plate is seen to be curiously sinuous and thin at parts. I shall not wait to describe the muscles which arise from the arch and which have partly been figured.

The next section (fig. 5) is the *front* view of a solid slice made through the "mesencephalon" (C 2) and infundibulum (*inf.*); the razor passed close in front of the auditory capsules (*pro.*), *behind* the basibranchial (see Plate V. fig. 5, *b.br.*), and in *front* of the pericardium (*p.c.d.*).

Here is the upper part (the so-called origin) of the temporal muscle (*t.m.*), the fibres being attached to the anterior face of the auditory capsule and to the upper surface of the "metapterygoid connective."

The Gasserian ganglion (5) is seen lying on that band close inside the anterior face of the auditory capsule; its fibres pass *over* and in front of this mandibular *root*. The cartilage of the ear-capsule (*pro.*) is seen to be ascending on each side of the cranial sac; *behind* this view it is the pterotic, in *front* it is alisphenoidal. With regard to the

ract of cartilage running from side to side, this view is explained by part of the lower view (fig. 8); it shows the connexion of the "mandibular pier" with anterior part of the investing mass (*i.v.*).

The thick outer edge of this bar is the opercular process of the hyo-mandibular; mesiad of that the rest of the nearly horizontal part is the topmost portion of the primary mandibular rod, its metapterygoid region. The rapidly ascending part is the "metapterygoid connective," the *secondary growth* which binds this arch to the investing mass *behind* and to the "trabecular connective" (fig. 8, *tr.c.*) in *front*. The cartilage on which the infundibulum (*inf.*) rests is the *connective* or secondary portion of the investing mass (fig. 8, *i.v.c.*); this is underlain by the broadening or basitemporal portion of the "parasphenoid" (*p.a.*).

It must ever be kept in mind that a large triradiate tract of cartilage here, on each side, is entirely secondary in its growth, only appearing in the third stage; the branches of this tract may therefore be called the "connectives" of the trabeculæ, metapterygoids, and investing mass (fig. 8, *tr.c.*, *m.p.g.c.*, *i.v.c.*).

Here the mouth (*m.*) is at its widest part, and looking backwards at this *front* view we see the pericardium (*p.c.d.*) overlain by the "hypobranchial" plates (*h.br.*), which nearly meet. Here also parts of the first and second branchial arches (*br.* 1, *br.* 2) are cut through, and the rich crop of *inner* branchial tufts are shown; the branchial cavity is only partly shown, the lower part of the section not being drawn, and the external branchial tufts are cut away.

The next view (Plate V. fig. 7) is a little *behind* the last; it is somewhat oblique, and catches only a small part of the infundibulum (*inf.*) beneath the "mesencephalon" (C 2); the "foramina ovalia" and the right Gasserian ganglion (5) are shown in this back view of a solid section. The fore end of the "investing mass" (*iv.*) is here seen distinct from the auditory capsule on the left side, and from the "metapterygoid connective" on the right: on this side the auditory sac (*pro.*) is slightly laid open; but on the left we see the ampulla and part of the arch of the "anterior semicircular canal" (*a.s.c.*): here the *angular* height of the capsule is seen to be due to the arch of this canal. The thin, free, hypobranchial horns (figs. 5 & 7, *h.br.*) are seen investing the pericardial roof (*pcd.*); this sac is laid open, and the heart is seen *in situ*. Part of each of the four branchial arches (*br.* 1–4) appears in this section, and their convergence from before backwards is illustrated (see also fig. 5); the external branchial tufts (*e.br.*) are indicated by outlines: the whole of this section is not shown below.

The next view (Plate VI. fig. 6) is the *front* of a solid piece still further back; it is slightly oblique, and thus the right side of the mouth is seen to be becoming narrower. There is a perfect roof of cartilage over the massive "medulla oblongata" (*s.o.*, *m.ob.*); this is the superoccipital; for the occipital ring is very obliquely placed, looking forwards. The membranous cranium, now serving as perichondrium to the enclosing cartilage, is some distance from the cerebral mass, the space being filled with a very watery tissue. softer than the subcutaneous stroma.

All trace of the distinctness of the auditory capsules and the investing mass is now lost, and the occipital ring is everywhere confluent with these organs. On the left side there is a section of the ampulla of the horizontal canal, and the crown of its arch is also shown (*h.sc.*). Above and mesiad of this is part of the posterior canal (*p.sc.*). The left "fenestra ovalis" is laid open (*fs.o.*), and the right stapes (*st.*) shown in section.

The form of the "tegmen tympani" (*t.ty.*) is shown on the left side of the figure (above the stapes). The walls of the periotic capsule are very variable in thickness, the inner being thinnest; this is only apparently separate from the "investing mass," the section being made through the "meatus internus."

The oval section of the gelatinous notochord is seen to be invested below by the basal cartilage, but it is naked above (see also fig. 6[a], *nc.*, *b.o.*). A posterior view of the same slice (fig. 7) passes through the epiotic region (*ep.*), and lays bare the hinder part of the arch and the ampulla of the posterior canal (*p.sc.*). The obliquity of the occipital ring of cartilage is such that this section, although passing through the periotic mass, is yet *behind* the occipital roof.

The section has been made in front of the interspace for the "vagus" nerve; but the glosso-pharyngeal (8[a]) is seen to pierce the cartilage at the junction of the auditory capsule with the investing mass. This cartilage now entirely surrounds the notochord (see also fig. 7[a], *b.o.*, *n.c.*). The oral cavity (*m.*) is now much narrowed, and is becoming the œsophagus.

Fifth Stage.—Tadpoles with hind legs reaching to end of tail.

Before the fore legs of the larval Frog appear, and whilst the tail is still of undiminished size, great changes take place in the actual and relative size of the parts composing the skull and face.

The cells of the upper and lower labials (Plate VII. fig. 1, *u.l.*, *l.l.*) have begun to pass, by proliferation, into fibrous tissue; the horns of the trabeculæ cranii (*c.tr.*) are becoming more expanded and adze-shaped, and their inner edges are growing mesiad to form the nasal floor. The upturned superior edge of the interorbital portion of the trabeculæ is growing further upwards into the side of the membranous brain-sac; and the same upgrowth of cartilage has appeared over the auditory sac in the pterotic region (*p.t.o.*).

The trabeculæ (Plate VII. fig. 2, *tr.c.*) are still distinct to within a short distance of the commissure, which is now more extended antero-posteriorly (fig. 1, *eth.*).

The nasal sacs (*ol.*), approaching the mid line, lie on the trabeculæ; at present they are membranous, but they already have a *floor* formed by the trabecular horns, and a *roof* will soon be developed.

Anteriorly the nasal sacs are only separated by soft, nascent tissue; but there is a dividing wall behind; this is the rudimentary "septum nasi," which has arisen like a rostral outgrowth from the middle of the rudimentary ethmoid (Plate VII. fig. 2, and Plate VI. figs. 9, 10, *eth.*, *s.n.*).

The relations of these parts are shown in Plate VII. fig. 1 from the side, in fig. 2 from above, and in Plate VI. figs. 9 & 10 sectionally, vertically transverse.

The broad ethmoidal wall, which had as a foundation the trabecular commissure, when seen from the front aspect (Plate VI. fig. 9), has this rudimentary "septum nasi" (*s.n.*) projecting directly from it.

The cartilage that lies between the olfactory foramina (1) answers to the "lamina perpendicularis" of the ethmoid; it is, however, very short and very thick. The lateral ethmoidal region, partly cut away in the figure (Plate VI. fig. 9) is curved gently round on each side, and passes continuously into the lateral skull-wall (Plate VII. fig. 1, *o.s.*, fig. 2, *eth.*). It also passes into the palatal bar (*pa.*), at the root of which is the curious pterygo-palatine process (Plate VII. fig. 2, *p.p.p.*), a part well seen in Tadpoles of *Rana pipiens* and *Pseudis paradoxa*; in them it has all the appearance of being a rudimentary "antorbital" or "pars plana."

A section made in front of the ethmoid, but through the growing "septum" (Plate VI. fig. 10, *s.n.*), shows the relation of the olfactory passages to the parts that are walling them in.

The trabecular horns (*tr.c.*) form a convex floor; these are connected by fibrous tissue to the partition; and a dotted line (*al.s.*) shows where the cartilaginous roof or aliseptal plates will appear. At present they are mere indifferent tissue; but soon, as the septum elongates, a laminar outgrowth of hyaline cartilage will proceed *forwards* from the anterior edge of the transverse ethmoidal wall, and *outwards* from the crest of the septum. The optic foramen (Plate VII. fig. 1, 2) and the "foramen ovale" (5) are now enclosed in cartilage; in the last stage they were passages in the membranous cranium.

The pterygo-palatine bar, which in the former stage was a mere bridge connecting the quadrate region with the prefrontal, has now acquired a length equal to that of the suspensorium; so that the subocular arch is now V-shaped.

Meantime the anterior crus of this arch has begun to lose its straight form, and to be bent a little forwards at its upper third (*pa.*); here it is partly constricted off: this transverse segmentation, although imperfect, marks where the pterygoid (*pg.*) begins; this latter part is broader than the palatine.

The metapterygoid root (*m.pg.c.*) of the mandibular arch is fast diminishing, and has begun to cling to the fore part of the ear-capsule, with which it will eventually coalesce. The temporal muscle (*t.m.*, indicated by dotted lines) still passes downwards and forwards, but much more *downwards* than before, on the inside of the orbitar process (*or.p.*); but this outgrowth is now much less, has become four-square, and is now *beneath* and *behind* the eye, not beneath and in front; it has been brought very near to the auditory sacs.

There is now a very sharp distinction between the supra- and the infrahyomandibular (*s.h.m.*, *i.h.m.*); and the upper region has become really as well as relatively less. The lower portion has grown into an ear-shaped lobe, scooped on the outside and convex within: the base of this lobe is now only half its former distance from the periotic mass.

This relative change of the infrahyomandibular has affected the hyoid cornu (*hy.*), which has been carried, skullwards, with it.

Every one is familiar with the manner in which the lower part of the first cleft is embraced entirely by the second arch in the Osseous Fishes, the *symplectic* continuation of the hyomandibular bounding this space in front, as the hyoid cornu does behind.

In this stage of the Tadpole there is a partial separation of the symplectic bar, the cartilage-cells receding from each other fore-and-aft, leaving a considerable space of clear intercellular substance entirely free from cells (Plate VII. fig. 1, *sy.*).

The segment thus partially severed is club-shaped below, and pointed above, where it reaches the cupped surface for the hyoid cornu.

Although there is a rudiment of the symplectic in the third stage (Plate IV. fig. 7, the lowest end of *h.m.*), yet I cannot find that it has developed to any extent, except to coalesce completely with the quadrate, even in the fourth stage (Plate V. fig. 1, *sy.*).

The free rays which are developed from the first and second poststomal arches to form the mandibular and hyoid arches are now much elongated. In the last stage MECKEL'S cartilage was not half so long as its suspensorium; they are now equal (Plate VII. fig. 1, *qu.*, *mk.*). The free rod itself also has undergone a great change of form, being thickest now at its articular end; its fore part was very clubbed in the fourth stage. The hyoid cornu (*hy.*) is not only twice as near the auditory capsule, but it is much narrower, and has coalesced with the *soft* basihyal (*b.h*).

A deposit of calcareous matter begins now in the roof of the skull in the outer layer of the membranous brain-sac; other tracts appear in the face. These will have a definite form in the next stage, and will then be described.

Sixth Stage.—Tadpoles with fore legs, and with tail reduced to one-half its former size.

The last moulting of the epidermis was accompanied by so much absorption of the opercular folds as to set the fore limbs free; and now, the lungs being in function, the branchial system is fast diminishing. In this stage I will begin with that which is most apparent. We see now a pair of membrane-bones on each side of the skull-roof, closing in the great "fontanelle;" these are the frontals and parietals (Plate VII. fig. 3, *f.*, *p.*); nor are these the only membrane-bones which have been added to the "parasphenoid;" for in the upper nasal region there is a pair of small semilunar patches, the nasals (*n.*); in front the upper labial cartilages have disappeared, and in their place, but not by immediate ossification of them, the premaxillaries (*p.mx.*) have arisen; these are short thick bars of fibrous bone, having an ascending process, the nasal process, above, whilst on the palatal region each bone is knobbed close to its fellow of the other side; this knob becomes the palatal process.

Behind the premaxillary, and below and outside the palatal bar, is a calcified web, of a styloid shape, and having an ascending process behind its first third; this is the maxillary (*m.x.*).

Within the nostril there is a small point of bone, the septo-maxillary (*s.mx.*); this will be better seen in the next stage.

The vomers have not yet appeared, but delicate spiculæ of bone are found in the fibrous stroma, lining the inner face of the suspensorium and of the pterygoid bars; these will be best described in the next stage; and when they are better developed they become the pterygoids, metapterygoids, and infrahyomandibulars.

On the outer face of the suspensorium a much better-developed plate of bone has arisen (*t.*); it is sigmoid in shape, broadish above, and pointed below; this resembles the "præoperculum," but it does not free itself from the common root of the mandibular and hyoid arches to grow backwards; it acquires a transverse "supratemporal" portion, a separate bone in *Triton*, and thus has a right to the title "squamosal" or "temporal."

On MEKCEL's cartilage, also, membrane-bones have appeared, these are the dentary (*d.*) and the articular (*ar.*); at this stage there is no difference in the character of these bones; they are thin fenestrate plates lying in contact with the cartilage, or at least having only a slight layer of fibrous stroma between them and the rod.

The development of the cranium has been very rapid. The trabecular "horns" have united in front to form the *lower* nasal alæ; the lateral cranial walls, forming the ethmoidal, orbito-sphenoidal, and alisphenoidal regions, are now well grown (Plate VII. fig. 3, *eth.*, *o.s.*, *al.s.*).

The cranial roof, also, under the parietals, and beneath them and the frontals, is now roofed-in with an extension forwards of the pterotic cartilage (*pt.o.*): here we see the posterior half of the great upper *fontanelle* covered in.

The suborbital space or fenestra (*s.o.f.*) is greatly changed in form by a reversal of the relative length of the crura which bound it (compare Plate V. figs. 1–4 with Plate VII. fig. 3); it is also much longer from end to end. Anteriorly, the palatal region (*pa.*) is nearly segmented off by a transverse cleft from the pterygoid bar; and in front of this joint it has developed a knee-like enlargement. The pterygoid bar (*pg.*) is a very uniform flat band until it reaches the suspensorium, which part is rapidly undergoing metamorphosis. The metapterygoid root has now coalesced with the fore part of the periotic capsule. The "orbitar process" (*or.p.*) is now reduced to a gently convexo-concave expansion of the fore part of the suspensorium; below, it belongs to common ground between the metapterygoid and quadrate regions, and is almost entirely *behind* the temporal muscle.

These parts are well seen from the inner side (Plate VII. fig. 4). The lower metapterygoid region as it passes into the quadrate (*m.pg.*, *qu.*) is of considerable thickness; but the starved orbitar process grows out of it in front like a thin shell, its concave face inwards. But behind the middle thick part there is another shell-like plate of cartilage (*i.hm.*); this also is thin, but it has its *convex* surface on the *inner* side, and is deeply scooped externally; this is the "infrahyomandibular," between which and the metapterygoid band there is still the remains of the upper part of the first *cleft*.

The "suprahyomandibular" (fig. 3, *s.hm.*) has become a free plate of cartilage of a trifoliate form; and the hyoidean cornu, besides being longer and narrower, is acquiring a greater projection of the supero-posterior angle, whilst the condyloid part is becoming more loosely attached to the glenoid facet on the postero-internal face of the suspensorium.

But the most remarkable changes have been suffered by the branchial arches, as may be seen by comparing their condition in the fourth stage with what is depicted in this (Plate V. figs. 1 & 5, and Plate VII. fig. 3). The first and fourth arches are no longer carpelliform pouches, but are narrow bands, and all four of these bars are much attenuated. The continuous "pharyngo-branchial" tract now sends upwards six spurs or lobes; the third and fourth arches are quite free below, and the "hypobranchial" base of the first and second arches has become narrow and thick (*h.br.*). Moreover the hyoid cornua, the basihyal, the basibranchial, and the hypobranchial bars have all coalesced. These are some of the most important modifications of the branchial system preparatory to its thorough abortion.

Seventh Stage.—Young Frogs with tails reduced to a mere stump.

The metamorphic changes which for the past few days had gone on with intense activity have now produced most interesting results, many of the morphological conditions being now entirely removed from what can be seen in ichthyic types.

Many of these changes were fairly begun in the last stage; but in this further condition they make a very near approach to what is essential to the adult Frog only, or principally, being modified in the old Frog by increased size.

Beginning, as in the last stage, with the osseous plates that surround the cartilaginous basis, we find that the parietals and frontals (Plate VII. fig. 5, and Plate VIII. fig. 1, *p.f.*) have begun to coalesce; the frontals do not quite roof-in the great fontanelle (*fo.*) in front; and the parietals seem entirely to overlie a cartilaginous roof; the cartilage, however, is imperfect beneath the centre of each bone. The nasals (*n.*) are now much longer semilunar plates; and the premaxillaries and maxillaries have equally advanced: the former (*pm.x.*) have now a well-developed nasal and palatal process; and the latter (*mx.*) is forming its ascending or facial plate. Above the plate is seen a small shell-like plate of bone which helps to form a floor to the nostril (Plate VII. fig. 5, and Plate VIII. fig. 1, *s.mx*); this is the so-called turbinal, my "septo-maxillary."

In the Frog this bone is severed from the vomer by the subnasal cartilaginous plate (Plate VII. fig. 6, *s.n.l*). In the adult the "septo-maxillary" finds its way nearly through the nasal passage; but in the young Frog it is best seen from the outside. The squamosal (Plate VII. fig. 5, *sq.*) begins to take on its typical character, the upper part having acquired more of a transverse direction; it is also much thicker.

The upper or supratemporal region of this bone is accurately moulded upon the shell-like remnant of the "orbitar process," its fore edge overlapping the margin of the cartilage (Plate VII. fig. 5, *sq.*, *or.p*). The descending process binds down upon the suspensorium.

The bony plates which arise in the perichondrium on the *inside* of the suspensorium and pterygoid bar can now be well seen. In the side view (Plate VII. fig. 5, *p.g.*) the pterygoid can be seen mounting above the edge of the bar; but it is best seen from within (Plate VIII. figs. 2 & 4, *p.g.*); it is a very delicate tract, much like a few threads of the skeleton of a Siliceous Sponge (fig. 4).

In front the bony pterygoid is a mere needle; but it expands at the root of the bar, then contracts, and expands once more above. The upper portion is semidistinct from the lower; it is quite distinct afterwards. Behind the metapterygoid bar another ectosteal plate has appeared, taking the form of and closely embracing the inner or convex face of the "infrahyomandibular;" this plate of bone (Plate VIII. figs. 2 & 4, *i.hm.*) answers to the inner face of the lower half of the ichthyic hyomandibular (see HUXLEY's Elem. p. 176, fig. 71, *HM.*). At present there is no bony palatal on the anterior part of the subocular arch; but at its keystone there is a small style (Plate VIII. figs. 1, 2, 3, 4, *qu.*); this lies on the outside of the quadrate condyle, and projects forward; it is the ectosteal plate of the quadrate sending forward a quadrato-jugal process.

Two other bones have now appeared on each side, embracing the fifth and the eighth nerves: the foremost of them is the prootic (*pro.*); and the other is the exoccipital (*e.o.*). These patches lie principally on the underside (Plate VIII. figs. 2 & 4, *pro.*, *eo.*), but they are also lateral (Plate VII. fig. 5, *pro.*, *eo.*).

They do not commence in the perichondrium, but in the superficial cells of the cartilaginous cranium ("superficial endostosis"); but, unlike endosteal tracts hereafter developed, they ossify the cartilage throughout. Each tract is sickle-shaped; the prootic commences external to the nerve outlet, soon to embrace it; the exoccipital begins on the inner side. Now also can be seen a pair of fibrous bones, which never, I believe, graft themselves on cartilage or take on an ectosteal character; these are the "vomers" (Plate VIII. fig. 2, *v.*). They are, at this stage, little spicular radiating tracts of ossified fibre, lying somewhat mesiad of the inner nares. The great basicranial splint, the parasphenoid (Plate VIII. fig. 2, *pa.s.*), has now become cross-shaped by acquiring *basitemporal wings*; in front it reaches to between the palatal roots in the prefrontal region; behind it nearly reaches to the end of the basioccipital region; and on each side its wings almost touch the edge of the "fenestra ovalis." This bone is convex and subcarinate below, and scooped above, where it undergirds the basis cranii. The dentary and the articulare are both easily separable from the cartilage (Plate VII. figs. 5, *d.*, *ar.*, *m.k.*); they are fast increasing in size and density.

The relation of the membrane-bones to the cartilaginous cranium is well seen in sections. The parasphenoid is seen as cut through at various points in Plate VII. figs. 7–10, *pa.s.*, the frontals in figs. 7–9, *f.*, the parietals in fig. 10, *f.*, the nasals and vomers in fig. 6, *n.*, *v.* They are so delicate at first, and so closely in contact with the face of the cartilage, that here at least the parosteal tracts have nothing to distinguish them from the ectosteal. From the rudimentary condition of the bony tracts it results that as yet the

zygoma is imperfect, a tract of fibrous tissue tying the minute quadrate style to the extremity of the maxillary (Plates VII. & VIII., *mx.*, *qu.*).

If the relations of the maxillaries to the endoskeleton be considered (Plate VII. & VIII., *m.x.*), it will be seen that they are rather applied as splints to the nasal capsule than to the pterygo-palatine bar; this relation to the nasal sacs is still more evident in the pre- and septo-maxillaries.

But even the maxillary itself is applied persistently as a splint, in front, to the posterior spur of the bifid "snout-cartilage." Since the last stage but one (fifth) the nasal region has ungone a remarkable series of changes, and these not easily understood. I have, however, studied this metamorphosis very carefully, both in the small common species and in other and larger types. The condition of things in the fifth stage must be kept before the eye, and then the various illustrations in this (seventh) can be read off.

The more enlarged figure of the anterior half of the primordial skull (Plate VIII. fig. 11) shows the appearance of the nasal labyrinth from above, after the removal of the bony plates. Here, this part of the skull shows that the trabeculæ have become nearly lost in the general cartilaginous growth, and that an inturned lamina from the lateral walls is continuous with a growth from the posterior edge of the ethmoid (*eth.*). But if this figure be compared with its counterpart of the fifth stage (Plate VII. fig. 2), it will be seen to what an extent cartilage has been developed around the nasal sacs. The septum nasi (*s.n.*) has now grown to the front of the face; and from its right and left edges above, and also from the anterior edge of the transverse ethmoidal wall, there has been developed a continuous *roof* of cartilage to the nasal sacs. Sectional views explain this; for the dotted line of the fifth stage (Plate VI. fig. 10, *al.s*) is now an elegant arch of cartilage (Plate VII. fig. 6, *al.s.*).

But the septum is not only continuous with the *roof*; a *floor* has been formed by the thinning out of the trabeculæ and *their coalescence with the inferior edge of the septum* (Plate VII. fig. 6, *tr.*, *s.n.*). In this latter figure the septum is thickened below its middle; this is where the septal outgrowths (to be described in the last stage) terminate posteriorly.

The ends of the trabecular horns can just be traced in this more advanced stage as a pair of emarginate outgrowths, "snout-cartilages" (Plate VII. fig. 11, *sn.c.*), each projecting outwards from the front angles.

The position of these processes is at the upper surface, although in the former stages the trabeculæ turned downwards (Plate VII. fig. 1, *tr.*); but they have changed their direction, as may be seen in the palatal figure (Plate VIII. fig. 2, *s.n.l.*), where the subnasal laminæ, or outspread trabecular horns, are seen to turn upwards behind the premaxillaries (*p.mx*). Each snout-cartilage forms a valvular process over the external nostril (*v.e.n*) in front of which it lies. But the external nostril (Plate VII. fig. 11, *e.n.*) is largely embraced by a peculiar development of the laminar roof. On the inner edge of the opening the aliseptal plate (*al.s.*) ends free; that free edge is then curled

round the greater part of the opening behind; and then the cartilage ends in a broad process which abuts against the snout-cartilage (*sn.c.*). If we now consider the manner in which the *roof* grows down into the *side wall*, it will be observed that there is a large swelling pouch on each side, connected by a broad tract with the aliseptal plate (*al.s.*), but sharply distinct from the aliethmoid (*al.e.*). Indeed, close to the ethmoid this pouch seems to be undergoing dehiscence, so as to separate the side-wall from the roof. These lateral pouches are the "alinasal" cartilages (*al.n.*); they terminate on the underside by curving in a falcate manner round the internal nostril (Plate VIII. fig. 2, *i.n.*).

The falcate process nearly applies itself to the elegant semicircular space between the outer edge of the trabecula, and the anterior margin of the palatine bar (*s.n.l.*, *pa.*); this is the normal position of the internal nostril (see Plate IV. fig. 10, and Plate V. fig. 4, *ol.*). A triangular fibrous valve is seen within the internal opening (Plate VIII. fig. 2, *i.n.*). The passage itself is very correctly circular. The alinasal pouches are very large (relatively) at this stage.

Mesiad of the inner nostril there is a delicate tract of ossified fibrous tissue, spreading like a patch of small crystals; this is the "vomer," (*v.*) related by its outer edge to the inner nostril (*i.n.*).

The peculiar position of the symmetrical vomers and the extreme distance of the inner nares, depends upon their relation to the "trabecular horns." If these coalesce to form merely the thickened base of the septum nasi, then the vomer is generally azygous, and the palatal openings of the nasal labyrinth are merely separated by the thickness of the septal base and the width of the azygous vomer. In the Frog, however, we have not merely the primary width of the diverging trabecular horns, but also their after-spreading, which gives still further lateral extension to the basinasal region (Plate VIII. fig. 2). The olfactory crura (Plate VII. fig. 11, 1) are now much nearer together; for the "perpendicular ethmoid" (*p.e.*) has grown into a definite (posterior) part of the general internasal septum, whereas formerly (Plate VI. fig. 9, 1) it was merely so much cartilage as lay between the olfactory openings in the low, thick, transverse cranio-facial wall.

The sides of this *barge-shaped* skull are now well chondrified, each ascending laminæ ending in a neat selvedge above (Plate VIII. fig. 1); the *regions* of this wall are the "ethmoid," "orbito-sphenoid," and "alisphenoid" (Plate VII. fig. 5, *eth.*, *o.s.*, *al.s.*); the last of these has already received some bony deposit from the prootic centre (*pro.*).

Sections made transversely through the skull show well the relation of the various parts. The first, through the anterior part of the "hemisphere" (Plate VII. fig. 7), shows the cartilage of the ethmoidal region where it begins to be orbito-sphenoidal; a bulging floor connects the trabecular thickenings; and beneath this is the fore end of the parasphenoid (*pa.s.*); the ascending plate is first thin, then thick, and is then bevelled before it terminates above.

The "fontanelle" has here the tips of the frontals (*f.*) partly enclosing it: on the sides

part of each eyeball (*e.*) is shown. The next section (fig. 8) made through the posterior part of the hemispheres shows a flatter and thicker floor, and diminished trabecular and upper thickenings; the frontals (*f.*) nearly cover the top; and the parasphenoid (*pa.s.*) largely underlies the floor.

Another section (fig. 9) passes exactly through the middle of the posterior sphenoidal region, cutting through the optic lobes and infundibulum (C 2, *inf.*); and here we see that the "infundibulum" rests upon a cartilaginous floor, the original hypophysial space being filled-in by a long commissure. Here also we see an ichthyic condition—namely, perfectly continuous cartilage, from base to top, overlain by the bony matter (*f.p.*) of the "coronal" synostosis. Here, again, the trabecular thickenings (*tr.*) are evident; but on the whole the cartilage is thin.

If we look at the sectional view of the Tadpole's skull (Plate V. fig. 2), *below* and *behind* the infundibulum (*inf.*) the skull floor is imperfect. This is a remnant, partly of the original pituitary space, and partly of the fissure between the moieties of the investing mass in which the notochord terminated (Plate V. figs. 3 & 4).

In this stage it has expanded (Plate IX. fig. 10, *p.b.c.f.*) into the "posterior basicranial fontanelle" of RATHKE; it is only transitory in the Frog.

The "stern" of this barge-shaped skull keeps its "deck" now to the end; it becomes extremely wide, having the ear-capsules impacted in its sides. In the sectional view (Plate VII. fig. 10) the thin cartilaginous roof, cut through where the supraoccipital region passes into the "pterotic," is seen to pass directly into the ear-capsule, close above the arch of the posterior canal (*p.s.c.*). The apparent want of continuity in the rest of the section arises from the fact that it has been made through the "fenestra ovalis" (*fs.o.*) and the "meatus internus" (7^b). The basitemporal wings of the parasphenoid (*pa.s.*) underlie most of the width of the broad basicranial floor in which the notochord (*n.c.*) lies imbedded; it is here free above, and, below, lies on a cartilaginous commissure. Part of the stapedial plate (*st.*) has been left to this section, which also shows the posterior half of the horizontal canal (*h.s.c.*), the vestibule (*v.b.*), and the huge "medulla oblongata (*m.ob*):" large as this is, it does not fill the cranial cavity. The two remnants of the notochordal fissure are seen from above in Plate IX. fig. 1 more highly magnified (30 diam.). The gelatinous remnant of the notochord (*n.c.*) is seen in the posterior space, which is a deep groove; the anterior space is the triangular "posterior basicranial fontanelle."

The changes that have taken place in the subocular or palato-quadrate arch are very instructive.

The anterior crus is becoming very long in proportion to the posterior, and the palatine portion (*pa.*) is very nearly constricted off from the pterygoid (*p.g.*).

The small knuckle in front of the bend in the palatal has now become a spur (Plate VIII. figs. 1 & 2), so that there are now three regions to the palatal, namely the suspensorial or "suprapalatal," the "prepalatal," and the postpalatal (Plate VII. fig. 11, *e.pa.*, *pr.pa.*, and *pt.pa.*).

The pterygoid bar (*pg.*) is very uniform in size, but becomes broader as it passes into the suspensorium; the orbitar process of the metapterygoid (Plate VIII. figs. 3, 4, *or.p.*) is now overlapped by the temporal or squamosal (*sq.*); it is a mere semielliptical expansion, convex externally and concave within.

MECKEL's cartilage (*mk.*) has greatly increased in length, like the palato-pterygoid bar. The quadrato-symplectic half-cleft (*qu.sy*) is still quite visible, the infrahyomandibular (*i.hm.*) is turning its inner convex face forwards, which it will do more and more. Between its upper margin and that of the sessile "orbitar process" is seen the narrow metapterygoid connective (Plate VIII. figs. 3, 4, *m.pg.*), the upper part of which has now coalesced with the prootic region of the ear-capsule. Mesiad of the infrahyomandibular (Plate VIII. figs. 4, 7″), the "portio-dura" nerve escaping from the posterior angle of the "foramen ovale" passes in front of the periotic cartilage; it then runs downwards and forwards (Plate VII. fig 5, 7ᵃ) on the inside of the posterior margin of the suspensorium, and halfway down divides into a large branch, the "chorda tympani," for the mandible, and two lesser "hyoid branches"*.

For the sake of clearness of ideas we may now recapitulate the changes undergone by the first and second postoral bars. The mandibular pier is now scarcely two-thirds the length of the free mandibular ray or articulo-meckelian bar (Plate VII. fig. 5, *ar.*, *mk*); and the angle formed by the upper or suspensorial part with the basicranial axis is becoming almost a right angle: thus the distance between the two points of the trabecular and mandibular bars, which was at first so small as to be joined by a very narrow connective, are now united by the long pterygo-palatine rod. That part of the hyoid pier which first coalesced with the mandibular is still one with it; but the free upper head has now become segmented off as completely as the recurved lower half of the arch, namely the "stylo-cerato-hyal." There is therefore now a free "suprahyomandibular," an "infrahyomandibular" confluent with the mandibular pier, and sending down behind that bar a secondary *symplectic* selvedge. Finally, there is the proper hyoid cornu comprising the styloid and cerato-hyal regions, the latter coalesced now with its basal element.

The hyoid now becomes a long narrow bar, very unlike its earlier conditions, and becoming not only loosely attached to its original fossa, but also gaining continually a more basicranial position (Plate VII. fig. 5, *hy.*, *st.h.*). Towards its base the bar thickens and projects forwards as a hypohyal process (*hy. h*) before it is bent backwards to unite with the still soft basal element; the latter part is confluent with the basibranchial (*b.br.*). The upper part of the hyoid pier is that in which the metamorphic changes are most remarkable and of greatest interest.

The more highly magnified views (Plate VIII. figs. 3, 4) give the best idea of these parts. The "suprahyomandibular," losing all relation to the hyoid arch, becomes now part of the *middle ear*; and a cartilaginous "opercular" of a crescentic shape (indicated by a dotted outline in Plate VIII. fig. 3, *a.t.*) forms the outer part of the auditory appa-

* I have carefully studied the distribution of the "portio dura" in the Bull-frog.

ratus, namely the "annulus tympanicus." The *essential* element of the middle ear, the stapes (*st.*), was seen in the fourth stage; the condyles and opercular process of the hyomandibular are now being prepared to form an osseo-cartilaginous chain from the "membrana tympani" to the stapes. Under these conditions a new nomenclature will be required; and this will be made to depend upon the *stapedial* relationship of the chain, notwithstanding its different morphological origin*.

I shall now call the lobes of this trifoliate plate of cartilage as follows—namely, the antero-superior "suprastapedial," the postero-superior "medio-stapedial," and the freed opercular process "extrastapedial (*s.st.*, *m.st.*, *e.st.*).

The stapes (*st.*) sends no *stalk* forwards to meet the new elements, but they grow towards it; this will be seen in the next stage. It may be remarked, in passing, that the topmost part of the second postoral arch, in becoming *free* from the periotic capsule, is merely *reverting* to its primordial condition.

The branchial arches are beginning to be atrophied (Plate VII. fig. 5); the cells forming the pharyngo-branchial spurs (*p.br.*) have proliferated into connective tissue; the arches have become very slender; and the small hypobranchial spur on each side has grown into a solid diverging horn (*h.br.*); the whole basi-hyobranchial bar has enlarged into a flat plate of nearly even breadth.

Eighth Stage.—Young Frogs of the first early summer.

During the next six or eight weeks (that is, by the commencement of summer) the young Frog has become larger by one-half, has acquired much more of its *specific* character, is more elegant in form, and much less like the Toad than in the newly metamorphosed condition.

Similar modifications have taken place in the skeleton, some of which are of the highest morphological interest; this is especially the case with regard to the "middle ear."

The flat plate which was segmented from the top of the second poststomal arch has become a narrow ray with two terete branches as bifurcations, one large and the other small (compare Plate VII. fig. 12, *s.hm.*, with Plate VII. fig. 13, *e.st.*, *s.st.*, *m.st.*). The free antero-inferior bar is spatulate, flat within, rounded without, a little bent and dilated at the end, and serves, like the "manubrium mallei" of the Mammal, for the tension of the membrana tympani, the edges of which are attached to the almost circular cartilaginous opercular. This process, the extrastapedial, is not, however, the representative of the "malleus," but is the liberated "opercular pedicle." The antero-superior lobe (Plate VII. fig. 13, *s.st.*) has now grown into a delicate terete cartilaginous ray, directly continuous with the free descending extrastapedial, but much smaller, and occupying only *one-third* of its top where it passes into it; this is the "suprastapedial;" it is attached loosely by fibrous tissue to the fore part of the tegmen tympani at the junction

* These terms are partly from Professor HUXLEY's paper "On the Representatives of the Malleus and the Incus of the Mammalia in the other Vertebrata (Zool. Proc. May 27, 1869, pp. 391–407), and are partly terms suggested to me by him in conversation.

of the prootic and pterotic regions, and therefore still retains the relation of the anterior condyle of the hyo-mandibular.

The remaining two-thirds of the top of the "extrastapedial" is occupied by the commencement of a bony ray, which has been developed out of the postero-superior lobe; the posterior hyo-mandibular head (Plate VII. figs. 12 & 13, *m.st.*); this is the "mediostapedial" (*H.* & *P.*).

This *apparent* "columella auris" is a somewhat crooked rod of solid bone; it is bony up to the extrastapedial and nearly to its base, which, however, is never entirely ossified. Its larger posterior end is greatly bevelled towards the skull; and a little while since, it did terminate in a large snail's-foot-like expansion of cartilage, which was growing backwards towards the stapedial plate. This expansion, formed out of the posterior margin of the third lobe in the trifoliate "suprahyomandibular" (Plate VII. fig. 12, *s.hm.*), has become a perfectly distinct segment having a cordate form, the *apex* of which is directed backwards (Plate VII. fig. 13, *it.st.*); this is the "interstapedial" (*H.* & *P.*); it now slightly overlaps the elegant oval valve of the "fenestra ovalis"—the "stapes" (*st.*).

In the figure (Plate VII. fig. 13) the stapes is slightly dislocated from the fenestra to show how little this *periotic segment* has changed since it was evident for the first time (Plate V. figs. 1 & 4, *st.*).

In the Frog, at any rate, there is no tendency in the stapedial plate to send a *connective process* towards the metamorphosed parts of the top of the second poststomal arch.

In the last stage the dentary was still distinct from MECKEL'S cartilage in front (Plate VII. fig. 5, *d.*, *mk.*, & Plate VIII. fig. 5); there is now another stage of things.

In this more advanced condition the *chin-end* of MECKEL'S cartilage has been ossified into a "mento-meckelian" rod of bone (Plate VIII. fig. 6, *mk.*, *m.mk.*); this is the "lower intermaxillary rudiment" of REICHERT. This part is very short, and is strongly united by a fibrous ligament to its fellow. The dentary (*d.*) has already coalesced with it; but the ectosteal plate *did not form* it; it is a proper endosteal bone, rapidly ossifying through the cartilage, like the prootic and exoccipital. Originally MECKEL'S cartilage was much clubbed at its distal end; even *now* the enlargement is shown, as this terminal bony part is thicker at its end than subterminally.

The branchial arches have almost entirely disappeared (Plate X. fig. 1), and the skeletal parts of the tongue are now nearly like those of the adult (fig. 2). The "stylo-cerato-hyal" (*st.h.*, *ch.*) is now a narrow band of cartilage, equally attached by fibrous tissue to the suspensorium and to the opisthotic region. This band, still sending forwards a "hypohyal lobe" passing into a cartilaginous plate, belongs both to the basihyal and basibranchial, which is now entirely composed of hyaline cartilage. The whole structure is apron-shaped, the hyoidean cornu forming the upper strings. Between its upper and broadest part it has a concave *throat* edge; and below, it gives off two pairs of shorter strings. These are very different morphologically; for the first, which are feeble and unossified, are the remains of the first and second branchial arches, coalesced and almost absorbed, whilst the hinder pair are the "free hypobranchial horns" (see Plate V.

fig. 5, and Plate X. figs. 1 & 2, *h.br.*), which have now become solid bony rays, soft at the end and diverging at nearly a right angle to each other, so as to enclose the larynx (*l.x.*); and thus they form a pair of true "cornua majora ossis hyoidei."

Ninth Stage.—Frogs of the first autumn.

In the course of the first summer the Frog more than doubles its bulk; and these more developed individuals present several important changes. As the head at this stage is still, when partly macerated, a small, flat, and compressible object, it can be mounted in glycerine after being treated with caustic soda, and thus have all its parts brought into view under different magnifying powers; the figures (Plate VIII. figs. 7, 7^a, 8, 8^a) are taken from such a preparation.

The premaxillaries (figs. 7, 8, *p.mx.*) are now well developed in both their processes, nasal above and palatine below, and have become dentigerous.

The "septo-maxillary" (fig. 7, *s.mx*) is now better seen as a notched and grooved plate of bone lying on the floor of the nostril. The maxillaries (*mx.*) are now dentigerous, and their ascending facial plate and zygomatic process are now much advanced; the latter, however, does not yet reach the quadrato-jugal process of the quadrate (*q.*).

The nasals (*n.*) are much altered in shape: they have retained their curve; but the upper part is now a broad semioval shell, sending down the lower as a mere spur overlying the palatal suspensorium.

The "coronal suture" is still visible between the frontals and parietals (*f.*, *p.*); and the former are making a more perfect roof to the front end of the great fontanelle (*fo.*).

"Below, the vomers (fig. 8, *v.*) have acquired their trifoliate shape; the *teeth* easily peel away *from their stalk or posterior process;* the internal nostril lies between and external to the second and third lobes; *the first lobe grows forwards and nearly touches the premaxillary. The "parasphenoid" (fig. 8, pa.s.)* has become much denser; it is also broader in front. Laterally, we see that the *temporal bone (figs. 7 & 7^a, s.t., sq.) has become typical; for now the supratemporal longitudinal bar (s.t.) is well developed, clamping* the extended periotic capsule behind, and running forwards into the postorbital region. Its descending bar is now very characteristic, being stout and expanded below as a strong suspensorial splint. The quadrate ectosteal plate (*q.*), with its long quadrato-jugal process, is now more easily to be understood; for it has fairly grafted itself upon the quadrate cartilage, which the mere "quadrato-jugal" never does.

On the under or inner side (fig. 8) we see a new bony bar; it is *f*-shaped, delicate and narrow, and is immediately applied to the suspensorial part of the palatal cartilage; this is the Batrachian counterpart of the perfect bony sheath of the Fish's palatal; it does not yet touch the pterygoid. This latter ectosteal plate (*pg.*) is now well developed, although it does not quite embrace the cartilaginous rod (fig. 7^a & 8^a, *p.g.*). Its posterior descending process is very *Reptilian*; its metapterygoid process (*m.p.g.*) is now a mere spur of bone binding the front of the "infrahyomandibular," a bone which

retains its separateness, and is now seen as an ovoidal shell lining the lower two-thirds *within* of the *free* "infrahyomandibular lobe" (fig. 8ª, *i.h.m.*).

The "mento-meckelian" bone (figs. 7 & 8, *m.m.k.*) is now in a very characteristic state, with its dentary continuation (*d.*) along the *outside* of the Meckelian rod; the "articular ectostosis" (*ar.*) is now seen to be of great length, reaching on the *inside* nearly to the "mento-meckelian shaft."

The prootics and exoccipitals (*pro.*, *e.o.*) are much extended into the surrounding cartilage, and form complete rings to the fifth and eighth nerves.

A new osseous centre has also made its appearance, and one of extreme interest, as it is especially a *Batrachian* bone; this is the "ethmoid," or "os en ceinture." It commences where the ethmoidal cartilage forms a narrow transverse ledge to the front of the boat-like skull, the anterior boundary of the great fontanelle (fig. 7, *eth.*). This bone appears to have both its ectosteal and its endosteal portions formed quite synchronously; at present no trace of it can be seen from the sides or from below (fig. 8).

After the fashion of the "Placoidei," the cartilaginous skull has begun to acquire bony matter; but this *direct calcification* of the peripheral cells of the hyaline cartilage does not take the form of *tesseræ* as in them, nor of proper morphological regions as in the higher types; it does, however, tend to *crowd itself* where, in another type, a proper bone-territory would be established.

This initial allotment is best seen in the epiotic region (*ep.*), but it can be traced on the superoccipital (*s.o.*), on the pterotic, on the basisphenoidal, and on the upper and lower nasal regions; this "superficial endostosis" can now be beautifully seen, the bony deposit being in very small grains—in semilunes half round a cell, and in separate and connected rings.

Looking at the primordial skull as a whole, we see at this stage (compare figs. 7 & 8 with figs. 1 & 2) that the cranial cavity is less oblong (it is now broader behind), and that the more extended periotic masses are relatively shorter antero-posteriorly.

The "alinasal" folds (fig. 7, *al.n.*) have lost their articulation with the ethmoid; the "prepalatal" bar is more extended and acute; the whole subocular arch is now more elegantly arcuate, and extends much further backwards. But the suspensorium has changed most; for it now turns *backwards* instead of forwards, forming an obtuse angle with the basicranial line. The extension backwards of the quadrate condyle is accompanied by much elongation of the mandible (figs. 7 & 8, *m.k.*), and the gape is now very similar to what is seen in the Crocodile.

The upper part of the suspensorium is still very distinctly seen to be double (see Plate VIII. fig. 8ª, where the parts are somewhat drawn out for display, and the retraction, so to speak, of the quadrate angle, has brought the two heads of the suspensorium into the same vertical line), the infrahyomandibular (*i.hm.*) now lying directly *inside* the metapterygoid band (*m.pg.*). But it is the metapterygoid which has changed its place, by what I have already described, namely by fusion of the upper part of its root with the fore face of the auditory mass, whilst the "infrahyomandibular" has attained such

a position as that of a Teleostean Fish would if a similar segmentation were to take place—if the hyo-mandibular were cut away below the opercular knob, and the lower half of the bone brought up to the prootic region. The bony metapterygoid process is now a mere spur clamping the *front* of the infrahyomandibular (fig. 8^a), which has now grown into an elegant hatchet-shaped plate, the sharp bevelled edge of which glides over the smooth unossified part of the prootic at some distance outside the prootic and parasphenoidal ossifications. The deep fissure between the continuous metapterygoid and the free infrahyomandibular is the persistent upper half of the first cleft (see Plate VIII. fig. 8^a as compared with Plate VI. fig. 8, *mpg.*, *hm.* *). A specialization of a lower part of this primordial cleft is seen (shown in dotted outline) in the rounded angle between the infrahyomandibular and the descending part of the pterygoid (Plate VIII. fig. 8^a, *eu.*, *i.hm.*, *pg.*): this is the "Eustachian tube" or passage; and the remainder of the cleft has a tympanic function, and is bounded in front by the posterior margin of the suspensorium (*qu.*, *sy.*, *pg.*), and behind by the stylo-ceratohyal (*st.h.*, *c.h.*). I have already shown that the lower half of this cleft is partly occluded by the symplectic (*sy.*), which now has become completely fused with the quadrate. This stage is excellent for showing how the stylo-hyal effects its transit from the infrahyomandibular. First we saw this part articulating by a solid condyloid head with the middle of the very elongated suspensorium behind (Plate V.); then, as the upper half of the suspensorium became relatively shorter and shorter, thus carrying the stylo-hyal nearer and nearer to the skull, we also saw that the condyloid head became shorter, the joint-cavity lost, and the fibres of the capsular ligament elongated to form a lengthening fibrous band (Plate VII. figs. 1, 3, 5, *i.hm*, *st.h.*).

In this stage (Plate VIII. fig. 8^a) the "stylo-hyomandibular ligament" is a long wavy band of fibrous tissue, still connecting the stylo-hyal region with the fossa, which was originally a "glenoid" facet.

The two heads of the suspensorium are hidden on the outer side by the supratemporal portion of the temporal bone, and by the large semilunar cartilage, the "annulus tympanicus." This curved plate does not yet meet at its two ends; its inner or upper edge is thin, and its outer or lower has a thickened and strong selvedge (Plate VIII. fig. 8^a, *a.t.*): the "extrastapedial" bar (*e.st.*) is seen through the transparent "membrana tympani."

The structures of the middle ear are best seen from below (fig. 8^a). The "suprastapedial" (*s.st.*) is but a feeble ray, and is loosely attached to the periotic where it is projecting outward and passing into the metapterygoid root. The "medio-stapedial" (*m.st.*) has grown longer, relatively, and is very similar to the columelliform bar of the Lizard and Bird. The interstapedial (Plate VII. fig. 14, and Plate VIII. fig. 8^a, *it.st.*) is now broadest behind, but it still retains its heart-shaped outline and is rather thin.

* This space is very large in the adult Bull-frog, and is partly filled up by a membrane-bone, the counterpart of which I have not seen in any other type.

The "stapedial," proper (*st.*), seen edgewise in Plate VIII. fig. 8^a, is in shape not unlike a mussel-shell (its front portion is shown in Plate VII. fig. 14); it is still thin, and not much larger than the fenestra ovalis (*f.o.*). The figures show these parts from below (Plate VII. fig. 14 is a more highly magnified view of part of what is seen in Plate VIII. fig. 8^a); and it is seen that the bevelled end of the "medio-stapedial" overlaps the "inter-stapedial," which in its turn overlaps the "stapedial." This takes place to a greater extent than in the Seventh Stage, which was a large advance upon the stage before it; in the adult these segments are much more compactly arranged (Plate VIII. fig. 9).

Tenth Stage.—The adult Frog.

The structure of the skull in old specimens can now be better understood. I shall begin with what first strikes the eye, namely the outer bony plates.

These have now become fairly specialized into two varieties:—first, those which refuse to enter into combination with the endoskeleton, the parostoses; and secondly, those which either directly combine with the cartilage or *lie bare* upon it, having ossified the perichondrium which separated them. Yet in no other group is the difficulty of separating parostoses from ectostoses so great as in the Amphibia*.

The premaxillaries (Plate IX. figs. 1–3, *p.mx.*) have not merely increased in size since the last stage; for the nasal processes (*n.px.*) are longer and more decurved, and the palatal processes (fig. 2. *p.px.*) are delicate and sharp. The maxillaries (*mx.*) have acquired a better facial plate (fig. 3, *f.mx.*), which articulates largely with the nasals (*n.*); and the zygomatic process (*z.mx.*), most of which is edentate, reaches to within a short distance of the quadrate, and largely overlaps its quadrato-jugal process; this is a thoroughly Batrachian condition; I am not aware that any other type has this kind of zygoma. The "septo-maxillary" (*s.mx.*) is not easy to understand; it appears *above* as a little grain of bone jammed in between the nasal process of the premaxillary and the facial plate of the maxillary (figs. 1 & 3, *f.mx.*) in front of the outer nostril. *Below*, it is seen as a little curled spicule of bone in the inner nostril (Plate X. fig. 5, *s.mx.*). A transversely vertical section, seen from behind, shows that it is more than half a tube, lining the front of the nasal passage and sending down a curled process which can be seen from the palate (Plate X. fig. 4, *s.mx.*). Another section (Plate X. fig. 6, *s.mx.*) shows how it rests upon the nasal cartilage, outside, close behind the little valvular flap (*v.e.n.*) which partly occludes the external nostril. The nasal (Plate IX. figs. 1 & 3, *n.*) is largely

* In my figures I have given an ochreous tint to those bones which *become* endoskeletal, if even they were developed from fibrous tissue at first, and have left those bones uncoloured which, *as a rule*, continue distinct from the endoskeleton of the Vertebrata generally.

There are, however, *four* varieties of bone in the adult Frog's skull, namely:—first, the unmistakable parostoses, such as the maxillaries; second, the loosely applied ectosteal patches, which may or may not become grafted on to the cartilage; third, bone-territories which affect the whole thickness of the cartilage, and in which the ectostosis and endostosis is nearly synchronous, as the ethmoid, prootic, and exoccipital; fourth, superficial endostosis with no fibrous bone grafted upon it, as in the epiotic and superoccipital regions; this last is the dying out of the chondropterygeous type of endoskeletal bone.

developed; for there is here no preorbital or lacrymal to set bounds to it*; it does not nearly meet its fellow on the broad, flat nasal cartilage; just like the vomer below. The nasal is scooped where it lies above the external nostril, and then it becomes narrow where it joins the facial plate of the maxillary, and ends in a blunt-pointed preorbital process. The roof-bones of the skull (*f.p.*) have become thick, smooth, and bevelled towards the sagittal suture, which is persistent. Not so the coronal; it is all obliterated save a small notch on the sagittal edge. These compound bones *dip* towards each other considerably; they are scooped behind above the prootics (*pro.*), to which the parietal portion is joined. Each parietal region diverges from its fellow behind, exposing the cartilaginous roof. The thick outer edge overlaps the cranial wall, and, descending, is scooped to form an "orbital process of the frontal" (Plate IX. fig. 3, and Plate X. fig. 8, *f.*). The parietal region is higher than the frontal. The relation of the nasals and fronto-parietals to the endoskeleton is shown in the sectional views (Plate X. figs. 3, 7, 8, 9). The Frog is not much more liberally supplied with bony plates than the *Lepidosiren*; for we everywhere find large bones occupying a *double* territory, if not taking the place of *three*. The so-called "temporo-mastoid" (Plate IX. figs. 1 & 3, *t.*) would seem to combine the supratemporal and preopercular of the *Triton* or of the Siluroid or Ganoid Fish; it has also a process which represents the "postorbital" of the Lizard. The manner in which the squamosal or supratemporal bone applies itself to the protruded prootic is seen in the sectional view (Plate X. fig. 8, *pro.*, *t.*). The descending portion speaks strongly for the subdivision of membrane-bones into parosteal and ectosteal; for it is separated from the endoskeleton by a stratum of perichondrium, and, *overlapping* the quadrate bony plate, it is separated from it by a layer of periosteum (Plate IX. figs. 1 & 3, *t.*, *q.*).

The vomers (Plate IX. fig. 2, *v.*) are very elegant trifoliate plates of bone, wide apart, like the nasals, on account of the outspread form of the face, and correlative to the broad "subnasal laminæ." The middle leaf, which is emarginate, and the narrow falcate posterior leaf, together largely surround the internal nostril; the pointed front leaf nearly reaches the suture between the maxillary and intermaxillary; and the rounded *stalk*, which converges towards its fellow, is dentigerous. The "septo-maxillary" (Plate IX fig. 2, and Plate X. fig. 5) can just be seen between the middle and posterior leaflets of the vomer.

The "parasphenoid" has not lost its ichthyic dimensions (Plate IX. fig. 2, *pa.*); but it has become more elegant in form. The fore part is the "rostral region" (*r.*); the transverse bars which undergird the expanded auditory masses are the "basitemporal processes" (*b.t.*). The rostrum is subcarinate, the basitemporal plates are obliquely truncated postero-externally; the median part ends behind the transverse bars, and supports the narrow unossified basioccipital cartilage (*b.o.*). Sections involving the parasphenoid are figured (Plate X. figs. 7–10, *pa.s.*), and show the differentiation of a definite perichondrial layer between it and the endoskeletal basis cranii, from the ethmoid (*eth.*) to

* In the Bull-frog there is a lacrymal.

the basioccipital (*b.o.*); so that during development it has remained true to its original character.

The rest of the bony centres may be studied as part of the endoskeletal skull; they are shown thus in the figures (Plate VII. fig. 16, Plate VIII. figs. 9, 10, Plate IX. figs. 1–9, Plate X. figs. 7–10), which represent them both from the surface and in sections.

When the parostoses are removed from a partially macerated skull, a very remarkable osseo-cartilaginous structure remains (Plate IX. fig. 6, upper view, and fig. 7, lower view); for whilst the cranium itself is like a flat-bottomed boat, the superpalatals near the front, and the auditory masses behind, project like two pairs of *out-riggers*.

These projections are connected together by the bowed pterygo-palatal bars; and the quadrate *dips* its condyle outwards, backwards, and downwards. Forthstanding from the fore end of the skull we see a symmetrical pair of chambers; these are continuous above and below, and are parted by a septum (Plate X. figs. 3 & 7, *s.n.*), the "septum nasi;" the whole structure, seen from above and below, is ox-head-shaped, the "horns" being, each, the main lobe of the right and left emarginate snout-cartilage, the axis of the premaxillaries, the only *free* part of the "cornua" of the trabeculæ.

Both above and below (figs. 6 & 7, *su.c.*), these sessile snout-cartilages send back a retral process which has a valvular relation to the nasal openings, outer and inner, partly closing the opening on its outer side (Plate IX. figs. 6, 7, *v.e.n.*, *v.i.n.*).

On the inner side of the inner nostril there is another valvular process (fig. 7, *i.v.p.*); this is better seen in the more enlarged figure and in the sectional views (Plate X. figs. 3, 4, 5, *i.v.p.*); it is a small curved horn of cartilage with a broad fixed root in front.

The transverse anterior margin of the nasal labyrinth is elegantly crenate, and the cartilage, both above and below, is partially hardened by endosteal bone on its superficies; the mid line is gently concave, both above and below (Plate X. fig. 7). The nasal roof is made by the "aliseptal" laminæ (*al.s.*), the floor by the "subnasal" or trabecular laminæ (*s.n.l.*).

The external nostril is surrounded below and behind by the "alinasal lamina" (*al.n.*), which has greatly changed from its condition in the 7th stage (compare Plate VII. fig. 11, and Plate IX. fig. 6, *al.n.*); for this curled cartilage was relatively very large, and articulated behind with the ethmoid above the superpalatal bar; but now it is small, has become more freely segmented from the nasal roof, and the part which articulated with the ethmoid turns forwards and has a free notched end. There are no turbinal outgrowths in the Frog's nasal capsule, and yet it is very complicated anteriorly; this complication is *septal*, and does not arise from the "alæ," as is the case with true *turbinals*. A transversely vertical section (Plate X. fig. 3) through the outer nostrils displays an unexpected number of laminæ, which I have only been able to understand by reference to what takes place in higher types. The section has been made exactly through the external nostril (*e.n.*), and close to the fore edge of the internal nostril (see Plate IX. fig. 7, *i.n.*); it has passed through the nasal process of the premaxillary (*n.px.*) above, and through the palatal process (*p.px.*) below. The aliseptal lamina (*al.s.*) is here at its narrowest,

and the subnasal (*s.n.l*) at its greatest width. At the outer angle, below, the maxillary (*mx.*) has been cut through, and mesiad of it the vomer (*v.*) at a great distance from the palatal processes of the premaxillaries (see also Plate IX. fig. 2). Here we see that the septum nasi (*s.n.*) gives off three pairs of alæ—the aliseptal above, the subnasal below, and between them, near the top, another, which passes downwards and outwards, forming a floor to the external nostril (*e.n.*). The alinasal is at this part *fixed*, although it is *free* behind; and it is continuous with the intermediate cartilaginous ala, which may be called the "transeptal lamina." Although there is no free "alinasal turbinal," there is nevertheless a fixed lamina answering in some degree to it (compare Plate X. fig. 3, *al.n.*, *t.s.l.*, with that of the *Tinamou*, 'Skull of the Ostrich-tribe,' plate 15. fig. 11, *n.t.b.*); for the "transeptal lamina" splits, when nearly half across, into two nearly equal layers, both of which terminate externally; the lower of these is evidently the *fixed* counterpart of the free "alinasal turbinal" of the Bird.

That the valvular processes are free "horns" is evident from this view, which shows the valvular process of the external nostril (*v.e.n.*), and the inner valvular process of the internal nostril (*i.v.p.*), both quite separate as sections.

A section made through the anterior margin of the annular ethmoid (Plate X. fig. 7. *eth.*) shows the perfect simplicity of the *true olfactory region*, which is of great width. and is closed externally where the upper and lower alæ pass into the "superpalatal" (*s.pa.*). Above (Plate IX. fig. 6, *s.pa.*, *al.e.*), the "ala," now "aliethmoid," passes directly into the "superpalatal" without any notch; but below, the "subnasal lamina" becomes narrow, and has a deep notch separating it from the thin lower flap which thickens to pass into the "superpalatal" (Plate IX. fig. 7). The ethmoidal roof, even a little in front of the annular bony mass, is the part from which the upper turbinals arise in higher types; the foundation for the middle turbinals must be sought for in the prefrontal wall on the posterior surface of the two laminæ which pass into the "superpalatal." The sectional view does not show this, for the razor passed immediately in front of it; but in the bird's-eye view (Plate IX. fig. 6, *p.r.f.*) the back of the *folded* lamina is seen, and on the postero-superior surface, which is bevelled, sloping down to the orbit, there is a very notable prefrontal patch of *endostosis*, the *morphological counterpart of the "prefrontal" of the Teleostean fish, the Crocodile, and the Monitor, the "antorbital" of the Bird, and the* "pars plana" of the Mammal; this would be the root of a "middle turbinal," if such there were.

The "notch" between the "subnasal" and the foundation of the palatal pier is cut through in the section (Plate X. fig. 7, *no.*); mesiad of this notch the dentigerous part of the vomer (*v.*) is severed, and outside it the thin ectosteal palatal (*pa.*), whilst on the roof is seen the nasal (*n.*). The bony substance of the ethmoid is seen both above and below the section, which has been made through the pointed middle portion (Plate IX. figs. 6, 7, *eth.*); this is the part where the "perpendicular ethmoid" ends and the proper "septum nasi" begins (see Plate VII. fig. 11, *p.e.*). A section made immediately behind the olfactory passages (Plate IX. fig. 8, 1) shows the fossæ in which the "rhinencephala"

rest, separated by the free end of the perpendicular ethmoid; this part is a feebly carinate projection, and sends backwards nothing that can be called a "crista galli." The nerve-outlet is single on each side; but this passage answers to the slit on each side the "crista galli" in the Mammalia, which in them is bridged over by many bars of bone, and thus converted into a series of foramina.

The ethmoid is *cranial* as well as *axio-facial*, and just so much room as the "rhinencephala" need belongs to it by right, so that the "girdle-bone" is not the absolute measure of this region; it does not reach far enough in the young of the Common Frog, nor in the adult of some species, whilst in others it trespasses on the surrounding territories when they are unprovided with their own bony centres.

Here, indeed, the "rhinencephalic lobes" do not reach so far back as the ethmoidal walls would indicate; for these are partly built upon orbito-sphenoidal ground. There is scarcely any tendency to the production of a *subcranial keel* as a continuation backwards of the perpendicular ethmoid; but the basis cranii is flat and shark-like, only modified by a gentle bulging below (Plate IX. fig. 9). Where the basal part of the ethmoid joins the presphenoid its margin is gently concave; it is more scooped above, where the bony matter first began; for here it forms the front margin of the great fontanelle (*fo.*).

To sum up the characters of this simple ethmoid, let it be noted that the "*rhinencephalic fossæ*," the *roof* upon which the frontals and, in most types, the top of the nasals rest, the *floor*, which in many types of Vertebrata joins a similar presphenoid behind, and the "pars perpendicularis"—all these are formed from one bony centre in the back wall of the nasal cavities, and the front end of the cranium. The cartilaginous foundations of this structure were laid in that part of the "trabeculæ" which was the first to coalesce, in front of the original pituitary space, and from which coalesced portion there grew, at a very early period, a transverse wall separating the brain from the nasal sacs (Plate V. figs. 1, 2, 3).

With regard to the succeeding regions, which have received anthropotomical names from the well-known bony centres, these must be surveyed from strictly *neural* landmarks, and not measured by the very indefinite bony tracts. The anterior sphenoidal region extends from an ideal line drawn circularly round the cranium outside the posterior margin of the "rhinencephala" to the "foramen opticum" (2). A section through the middle of this region shows the large orbital alæ, "orbito-sphenoids," as a rather thin lamina of hyaline cartilage, which ends by a sharp edge above (Plate IX. fig. 9, *os.*).

These alæ pass into the "presphenoidal" region below, which is somewhat thicker where it bulges down upon its undergirding bone, the "parasphenoid" (*p.s.*, *pa.s.*): these alæ, thus united below, enclose the "prosencephalon" (C^1), the roof being strongly finished by the frontals, which send downwards rudimentary orbital plates. The anterior margin of the orbital alæ has been ossified by the ethmoid; for the rest there is only incipient "endostosis" below, in the presphenoidal region; and this is not divided off

from a like deposit in the basisphenoidal, whilst, common to both, is the permanently separate cortical plate, the "parasphenoid."

The posterior sphenoidal region, like the anterior, is devoid of any distinct bony centre, and must likewise be surveyed by the primordial land-marks, the nerve-passages. Thus the lines that fall unto it are from the "fenestra," in which the optic passage lies in its fore part, to the great "foramen ovale" behind. *Apparently* its landmark has been removed; for the "prootic" stretches forwards over two-thirds of this space (Plate VIII. fig. 9, *al.s.*, *pro.*).

Below, in the basisphenoidal region, and on the sides, as far as they are unaffected by the prootics, the posterior resembles the anterior sphenoid in all that is essential; above, however, we note a difference; for we are now *behind* the great fontanelle, and the alisphenoids end above in a roof-plate, a rudiment of the very perfect cartilaginous roof of the Shark, in which type the "fontanelle" looks forwards at the fore end of the cranium. In Osseous Fishes the thick upper edge of the alisphenoidal cartilage adjoining the roof is separately ossified, and forms their large "postfrontal," the cartilaginous basis of which may, or may not, pass across the primary "fontanelle."

The postsphenoidal roof-plate is feebly ossified by endostosis, the rudiment of a "suprasphenoidal" bone. This upper tract is lozenge-shaped, but the posterior and lateral angles pass into the adjacent regions (Plate IX. fig. 6, *su.s.*).

The outstretched auditory region has acquired one large pair of bony centres, the "prootics" (*pro.*); but the roof-crest ("pterotic"), the supero-posterior ("epiotic"), and the infero-posterior ("opisthotic") regions are but little differentiated in this way; yet the auditory masses are largely ossified behind, having a borrowed source of bone, the "exoccipitals" (*e.o.*).

Seen from the inside (Plate IX. fig. 5), the periotic masses have a smoothly rounded face, which, projecting inwards, takes from the cranial cavity (Plate X. fig. 9); outside, as in the Lacertians, these masses project so as to increase the breadth of the skull threefold, thus throwing out the mandibular pier, and giving the mouth its enormous gape.

In a section made through the "foramen ovale" (Plate X. fig. 8, 5), the ampulla of the anterior canal is exposed with part of its arch; here the parietal (*p.*) is the only roof to the skull in one place, above the posterior fontanelle (*p.fo.*); externally the fore edge of the prootic region (*pro.*) is rather flat and largely unossified; this is the part where the "metapterygoid root" has coalesced with the front face of the auditory capsule. This narrow outer half of the projecting prootic region is strongly clamped by the supratemporal part of the squamosal (*s.t.*). Ossification has affected the rest of the cartilage throughout, and it reaches nearly to the "posterior fontanelle" above, and to the thick part of the parasphenoid (*pa.s.*) below. The "Gasserian ganglion" (5) is seen *in situ*, and above it a projecting spur of bone.

In the next section (fig. 9) the crown of the arch of the anterior canal is cut through (*a.sc.*), and the whole width of the labyrinth-cavity is exposed, near the middle of which

is seen the ampulla and part of the arch of the "horizontal canal" (*h.sc.*). This section has been made close behind the posterior fontanelle, between the prootic and exoccipital (see Plate IX. fig. 6, *pro.*, *eo.*, *p.fo.*), cutting through the posterior part of the parietal (*p.*), and the broad basitemporal portion of the parasphenoid (*pa.s.*); on the inner side the auditory nerve (7^b) is seen passing through the "meatus internus." A little of the prootic (*pro.*) is seen here, both in the cranial and in the auditory cavity; and above, it encloses the crown of the anterior canal, and reaches that of the horizontal (*a.sc.*, *h.sc.*). The outermost part of the projecting periotic mass is thicker here and is deeply scooped. If the section had been made a little further backwards, it would have passed through the "fenestra ovalis" (see Plate VIII. fig. 4, *fs.o.*); but in the adult the original fenestra is largely walled-in with cartilage, leaving only a small opening to the "vestibule" (Plate VII. fig. 16, *fs.o.*). The piece of cartilage seen in the fenestral pit is the stapedial (*st.*); and below the pit the unossified cartilage is "opisthotic" (*op.*), regionally considered. This section fortunately runs through a very important articulation, namely that of the stylo-hyal (*st.h.*) with the opisthotic cartilage.

The original ovoidal form of the periotic capsule is not entirely lost, as may be seen by comparing the inner view (Plate IX. fig. 5) with this section, which shows the form of the cavity, and the bulging inwards of the inner wall.

Only a narrow border-land is left above between the overgrown prootic and exoccipital bony centres (Plate VIII. fig. 9, Plate IX. figs. 1, 3, 4, 5, 6), and the occipital arch is, as in all the "Sauropsida," confused in its ossification with the posterior face of the periotic mass; nothing can more plainly bespeak caution in determining true morphological *regions* from encroaching bony growths; we never see the auditory sense-capsule perfectly differentiated from the occipital and sphenoidal regions, as a bony mass, until we reach the Mammalia.

A section made still further back passes through the epiotic eminence (Plate X. fig. 10, *ep.*), which is affected by endostosis at the surface; this eminence is caused by the development of the arch of the "posterior canal" (*p.sc.*), which passes downwards and forwards to reunite with the anterior canal. It is crossed close above its ampulla by the "horizontal canal" (*hs.c.*), the ampulla of which was shown in the last section (fig. 9); part of the stapes (*st.*) is seen in the fenestral fossa behind; and part of the parasphenoid (*pa.s.*) is seen towards the mid line of the skull. Across this part the skull is tolerably well ossified—not, however, by a "periotic" centre, but by the exoccipital. There the bone is so thick that medullary cavities are seen in it; it reaches to the narrow basioccipital region below (*b.o.*), and to the wider supraoccipital region (*s.o.*) above; this is ossified superficially, but the basioccipital region is not.

Further explanation of the transverse sections is given by a side view, in which the outer surface has been partly pared away (Plate VIII. fig. 9); in this way it is seen that the ampullæ of the anterior and horizontal canals (*a.sc.*, *h.sc.*) are imbedded in the large spreading prootic (*pro.*), whilst the ampulla of the posterior canal (*p.sc.*) is enclosed in the exoccipital (*e.o.*) Below the posterior ampulla is the exit for the compound 8th

nerve (8); below the two bony masses is the huge deep fossa for the stapedial and inter-stapedial cartilages (*st.*, *it.st.*); and below this hollow is the rounded unossified opisthotic region (*op.*), to which the stylo-hyal (*st.h.*) has at last become closely attached. The broad occipital condyle (*o.c.*) is capped with persistent cartilage; and the exoccipital bony mass mounts up above the condyle into the supraoccipital region, and descends below into the base of the skull.

The endoskeletal part of the premaxillary region has been described with the nose-capsule; we now come to the lateral regions. There has been no tendency to segmentation of the "superpalatal" bar from the prefrontal mass (Plate IX. figs. 6, 7, *s.pa.*, *pr.f.*); and the "prepalatal" (*p.pa.*) is now a flat triangular projection slightly affected by endostosis, continuous with that of the rest of the bar. The constriction between the "postpalatal" and pterygoid regions is now obliterated (*pt.pa.*, *p.g.*); the ectosteal palatal lamina (fig. 7) has not increased in size *relatively*; thus most of the palatal region is left unarmed with bony matter.

The "subocular fenestra" (*s.o.f.*) is a large ellipsoidal space made slightly reniform by the bulging of the cranial side-wall; outside it is bounded by the elegantly arcuate anterior part of the pterygoid (*pg.*).

There is still a very solid core of hyaline cartilage to the pterygoid, the "Anoura" being very remarkable in that the ectosteal plates show, for the most part, so feeble an affinity for the cartilage within, and even for the endosteal bone into which the cartilage is changed ('Shoulder-girdle and Sternum,' pls. 5–8, pp. 66–89). This unchanged core is most exposed on the upper surface (Plate IX. fig. 6). The metapterygoid bony spur (*m.pg.*) is now small and placed *anterior* to the triangular "infrahyomandibular" (*i.h.m.*); behind the latter is the deep recess which bounds the "Eustachian tube;" and then the pterygoid grows downwards, backwards, and outwards, strongly clamping the suspensorium (figs. 1–7); this posterior portion of the "ectosteal pterygoid" is a *supererogatory* growth of fibrous bone, its proper cartilaginous axis ending at the front margin of the suspensorium.

The posterior part of the bony pterygoid is best seen from behind (fig. 4, *p.g.*), in which view we can most readily understand the manner in which the suspensorium is continuous with the skull in its metapterygoid or antero-external portion, and how it glides on the skull by its postero-internal portion, the infrahyomandibular (*m.pg.*, *i.hm.*).

The metapterygoid portion of the suspensorium is overhung by an *eave* of cartilage from the projecting periotic capsule; this eave is cut through in front of the suspensorium in the section (Plate X. fig. 8); and under the shadow of this eave the "extrastapedial" *spatula* turns forwards to receive, on its outer surface, the fibres of the "membrana tympani" (Plate IX. fig. 3, *a.t.*, *e.st.*). The cartilaginous eave is *tiled* over by the embracing supratemporal part of the squamosal, the projecting edge of which forms a strong ridge for the attachment of the ends of the cartilaginous "annulus," which have now met each other (Plate IX. fig. 3, *a.t.*, *s.t.*).

The quadrate angle of the subocular arch now projects almost as much backwards as

it once did forwards, so that the angle formed by the suspensorium with the basicranial axis is very obtuse (see Plate IX. fig. 3, *q.*). The ectosteal plate of the quadrate is overlapped in its broad grafting portion by the spatulate lower end of the squamosal; in front of the cartilage the quadrate plate grows into a long quadrato-jugal process which largely overlaps the long jugal process of the maxillary.

The posterior lobe of the quadrate condyle is, in reality, the "symplectic lobe" (*sy.*); but, as in the palato-pterygoid, the cleft has filled-in again*, which for a time separated it from the mass of the quadrate: this return to a lower type seems to be peculiar, but must be caused by some general law.

The rest of the mandible has changed very little from its condition in the last stage. The "intermaxillary rudiment" or "mento-meckelian" (Plate IX. fig. 3, *m.mk.*) is permanently small, and the dentary runs back two-fifths of the length of the ramus; the "articulare" ensheaths the inner side completely, but a wide space of the permanently unossified core is bare on the outside. The articular condyle (*ar.c.*) is a smooth egg-like mass, with its long axis longitudinal; it rolls very freely beneath the smoothly scooped base of the quadrate.

The pier of the next arch has been differentiated into most of the structures of the middle ear, as well as the hyoid crus; it has been subjected to a large amount of special metamorphosis, the hyoid arch lending, as it were, much of its substance to the organ of hearing.

When the parts which have grown by modification of the suprahyomandibular segment are examined *in situ* (Plate IX.), it is seen that they reach from the stapedial plate (*st.*) to near the anterior edge of the membrana tympani (Plate IX. fig. 3, *m.t.*, *s.st.*). Seen from behind (fig. 4) the whole series is arcuate; for the "medio-stapedial" ascends a little, and the extrastapedial descends in the same degree; but the principal direction of the former is *outwards*, and of the latter *forwards*, whilst the feeble "suprastapedial" ray passes upwards, backwards, and inwards, and is loosely attached to the "tegmen tympani" by a delicate fibrous ligament. In the enlarged side view (Plate VIII. fig. 9) the whole series of "middle-ear" elements is drawn out, longitudinally, for display, the "tegmen" being pared away, the osseous medio-stapedial (*m.st.*) being bent towards the skull, and the cartilaginous parts freed from their attachments. The "suprastapedial" (*s.st.*) is a delicate rounded rod continuous with the "extrastapedial" below, and forming an acute angle with the medio-stapedial" (*m.st.*); it is the "anterior hyomandibular fork" modified (see Plate VI. fig. 8). The "extra-stapedial" (*e.st.*) is an elegant spatula; it is thin, convex on the outer and concave on the inner side, and on its *convex* side it receives the fibrous mesh of the "membrana tympani" (Plate IX. fig. 3). At first sight it would seem strange that this free cartilaginous *spatula* should answer to the outstanding condyle on which the "principal opercular" is articulated in the Osseous Fish; but a reference to the early condition of these parts in both Fish and Frog will make things clear. If my fourth stage (Plate V. figs. 1–4, and Plate VI. fig. 8) be

* Here we have what may be called *retrograde metamorphosis*.

compared with the embryonic skull of the Teleostean (HUXLEY, 'Croonian Lecture,' p. 29, fig. 8, and 'Elem. Comp. Anat.' p. 185, fig. 72), it will be seen that the hyo-mandibular runs almost directly *forwards*, and that the boss to which the "opercular" is attached grows out a little distance below the posterior hyo-mandibular fork. Now the segmentation of the hyo-mandibular in the Frog takes place immediately below this *boss*, leaving it free; and from the first it has a direction *forwards*, which is only intensified and not altered in the adult. The "medio-stapedial" (*m.st.*) has now all the appearance of the osseous auditory "columella" of the Lizard; it is twisted and curved somewhat, is very slender at first, and then bulges behind; it is bevelled at its end, towards the skull, and is slightly unossified at its edge behind. The bevelled inner face of the "medio-stapedial" articulates with the lower half of the outer face of the "interstapedial," the segment which was taken from its base. This intercalary piece, the undoubted homologue of the mammalian "os orbiculare," has now become a solid wedge of hyaline cartilage, and has found lodgement in the anterior part of the deep "stapedial fossa." The "interstapedial" is pointed in front, has a convex upper, and concave posterior and lower margins; it is thick and solid, especially behind, where it articulates with the stapedial plate. The homology of the "interstapedial" is with the *condyle* of the "posterior hyo-mandibular fork" of the Osseous Fish; its function is to connect the true "stapedial" (or auditory) segment with the medio-stapedial *hyoid* element. The periotic element, "stapedial" (*st.*), is elegantly elliptical in shape; but the anterior margin is shortened, where it fits to the subconcave face of the "interstapedial" (Plate VIII. fig. 9, *st.*, *i.st.*); it is gently concave on the inner face, and is very thick and convex on the outer (Plate VII. fig. 15, *st.*); like the other parts of the middle ear, with the exception of the medio-stapedial, it is wholly unossified. It is attached to the edges of the fenestral fossa by a delicate band of fibrous tissue (see Plate VIII. fig. 10, where it is seen from within); but much of the inner face is in immediate contact with the cavity of the vestibule.

The "fenestral or stapedial fossa" (Plate VII. fig. 16, *st.f.*) is beautifully egg-shaped and of considerable depth (see section, fig. 15); behind, the exoccipital (*e.o.*) keeps at some distance from its rim; but in front the prootic (*pro.*) sends a small wedge of bone into its fundus. The "fenestra ovalis" (*fs.o.*) takes up the postero-inferior third of the fundus; it is reniform, with the concave edge looking obliquely forwards and upwards; it nowhere reaches the edge of the pit; the main otoconial mass (*ot.*) can be seen through the "fenestra." This operculated cleft in the "periotic" wall was once the size of the stapedial plate; the floor of the pit has grown, whilst the stapedial plate has been thickening, as the creature has become full-sized. The "stylo-hyal" (*st.h.*) has its permanent attachment a short distance below the rim of the *fossa* at its interior third (Plate VII. fig. 15, Plate X. fig 9, *st.h.*).

The "stylo-cerato-hyal" band of cartilage has changed but little since the Frog was two or three months old; it has not become osseous (Plate IX. fig. 3, *st.h.* and Plate X. fig. 2). The "*hypohyal*" region (*hy.h.*) retains the lobe, both before and behind, which was seen in the Tadpole (Plate V. fig. 2): these lobes have now become

elegant leafy plates of cartilage; they both look forwards and are separated by a narrow band. The deeply concave anterior margin is accurately semielliptical in outline; behind this great "notch" the broad cartilaginous "basi-hyo-branchial" plate is continued backwards so as to have a length equal to its greatest breadth; it is narrower behind than before, and the sides are concave.

The single pair of remnants of the first and second branchial arches (*br.*) are curiously dissimilar; they are not much in front of the "free hypobranchial horns" or thyro-hyals (*th.*): these latter have become much more slender in the shaft, and are wide behind; they are well ossified by an ectosteal sheath or "shaft-bone."

The great basal plate has irregular patches of "superficial endostosis" on its surfaces; but its various outgrowths are soft; it has lost the symmetrical foramina (fig. 1). The upper surface of this evenly spread lamina of cartilage is gently concave; its long sigmoid suspensors, the hyoid cornua, give it its exquisite mobility (see Plate IX. fig. 3, *hy.*), a state of things of no little consequence to the Frog with its peculiar mode of respiration.

SUMMARY.

It appears to the writer that several things are *proved* in the foregoing descriptions which could only be *guessed at* if the subject had not been treated in a somewhat exhaustive manner. The most important part of this attempted demonstration of the morphology of the Frog's skull is that which treats of the first two stages; without these the labour would have been greatly in vain, as nothing could have been determined as to what is a *primary morphological element*, and what is a mere *secondary* structure, uniting together and, by that fusion, immediately *masking* the simple primordial structure.

But the processes of growth are so rapid, the *axis* of the embryo is subject to such peculiar bending and straightening processes, the relation of parts so greatly altered by that which was afar off being brought near, and *vice versâ*, that a complete mastery of the earliest differentiation is fundamental to the whole business of research. This teaching was long ago impressed upon me by the "Croonian Lecture," which contained the rudiments of this extended monograph; I can appreciate its value now that I have worked for a long while at the real objects.

In the skeletal parts I consider that to be a *primary morphological organ* or *element* which is first differentiated into a definite tract of tissue more solid than the surrounding blastema, and which continues thus distinct until it can be demonstrated to be cartilage by the formation of a concentric line which marks off the outer or cortical cells as the perichondrial layer. This can be done in the Frog-embryo when it is only one-sixth of an inch in length, two or three days before hatching, and when the cephalic and caudal extremities project very little beyond the yelk-mass: this is my first stage. This is not the point at which to commence *general* embryological research into the structure of the Frog, but is a very convenient stage for the *special* morphology of the skeletal parts.

Any *opening* found at this stage in the *walls* of the embryo is evidently *primary*; for it is anterior to that most remarkable *dehiscence* of the visceral walls by which the "facial

clefts" are formed; there is at this stage an opening of this kind, a *visceral fontanelle*; it is the opening of the mouth (Plate III. figs. 1 & 8, *m.*). The secondary openings (*clefts*) are of the utmost interest, as the two foremost of them are persistent in the highest Vertebrata, and form the passages necessary to the organs of *smell* and *hearing*.

The primordial cartilaginous bars which are so definitely separated by the clefts are modified to an unlooked-for extent themselves, and also serve as a foundation upon which, *continuously*, structures are built with which they could have had no primary relation.

The Cranium,

in the first place, may be remarked upon as being largely constructed upon facial bars, as well as having a true *axial* region, continuous with the vertebral column.

The neural axis, whilst passing from a grooved into a tubular form, acquires a membranous investment; this investment lies immediately upon the notochord and the symmetrical vertebral rudiments. In the vertebral part of the axis this is a mere cylinder; but in the cephalic region, where the vertebral rudiments are *continuous* to the end of the notochord, the neural mass becomes swollen into vesicles, and the membranous investment into *pouches*. This lobular bag, with its cerebral contents, does not derive the skull, which more strongly encloses it, from the continuation of the axis merely; for the axial part stops short behind the down-turned pituitary vesicle (Plate III. fig. 4).

The remainder of the skull has a *facial* foundation; it is built upon the first pair of facial arches, which straighten themselves beneath the membranous cranium, and then send outgrowths of cartilage upwards which form walls on each side and in front.

Not only so, but the auditory sacs imbed themselves in the lateral walls, which are really derived from the *axis*, thus forming a large "fenestra" on each side, which, however, is closed by the auditory sac.

Thus it is seen that the skull of the Frog is a morphologically compound structure.

The Sense-capsules.

Two pairs of these, the optic and the auditory, have their own cartilaginous *pouch*, and are, primarily, equally independent of all the surrounding skeletal structures, the *earball* as much as the eyeball.

The olfactory sacs, however, are entirely membranous; and the labyrinthic chambers in which they are lodged, and to which they form a lining, are developed independently as superstructures upon the first pair of visceral arches, and upon the *secondary* "connective" which binds them together.

If all this be true, then the terms *cranium* and *face* are commonly used in a very arbitrary manner; the terms *axial*, *visceral*, and *sensory* must be used as much as possible if we would speak correctly of these things.

But the term *skull* for the cartilaginous or bony box enclosing the whole brain can

never be misunderstood if its composite character be borne in mind; the sensory organs, as well as the jaws, cheeks, and parts of the throat, must all be included in the term *face*.

The Visceral (Facial) Arches.

The Frog developes seven pairs of visceral arches: the seventh is not differentiated until a few days after hatching; and these simple, free, subarcuate descending rods undergo an amazing amount of metamorphosis.

The last four, the gill-arches, are transitory, and their remains are of little importance in the adult; the first three concentrate nearly all the interest upon themselves.

The diagrammatic figures (Plate X. figs. 11–20) are intended to illustrate ten stages in the morphology of these three arches; and as the auditory capsule enters into such remarkable relation with two, and especially with the last, it is figured also.

The first arch is shown in dotted outline, the second in continuous outline, and the third is coloured.

In the first stage (fig. 11) we see three curved clubbed rods, slenderest above, where they end in somewhat twisted points, which turn a little forwards. The first (1 *tr.*) is the "trabecular" rod; it is *recurved*, and diverges from the next; for here is the oral opening. The other two (2 *mn.*, 3 *hy.*) are very similar; but the foremost is the thicker of the two: they are curved somewhat backwards as well as inwards, below. Here we have the rudiments of the first and second "postoral bars," or the mandibular and the hyoid. The auditory pouch is *above* and *behind* the third bar.

Fig. 12 represents the second stage, a few days after hatching; in this already there are some changes to be noticed.

The slight curve forwards of the narrowing upper part of the two foremost bars (1 *tr.*, 2 *mn.*) has increased, the lower part of each has expanded, and the second has formed a small inturned bud, the rudiment of Meckel's cartilage (*mk.*).

Another important change is the divergence, backwards, of the lower half of the third arch; and the approximation of the auditory sac to this arch above is noteworthy (3 *hy.*, *au.*).

Fig. 13 represents these arches in a Tadpole 5 lines in length; here the changes have been sudden and great.

The "mesocephalic flexure" is almost obliterated, and the first pair of bars (1 *tr.*) have also ascended; but they have likewise applied themselves more accurately to the base of the membranous cranium, and have coalesced with each other in front of the pituitary body and with the second arch in two places.

The second arch (2 *mn.*) has coalesced above with the investing mass and with the first arch; it has diverged in its descent much more from the trabecular bar; but at its lower third a connective bar has bound the two together (*p.pg.*); outside this connective there has appeared a leafy flap of cartilage which encloses the temporal muscle. The free leaf of cartilage is the "orbitar process" (*or.p.*), and the fixed secondary band is the first rudiment of the pterygo-palatine arcade; this therefore is not in the Frog a primary arch.

MECKEL'S cartilage (*mk.*) is now well developed as a free segment cut off from the second facial (first postoral) arch.

The auditory capsule (*au.*), now a closed pouch of cartilage, has overgrown the top of the third arch (3 *hy.*); and the upper part of this arch has applied itself close to the antero-inferior face of the sac. It is then free, and then applies itself closely, to beyond its middle, to the second arch; but the diverging lower part has now become segmented from the upper, an oblique joint-cavity has been formed, and a small angle of the upper part projects below the joint.

All this is typically *ichthyic*; the upper part, closely applied to but not coalesced with the auditory sac, is the "hyo-mandibular" (*hm.*); the small angular projection below the joint is the "symplectic" (*sy.*); and the free segment is the "hyoid cornu" ("stylo-cerato-hyal") (*hy.*).

Fig. 14.—In perfect Tadpoles an inch or more in length, the cranial cavity, its trabecular *rests*, and the suspensory part of the short Meckelian rod have all undergone great relative as well as real elongation. The free outgrowths of the trabeculæ ("cornua," *c.tr.*) now form large unsegmented counterparts of the Meckelian segments of the next arch, and behind the commissure they have become united by a thin connective tract, which forms the cranial floor. The pterygo-palatine band has not lengthened, so that the suspensory portion of the second arch now forms a very acute angle with the basicranial axis. It has become largely bowed-out, however, and has thus formed the large "sub-ocular fenestra" (*s.o.f.*), an expansion of the *true* first cleft, or that between the trabecular and mandibular bars.

The connective at the top of the second arch is now long, bows outwards and backwards, and is confluent with the investing mass behind and the trabecular root above. The second cleft (first "postoral") has its upper part still very evident in the space between the band in front, and the auditory sac and top of the third bar behind. This remnant of the second cleft is rounded below; and thence the rest of the hyo-mandibular ("infrahyomandibular," *i.hm.*) has become completely fused with the arch in front; this region also is greatly elongated, so that the elegant condyle of the "hyoid cornu" (*hy.*) is articulated to the lower third of the compound suspensorium, and the more mobile parts of the face and mouth are here as far from the ear-sac as in *Syngnathus*.

The free hyoid cornu is now of great breadth, but, as in the third stage, it has not coalesced below with the basihyal. The "suprahyomandibular" region (*s.hm.*) is now relatively much smaller; it has gained a more external position with regard to the ear-sac, with which it has coalesced by a pedate or sub-bifurcate head. The neck is narrow; and below the neck it bulges downwards so as to form an "opercular process," a truly ichthyic structure. The ear-sac has opened again; and its *own* opercular piece, the "stapes" (*st.*), is some distance behind, and more internal than, the head of the hyo-mandibular.

In the fifth stage (fig. 15) the mandibular pier (*qu.*) has receded from the trabecular

region (1 *tr.*), thus giving rise to an elongated pterygo-palatine (*pa.*, *pg.*), now as long as the suspensorium; this bar is semisegmented. MECKEL's cartilage (*mk.*) has similarly increased in size; the orbitar process (*or.p.*) is becoming starved, and is changing an anteorbital for a postorbital position; and the connective of the second arch is not so far from the front of the ear-sac.

But the most interesting changes are taking place in the third, or "second postoral arch." The "suprahyomandibular" (*s.h.m.*) has become nearly free; and the "infrahyomandibular" (*i.hm.*) has grown upwards into a large leafy lobe, which turns its convex side somewhat inwards. Moreover the whole upper region is only half as long as in the last stage, so that the free hyoid cornu is so much nearer to the skull; it is also much narrower in form.

The "symplectic" rudiment (*sy.*) has grown down the posterior edge of the much elongated quadrate region, and is partially segmented from it.

In the 6th stage (fig. 16), Tadpoles with stumpy tails, MECKEL's cartilages and the pterygo-palatine bars (*mk.*, *pa.pg.*) are both longer than the suspensorium, on the front face of which, near the top, is seen the remains of the "orbitar process" (*or.p.*). The elongating hyoid cornu (*hy.*) is becoming detached from its fossa, the joint-cavity degenerating into a mere loose ligamentous union. The "infrahyomandibular" (*i.hm.*) has its free lobular edge still higher up; and the *starved* "suprahyomandibular" (*s.hm.*) has become a free trifoliate plate of cartilage. It has now lost its ichthyic character, and is becoming *functionally related* to the stapes (*st.*) as part of the middle ear; and its lobes can now be identified with the processes of cartilage which are connected with the stapes in the "Sauropsida." The antero-superior lobe is the "suprastapedial," the postero-superior is the "medio-stapedial," and the free process, which grows downwards and *forwards*, is the "extrastapedial;" the whole piece is now detached from the auditory capsule above, and it is still some distance in front of the stapes.

Seventh Stage (fig. 17).—This is when the little Frogs have nearly lost the tail-stump; a cartilaginous "annulus tympanicus" can now be seen. There is not much difference between this stage and the last in many respects; but the "hyoid cornu" (*hy.*) has become very long and narrow, is quite loosened from its original attachment, and is gaining a new one above. In the sixth stage the palatine (*pa.*) was bending forwards; it is now pointed in front; in both the "trabecular horns" are almost entirely involved in the nasal labyrinth.

Eighth Stage (fig. 18).—In Frogs taken a month or two later, the pterygo-palatine and MECKEL's cartilage (*pa.pg.*, *mk.*) are still more elongated, and the relatively shortened suspensorium (*qu.*) forms a right angle with the basicranial axis. The metapterygoid root clings more closely to the periotic cartilage; and the "infrahyomandibular" (*i.hm.*), relatively smaller, is acquiring a more *inward* position. The *narrowing* hyoid bar is now quite loosened from its old attachment, and is very near to the opisthotic region. The detached "suprahyomandibular" has now undergone a more perfect metamorphosis than can be seen in either Lizard, Crocodile, or Bird. The anterior head has become a

delicate terete rod which passes upwards and backwards, and is attached to the "tegmen tympani" by a fibrous ligament; this is the "suprastapedial" (*s.st.*). The freed "opercular lobe" has become an elegant spatulate "extrastapedial" (*e.st.*); and it is tilted more upwards than when it was unsegmented in the fourth stage (fig. 14). The hinder head of the "hyo-mandibular" has bent itself downwards a little, has become relatively very long, and has become ossified into a little bony shaft. The "condyle" at its free end has become segmented off into an "orbicular element," the "interstapedial" (*it.st.*); the bony bar is the "medio-stapedial" (*m.st.*).

Ninth Stage (fig. 19).—In three months more the facial bars are becoming more and more like those of the adult, the pterygo-palatine and Meckelian rods larger still; and as a correlate of this, the suspensorium now forms an *obtuse* angle with the basicranial axis. The "infrahyomandibular" (*i.hm.*) now forms an elegant hatchet-shaped condyle, which glides over the antero-inferior face of the periotic cartilage directly *inside* the metapterygoid root. The elements superadded to the stapedial are lengthened, are more elegant in form, and more completely adapted to the proper stapedial plate. A long sinuous ligament still binds the stylo-hyal head to the inferior angle of the "infrahyomandibular;" but a new ligament ties it to the periotic capsule below the stapes.

Tenth Stage (fig. 20).—We now come to the skull of the old Frog, where we see what is typical in the *highest* form of the Batrachian. The first or trabecular bar (1 *tr.*) still shows the tips of these "horns" on the front of the nasal box; but, for the rest, they are completely amalgamated with the nasal and cranial structures. Between them and the second bars (first postorals, *qu.*) there now intervenes the long, elegantly arcuate, fore-spurred pterygo-palatines (*pa.pg.*); and yet this second bar, to which the *middle third* of the third bar has coalesced, has much the same curve in itself, and has a similar angular relation to the basicranial axis as in the first stage (compare figs. 11 & 20). Thus the mandibular pier, leaving out of consideration the free Meckelian rod, has gradually travelled so far forwards as to be nearly parallel with the basis cranii (fig. 14), and then has slowly grown backwards until it has more than regained its original position. The middle third of the third (second postoral, *i.hm.*) still retains its suspensorial function, although it has long lost its distinctness. Having been carried upwards during growth, it reaches just to the point where the *apex* of the arch was in the third stage (fig. 13, *h.m.*), when the auditory sac had grown so far forwards as to overshadow the bar. Having lost the upper third, which was segmented off to form the uniting chain, it now articulates by a gliding joint (Plate IX. figs. 2, 4, 7, *i.hm.*) with that same region of the ear-sac; and the mandibular pier, having clung to the much-projecting cartilage of that sac, is now exactly *outside* this remnant of the third bar. The 4-*cornered* free hyoid segment, once so far from the ear-sac, and being indeed the lowest *two-fifths* of the third bar (fig. 13), is now a *long band* articulating with the ear-sac a little way *behind* the *head* of the middle part ("infrahyomandibular"), from the *base* of which it was first taken (fig. 13 & fig. 20). The highly metamorphosed upper end of the second postoral bar now forms four out of five of the elements of the "middle-ear" chain. The stapedial plug (*st.*) and the ter-

minal segment of the superadded series, the "interstapedial" (*it.st.*), now lie in a deep fossa in the thick periotic wall; and the "fenestra ovalis" only occupies the posterior third of the fundus. Rounded in front, the stapes fits into the concave end of the parrot-beak-shaped interstapedial by a cup-and-ball joint; the piece in front, "medio-stapedial" (*m.st.*), is attached to the latter by the upper edge of its bevelled broad base, which is not entirely ossified; but there is no joint between this narrow-topped bony bar and the two unossified rays in front—namely, the delicate terete "suprastapedial" (*s.st.*), and the large spatulate extrastapedial (*e.st.*).

With regard to the *clefts* that part these foremost facial bars, it may be seen that although they are persistent yet they have undergone very curious morphological changes. The first cleft, that between the trabecula and mandibular rod, began to open *inside* the angle where the two clefts diverge (fig. 11) in front of the part where the transverse connective afterwards appears (fig. 13).

It slowly completed itself by dehiscence of the dermal tissue; and thus an external and internal nostril were formed. The pterygo-palatine bar filled up the next part of the cleaving space; and the remainder was filled in below by fibrous membrane, and above by fusion of the heads of the two arches.

The next *secondary* opening, the cleft between the first and second postorals, forms for a little while an obscure, imperfect opening through the skin, in the space in front of the divergence which forms the free hyoid cornu (figs. 1, 3). On the inner side, above, there lingers for a long while a mere membranous tract between the heads of the two arches; and this is in some degree persistent, for the joint-cavity between the "infrahyomandibular" and the ear-sac (Plate IX. figs. 2 & 7) is the remnant of this space.

But opposite the partial external opening, in the space formed by the divergence of the lower third of the bars (figs. 1, 2), there is no filling-in on the inner side. The angular space between the descending quadrate and the hyoid cornu (see figs. 13 & 14) becomes modified (Plate IX. fig. 7, *eu.*), and is completed into a ring by membrane; this is the outer end of the "Eustachian tube;" and this short tube, which opens into the throat within, expands externally into the tympanic cavity, between the "extrastapedial," the "stylo-hyal," and the "quadrate."

Thus the first postoral cleft, although showing so little on the outside, is yet largely developed on the *inner*, and, like the persistent portion of the *preoral cleft*, becomes a structure of the highest importance; the one helps to perfect the olfactory organ, the other the organ of hearing. Not only so, they are intimately related in their higher morphological conditions; they both open on the same plane into the great oral chasm; and the foremost pair form the *bellows* which supply with air the hindermost in their tympanic function. The process of dehiscence by which the nostrils, Eustachian tubes, and ear-drums were formed, was taking place in my first stage (Plate III. fig. 11), before the formation of the heart and liver, and a long while before the intestinal cavity.

I need not recapitulate what has been said of the transitory arches, the branchials: their *clefts*, although by far the most perfect, entirely disappear.

Bibliographical references and comparisons with other Vertebrate types.

Considering that the present paper is merely one of a series, and that its character is very *special*, there will be little need for reference to the invaluable works which have reference to the *general* structure and development of the Amphibia.

The most important help, indeed, has been derived from works treating of *other types*, especially where the *skeletal* parts have been worked out.

In my endeavour to make plain to others what to my own mind was for a long while a labyrinth of difficulties, I have had constant recourse to the works of Professor HUXLEY, especially his "Croonian Lecture," 'Elements,' and "Representatives of Malleus and Incus in other Vertebrata." But such reference is only part of the matter; for we have been workers together in this research.

Another most valued author, whose works have been always with me from my youth up, must be mentioned, viz. the late Professor JOH. MÜLLER; his Monograph on the 'Myxinoids' has been vital to this attempt to make plain the intricate morphology of the Batrachian skull.

I shall make comparisons of the present subject with the structure of various Vertebrate types, and, first, may give

A. *A comparison of the Skull of the Common Frog with that of other "Anura."*

In the Bull-frog (*Rana pipiens*, L.), with a perfect general agreement, there are variations in particulars of the utmost interest*.

In the Bull-frog the upper labial is both notched and fenestrate, but it is never cut through as in the Common Frog, in which it was single at first. This corresponds with what is seen in the Lamprey (MÜLLER, 'Myxinoids,' pl. 4. figs. 2–4, N).

The external angles, however, are almost segmented off, showing a tendency to form the "rod-shaped appendage of the labial ring" (MÜLLER, *op. cit.* pl. 4. fig. 2, Q); the lower labials are much more developed than in the common kind, and are relatively nearly as large as those of the Lamprey. Another myxinoid character in the Bull-frog is the large amount of free trabecular horns left in front of the nasal sacs after the latter are well formed; they persist also in the adult as free leaf-like flaps, the divided counterparts of the Lamprey's emarginate "ethmo-vomerine plate" (MÜLLER, *op. cit.* pl. 4. figs. 2–4, L).

Full twice as many bones, "ectostoses" and "parostoses," are found in the Bull-frog as compared with the common kind; these especially abound in and on the "suspensorium," and have been very instructive to me in working out the Grey Frog, enabling me to speak more confidently of *ichthyic parts* and *regions*.

One thing must be mentioned in: *R. pipiens* as in *Hatteria* (HUXLEY, "On the Representatives of the Malleus and Incus," p. 397, fig. 4), the "suprastapedial" retains its *secondary* fusion with the auditory sac. The cartilaginous "annulus tympanicus"

* I am now drawing from unpublished material: the dissections and drawings of the larval Bull-frogs and of a huge Tadpole of *Pseudis paradoxa* were made from specimens given me by Professor FLOWER; the adult Bull-frog I received from Dr. MURIE.

forms a perfect ring in the Bull-frog, its ends becoming completely confluent; its "pterotic" and "epiotic" regions, although soft, are as well developed as in the "Teleostean" Fishes.

One thing appears to be quite unique, although it will perhaps turn up in some other type, and perchance in the extinct "Labyrinthodont;" this is the presence of an *anterior* "parasphenoid," the fore part of the "rostrum" being separately ossified.

A similar breaking-up of centres is seen in the palatines, maxillaries, "infrahyomandibulars," &c.

My great use of the larval *Pseudis* has been to obtain a thoroughly satisfactory elucidation of the formation of the nasal alæ and septum; and in it I first clearly saw that the pterygo-palatine band was an entirely *secondary* growth: in this type there is another "connective" *in front of* the nasal opening, from the inner angle of the quadrate to the trabecular horn. In *Bufo vulgaris* the "extrastapedial" is shaped like a peltate leaf; its "suprastapedial" is small, terete, and *free*; its "medio-stapedial" bar is very long, and is ossified by a small shaft bone where it passes into the "extrastapedial," and the rest by a larger shaft.

The "stylo-cerato-hyal" is very large in the larval condition; it undergoes the same morphological changes as in the Frog, and ultimately *coalesces* with the auditory sac, as in the Mammalia.

B. *Comparison of the Frog's Skull with that of various ichthyic types.*

1. With *Petromyzon marinus*.—In the larval stage ("*Ammocœtes*") as given by Müller (*op. cit.* pl. 4. figs. 6–10), the "trabeculæ" have coalesced, but the "horns" have not budded out; thus it is later than my second stage, and earlier than the third. The author has not shown the other facial bars; the first pair (D), as here figured, have already become nearly straight, and, besides uniting by the "anterior commissure," they have coalesced with the "investing mass" (*d*).

In the adult (*op. cit.* figs. 2–4) an amount of morphological change has taken place wholly beyond what might have been expected in so low a Fish. The three foremost pairs of arches have coalesced with each other above, and also with the investing mass. In front of the "commissure" the trabeculæ have developed an azygous bilobate plate, the equivalent of the symmetrical "horns" of the larval Frog; and their immediate successors, the mandibular bars, have each sent forwards a styloid rudiment of Meckel's cartilage (fig. 2, *i*), which, however, like its trabecular counterpart, is not segmented off. The pterygo-palatine bar (*I*) is as large as in my "fifth stage." The hyo-mandibular (*i'*) diverges early from the mandibular pier, with which it is largely confluent above; it has cut off a "stylo-cerato-hyal" (*i''*).

The branchial arches (*d,d,d*) are tied together above by a continuous "connective," which runs into the fused roots of the mandibular and hyoid arches. They are converted into an exquisite piece of basket-work by being extensively bound together by similar *secondary* growths. (Owen, 'Lect. Comp. Anat.' p. 52, fig. 11.)

So much, at any rate, may be said of the skull and face of the Lamprey as read in the light of the development of the same parts in the Frog; they mutually illustrate each other; and in my ten stages, whilst the earliest is two steps lower than the Ammocœtine larva, as given by MÜLLER, yet, if the oldest stage of the Frog's skull (see Plate IX. figs. 6 & 7) be compared with that of the adult Lamprey, it is evident at once that, notwithstanding its almost *mammalian* metamorphosis of certain parts, much that is truly *petromyzine* remains in the old Frog.

2. *With the Skull of* Chimæra.—The skull and face of this remarkable type may be illustrated by the earliest conditions of those of the Tadpole.

In the Chimæroids the excessive growth of cartilage (MÜLLER, *op. cit.*; HUXLEY, Elem. pp. 195–197, figs. 77, 78) has obliterated all traces of distinction between the "investing mass," "trabeculæ," a large part of the first and second postoral arches, and the auditory capsules.

The space formed by divergence of the trabecular and mandibular bars is entirely filled in by cartilage, so that the "orbitar process" is not a distinct flap from the pterygo-palatine; and the "subocular fenestra" is also completely occupied by the same growth. Thus there is no distinction between the "metapterygoid" and "mesopterygoid" regions (HUXLEY, fig. 78, *D*); and there is no boundary between the prefrontal and palatal regions. The "trabecula rhorns" are thoroughly involved in the nasal walls; but they send out symmetrical and also single outgrowths as "snout-cartilages" (MÜLLER, *op. cit.* pl. 5, fig. 2, *i, h*). The mandibular arch also developes an outgrowth below as in the Lamprey; but this is segmented off as in the Frog: this thick mass of cartilage (C) is "MECKEL's rod." The third (hyoid) arch is largely confluent with the second; but in HUXLEY's figure of *Chimæra monstrosa* a groove is shown on the inner side (fig. 78); this is the remnant of the upper half of the first postoral cleft; yet the parts before and behind it, the metapterygoid and hyo-mandibular regions, are thoroughly confluent. As in the Lamprey and Frog, the lower diverging half becomes segmented off from the second postoral as the hyoidean apparatus (MÜLLER, fig. 2, *k,l,m*; HUXLEY, fig. 77, *o*). There are many upper and lateral "labials;" but there is a large single horseshoe-shaped lower piece (MÜLLER, *c*).

The trabecular floor of the skull (HUXLEY, fig. 78) is thin; and from the commissural region there proceeds a transverse wall separating the *cranial* from the nasal cavity, exactly as in the Tadpole.

The great "upper fontanelle" is entirely roofed-in with solid cartilage, which is continuous with the shelving nasal roof (*Na*): thus in *Chimæra* we have middle and lateral ethmoids, with their upper alar growths all marked out, and also the Batrachian *trabecular floor*.

3. *With the Skull of Sharks and Rays.*—In *Squatina* (MÜLLER, *op. cit.* pl. 5, figs. 5, 6; HUXLEY, Elem. p. 198, figs. 79, 80) the trabeculæ are entirely involved in the skull and nasal sacs, so that the front part is short and transverse as in the Frog. The labial plate was broken up into two rays on each side above, and one below. But the things

of interest are the large size of the pterygo-palatine band, the abortion of the first post-stomal arch down to the giving off of the pterygoid (Müller, *a*; Huxley, *h*), the immense Meckelian bars, and the equally immense "hyo-mandibular," which ends abruptly at the point where the lower half of the bar bent backwards, to be segmented off as the free hyoid crus (Huxley, fig. 80, *g*, *Hy*). The suspensorial part of the palatine rolls freely on the prefrontal part of the trabecula; and it sends forth a large "prepalatal" projection, like that of the Frog, but much larger.

In the Skate (*Raia*) the trabecular horns, after they have coalesced, give off a long "prenasal" or snout-cartilage, as in the Bird; in the Sharks, as a rule, there are besides this an additional pair growing from the nasal walls, and converging in front to the tip of the azygous rod, to form the skeleton of the "breakwater."

The metapterygoid region of the mandibular arch is not always absorbed; in *Narcine* (Müller, *op. cit.* pl. 5. figs. 3, 4) a slender *f*-shaped free metapterygoid exists above the pterygo-quadrate region, and the roof of the palate has in it a pair of *free* cartilages. Here the upper labials are single on each side* above and below.

My third stage well illustrates the "placoid" type of skull, especially if we suppose the metapterygoid to be either aborted or segmented off (see Plate IV. fig. 7, and Plate X. fig. 13). If this stage be examined, it will be seen that the tops of the trabecular and mandibular bars are as yet unconformed to the membranous brain-sac: compare this state of things with what is seen in *Carcharius* (Huxley and Hawkins's Atlas, plate 5. fig. 4), where a bowed band of cartilage grows out behind the optic foramen, and regains the skull in front of the hyo-mandibular. Here, I doubt not, the upper free ends of the two foremost facial bars had coalesced by a "connective," and had passed into the "investing mass" behind; the trabecular bar then grew inwards to form the subcranial beam, whilst a large portion of the second bar became absorbed, leaving only the quadrate angle with its huge forthstanding pterygo-palatine in front, and having the equally large Meckelian segment articulated to it below.

4. *With the Skull of the Sturgeon and* Spatularia.—In *Spatularia* (Müller, *op. cit.* pl. 5. fig. 7; Huxley, Elem. p. 202, fig. 81) the palatine freely moves on the prefrontal region of the highly modified "trabecula," and the metapterygoid region is wholly absorbed, as in the generality of the Placoids. But the second postoral arch is divided in a new manner. The entire bar is divided into three main portions, namely:—a "suprahyomandibular," which is very large; then a shorter piece, which is the infrahyomandibular" and "symplectic" in one; and then the free "hyoid cornu" (Müller, *op. cit.* pl. 5. fig. 8, *d, f, h*). The projecting lower end of the "suprahyomandibular" is the "opercular lobe," and carries the fan-shaped radiating "opercular."

On the whole the Sturgeon agrees with *Spatularia*; but there are important differences. I cannot find space to describe the immense but quite intelligible modification of the trabecular bands (Müller, *op. cit.* pl. 9. fig. 10). The pterygo-palatal con-

* These two pairs of cartilages appear to correspond to the azygous lozenge-shaped piece in the Sturgeon (Müller, pl. 9. figs. 10 & 11 A, *b*).

nective ($c\,e$) is wholly loosened from the prefrontal region and runs back into the severed quadrate angle, below which is articulated the short, thick Meckelian rod (p). But above the severed quadrato-pterygoid there is, not as in *Narcine*, two pairs of cartilages, but one lozenge-shaped piece (b), evidently the two "metapterygoïds" in one. This marvellously metamorphosed mandibular arch is followed by a most massive hyoidean apparatus, the result of segmentation and overgrowth of the second postoral bar. The little "suprahyomandibular" of the Frog is here represented by a thick rib of cartilage having an ectosteal sheath near its upper condyle (MÜLLER, fig. 10, M′, M, N; HUXLEY, f, g; HUXLEY and HAWKINS, pl. 5. fig. 3^b). The broad unossified part carries the "opercular" behind, and, below, ends in a round condyle, which is tied to the cup on the top of the next segment. This shorter but equally massive piece is invested with bone, and has articulating with it, behind, a nodule of cartilage; this is above the middle, and separates the "infrahyomandibular" and "symplectic" regions; it is the "stylo-hyal" (see the figs. in OWEN's Lect. Comp. Anat. vol. ii. p. 131, fig. 43, *between* Nos. 28 & 40). To the stylo-hyal is articulated the shortish, partly ossified, inferiorly segmented cerato-hyal. (OWEN, *ut supra*, No. 40; HUXLEY and HAWKINS's Atlas, pl. 5. fig. 3^b, *Hy*).

Here, in the Sturgeon, apparently for the first time, that segment is found in the second postoral bar which answers to the Mammalian "incus;" *its prototype* is the Sturgeon's "suprahyomandibular."

5. *With the Skull of* Lepidosiren.—Save for the addition of bony investing plates, the skull in this type answers very closely to that of the Chimæroids (see HUXLEY, Elem. pp. 207–210, figs. 84, 85). There is the same filling-in of the gaping space between the first two bars by the "pterygo-palatine connective;" and the piers of the first two post-orals are entirely confluent; the Meckelian and hyoidean rods are similarly free. I find a pair of oval upper labials attached to a short azygous "snout-cartilage" ("*prenasal*").

6. *With the Skull of Teleostean Fishes.*—As I hope to make the Teleostean my next subject, I shall merely refer to what Professor HUXLEY has done with regard to the structure of the skull in very young *Gasterostei*. In his earliest stage (Croon. Lect. p. 29, fig. 8, left-hand figure, and Elem. p. 185, fig. 72 A.) the trabeculæ, investing mass, and auditory sacs have become confluent, the commissure, which is the foundation of the ethmoid, is complete, and from this conjoined part of the trabeculæ there proceed a pair of short emarginate "cornua trabeculæ." The first postoral bar (mandibular), instead of attaching its free, rounded, upper end to the trabeculæ and investing mass, has retained its freedom, and has somewhat descended from the skull.

The auditory sac has grown over the top of the second postoral; and it has expanded into a pedate form above, to embrace the sac *antero-externally*. MECKEL's rod, not shown in this figure, has been differentiated from the quadrate end of the mandibular arch, which has sent forward a triangular *pterygo-palatine* "connective," the apex of which touches the prefrontal region of the "trabecula."

Let this stage be compared with my third (Plate IV. fig. 7, and Plate X. fig. 13), and

it will be seen that, whilst the first and second postorals are similarly distinct, yet there is a great difference between the Fish and the Frog. In the Fish the second postoral not only *diverges*, it *also bifurcates*, an anterior slender fork passing down close behind the preceding bar in front of the gaping space (*Sy.*), and a posterior stout fork passes behind this space (*Hy.*); the slender "symplectic" fork ossifies separately (Elem. fig. 72, B, C, *Sy.*); another bony sheath encloses the broad-topped upper piece, "hyo-mandibular" (*H.M.*); the top of the diverging bar becomes segmented off and ossified as the "stylo-hyal;" and the remainder becomes a thick rib of bone, from which a distal segment is cut clean off as the "hypohyal." Both this latter and the larger piece are ossified by two centres each. Above the process which becomes the "stylo-hyal" there is a knob, the "opercular process," the part which becomes the "extrastapedial" of the Frog. The synchondrosis between the "hyo-mandibular" and the symplectic of the Teleostean Fish does not correspond with the joint-cavity which *passes through* the "hyo-mandibular" itself in the Sturgeon and the Frog.

The symplectic is a mere rudiment in my third stage of the Frog, but it developes more afterwards; it is persistently free from the quadrate in most Teleosteans, but early coalesces with that part in the Eels (e.g. *Anguilla acutirostris*): in these Fishes the pterygo-palatine connective is early aborted, and is feebly indicated in very young individuals by a delicate rod of ossified membrane, the "pterygo-palatine" bone.

C. *Comparison of the Frog's Skull with that of the Urodelous Amphibia.*

These lower forms of Amphibia lie between *Lepidosiren* and the "Anura." They agree largely with the former; but the pterygo-palatine cartilage is very much aborted, as in the Eel. There is a "stapes;" but there is no metamorphosis of the top of the hyoidean bar to form any secondary elements to the "middle ear."

D. *Comparison of the Frog's Skull and Face with that of the "Sauropsida."*

1. *With the Reptilia.*—This comparison has been in some degree anticipated by my use of Professor Huxley's terms, as given in his paper on the prototypes of the Malleus and Incus (Zool. Proc. 1869).

My first and second stages throw an unexpected light on the *hyo-stapedial* structures of *Sphenodon* (*op. cit.* p. 397, fig. 4). The second postoral of this Lizard has coalesced and retained its coalescence with the auditory mass. It has not segmented itself into upper, middle, and lower parts, but the "supra-" and "infrahyomandibular" regions are permanently continuous with each other and with the "stylo-cerato-hyal." The semicircular scooped "extrastapedial" (*E.St*) is the "opercular process;" and from this there grows backwards a "medio-stapedial" bar, which is continuous with the cartilage that is ossified to form the columelliform rod of the "stapes:" here the stapedial plate early formed a union, by means of a *secondary connective bar*, with the continuous (unsegmented) hyoid arch.

With regard to the "pterygo-palatine connective," it may be remarked that the Reptilia develope but little cartilage, and that of a very simple type, in this region; it is

almost entirely formed of membrane, and then afterwards of membrane bones. Where, as in Chelonians, and especially Lacertians, something more solid is early developed, it soon ossifies, and becomes curiously modified as the "epipterygoidean columella." Not only the Reptilian group, generally, but the Lacertilians themselves, vary in their facial structures much more than the so-called Bird-Class; three of the "Families" may suffice for illustration. In the "Varanians" (e. g. *Psammosaurus*) the long outgrowth of the stapes is only differentiated from the top of the second postoral by stoppage of ossification, as in the Crocodile and *Sphenodon*; its *stapedius* muscle sends its tendon through the two forks of the "suprastapedial" to the end of the spatulate "extrastapedial;" and the cartilage sends down an "infrastapedial" close in front of the stapedial shaft. The long, flexible stylo-hyal is free as in the newly metamorphosed Frog, and articulates with the cerato-hyal below (HUXLEY and HAWKINS, "Atlas," pl. 8. fig. 15, *b*). Here the Ichthyic and Batrachian structures are but thinly veiled.

In the Chameleons (*C. vulgaris*) the "membrana tympani" is abortively developed; there is no "fenestra rotunda," and the stapedial connective *articulates* with the top of the second postoral, a joint-cavity intervening between it and the "medio-stapedial" process. Nearly all the suprastapedial is reduced to membrane; there is no distinction between the extra- and the infrastapedial regions.

In the Cyclodonts (*Cyclodus nigro-luteus*, and *Trachydosaurus rugosus*) there is a "fenestra rotunda;" but the "membrana tympani" is a mere band, as in *Sphenodon*. Here the unossified end of the stapedial shaft is bulbous, and is loosely attached to a feeble unossified "hyo-mandibular," which ends above in a very short, rounded "suprastapedial" process; a feeble aculeate "extrastapedial," grows from its side, and it ends below in a bluntly styloid "infrastapedial." These types are introduced to illustrate what I have said of "secondary connectives;" in the Crocodile, *Sphenodon*, and Monitor the stapes passes into the hyoid pier just as the quadrate region of the second facial bar passes into the prefrontal region of the first in the Frog.

In the Chameleon and Cyclodont the stapes sends forwards the "connective," which, however, does not pass into the hyoid, just like the pterygo-palatine connective of most of the Orders of the Fish-class.

2. *With the Skull and Face of Birds.*—The parts under especial consideration are described by Professor HUXLEY in his above-cited paper (pp. 398, 399); but I cannot agree with one expression—namely, that the "suprastapedial" is represented *only* by fibrous tissue; I would say *principally*.

The broad bent end of the infrastapedial (*op. cit.* fig. 5, *I.S*) is not unfrequently ossified, and much more developed than in the Fowl; this bony centre is not segmented off, and yet it is evidently the stylo-hyal: it is hatchet-shaped in *Gymnorhina tibicen*, spatulate in *Fruticicola rubetra*, and styloid in *Sula alba*.

In my last paper (Philosophical Transactions, 1869, pp. 755–807, Plate LXXXI.–LXXXVII.) I have not worked out the relation of the stapes to the hyoid arch in the first stage. In the second (Plate LXXXI. figs 5 & 9) the stapedial plate is already continuous with the aborted hyoid pier: that rudiment has already three processes, the first appearance of the

"supra-," "extra-," and "infrastapedials." At page 759 I have said of the "trabecular horns" that they "do not coalesce to form the intermaxillary axis or "prenasal cartilage." This is quite true; for the cornua have coalesced before that rod is developed. I did not, however, get a full insight into the formation of the nasal labyrinth from my researches into the structure of the Chick; the huge Tadpole of *Pseudis paradoxa* gave me the true insight. I here make correction of my imperfect and somewhat erroneous description of my first stage (Plate LXXXI. figs. 1, 2, p. 759).

The "first stage" in the Fowl answers to the third in the Frog-embryo, so that the parts are already highly modified. The first pair of visceral arches (trabeculæ, *tr.*) are by the "meso-cephalic flexure" bent over and backwards into the shape of hooks, so that two-thirds are seen above (fig. 2) and one-third below (fig. 1). The apex of each bar is seen projecting outwards at the part where it has coalesced with the investing mass (*i.c.*); this free apex becomes the "lingula" (Plate LXXXII. fig. 3, *l.g.*). In front of the oval pituitary space a large "commissure" has tied the two bars together for a large space, part of which can be seen from below (fig. 1); in this aspect we see the free "cornua" which have grown into the "naso-frontal process" (*f.n.*) from the conjugated part of these bars.

The pituitary space (*py.*) soon shortens, and the commissural region rapidly lengthens; the trabeculæ, themselves thus united, form the thick base of a large crest, which intervenes (fig. 3, *eth.*, *p.s.*) between them and the floor of the membranous cranium; this secondary crest is the "orbito-nasal septum;" and the roof of the nasal sacs grows from its top. The first visceral arch has become widely divergent from the second; and between them an arcuate "connective" ("pterygo-palatine," *pa.*, *p.g.*) has appeared; it is very slightly connected with either bar, and scarcely undergoes chondrification. The first, second, and third visceral bars neither coalesce with each other nor with the auditory sac.

E. *Comparison of the Frog with the Mammal.*

If the figures given by Professor HUXLEY (Elem. p. 143) from ECKER'S work be compared with my earliest stages, the complete correspondence will at once be seen between the human embryo and that of the Frog.

The "maxillary process" (*a*) seems, as in the Bird (*op. cit.* p. 139, fig. 57, F, *l*), to be a small additional arch, but it does not chondrify, and is merely a *process* of the second or mandibular bar, and is developed into the palatine, pterygoid, maxillary, and malar bones.

From the middle of the "sella turcica" to the end of the nasal septum, the craniofacial base is formed of the first (trabecular) pair of arches; and if the reader would see what these minute threads of soft cartilage may become by metamorphosis and growth, he should consult the posthumous work of Dr. ESCHRICHT on the Anatomy of the Cetacea, which has recently appeared under the editorship of Dr. REINHARDT* (plate 2 showing the primordial skull of *Balæna japonica*).

* "Ni Tavler til Oplysning af Hvaldyrenes Bygning, udförte til utrykte Foredrag af afdöde Etatsraad Dr. D. F. ESCHRICHT. Med tilhörende Forklaring." Copenhagen. 1869.

The space between the true cranial sac and the trabecular undergirders, which in the Lamprey (MÜLLER, *op. cit.* pl. 4. figs. 1, 3, 9, 10) permits the passage backwards of the azygous nasal pouch (fig. 1, *k''*), reappears by a sort of retrograde metamorphosis in the Mammal; here the pouch, or rather air-cavity, is common to two symmetrical nasal sacs, and not to an azygous sac as in the Lamprey. In the Aye-Aye (OWEN, Trans. Zool. Soc. 1863, pl. 20. fig. 6, *q*) this air-sac completely divides the proper cranial floor from its trabecular addition, up to the very point at which the trabeculæ became confluent with the investing mass in the early embryo. Another return to embryonic conditions is well seen in the complete separation of the "pars petrosa" from the sphenoidal and occipital regions of the skull; this is most exquisitely seen in *Scotophilus pipistrellus*.

In Mammalia the second visceral arch (mandibular) never undergoes segmentation above; the upper part of the head of the malleus answers to the "quadrate" (HUXLEY, "Malleus and Incus," p. 402, fig. 6, *M*), the rest of the head to the articular region, and the processus gracilis to the Meckelian bar*. The "articular" region gives off a manubrium similar to the "opercular process" of the "hyo-mandibular" ("extrastapedial"); and this serves the same purpose, namely to stretch the tympanic membrane.

The third bar (second "postoral") becomes very much subdivided; its segments are well shown in Professor HUXLEY'S figure. The upper part of the first postoral cleft is converted into the joint-cavity between the "malleus and incus;" the rest forms the tympanic cavity and Eustachian tube. The "suprastapedial" region is here represented by the head and "short crus" of the incus, the "medio-stapedial" by the "long crus," the "interstapedial" by the "os orbiculare;" there is no extrastapedial.

The remainder of the hyo-mandibular region or upper half of the primary bar is a small "infrastapedial" ("infrahyomandibular") rod (*a*), which runs parallel with the tendon of the "stapedius" muscle. The stylo-hyal" (*st.h*) is separated from this little rod, from the lower end of which it was segmented at an earlier period (Mr. HUXLEY'S figure is from a fœtus five months old); these segments are now separated by nearly all the length of the belly of the "stapedius."

The decurved and transverse position of the infrastapedial is evidently due to the upward movement of the freed "stylo-hyal;" most of the tissue of the bent and broken part of this visceral arch has been converted into the tendon and belly of the "stapedius." The head of the stylo-hyal, overgrowing the bent portion above it, has ascended to the "opisthotic" region, and has completely coalesced with it as in the Common Toad; this is a coalescence of two cartilaginous tracts totally unrelated morphologically.

In the Mole (*Talpa europæa*) the part of the malleus articulating with the incus is half severed from the rest, forming an upper *quadrate* lobe; the "short crus" of the incus is very feeble. A similar well-marked quadrate lobe is seen in the Shrew (*Sorex tetragonurus*); and it has a still smaller "short crus" to the incus; the "orbiculare" is, relatively, nearly as large as in the Frog.

* The fore part of the Meckelian rod is ossified separately as the "inferior intermaxillary rudiment" (see CALLENDER, Philosophical Transactions, 1869, Plate XIII. fig. 6, *a*, p. 170); there the Batrachian structure is repeated; the same "mento-meckelian" ossicle may be seen in old Sturgeons.

Concluding Remarks.

The foregoing comparisons are given under the impression that they may be of some immediate use to the student, although they must be given on a larger scale after several more types have been worked out.

Moreover it is no little relief to the worker himself thus to be able to rise from a task which has used the leisure of well nigh two years; such an expatiation is necessary before the renewal of a similarly persistent concentration to new work of the same kind. He contends, also, that the mind both of the reader and the writer will be strengthened as well as refreshed by a wider view, and that each separate type will then be seen in the light of many other types. Indeed thus alone will it be possible to obtain broad views in vertebrate morphology, "as a man conveniently placed in some eminent station may possibly see, at one view, all the successive parts of a gliding stream; but he that sits by the water's side, not changing his place, sees the same parts only because they succeed, and those that pass make way for them that follow to come under his eye."

I must confess to having subjected myself to this mole-like burrowing into so limited a territory that I may obtain fresh material for ratiocination—"that way of attaining the knowledge of things, by comparing one thing with another, considering their mutual relations, connexions, dependencies, and so arguing out what was more doubtful and obscure, from what was more known and evident."

To have worked out one single species in this way may seem to be but like the forming of a single track in a primæval forest; yet when well cleared, so perfect is the unity of each subkingdom, by such a narrow path the worker is "regularly led on through the labyrinths of Nature, when still new discoveries are successfully made, every further inquiry ending in a further prospect, and every new scene of things entertaining the mind with fresh delight." Leaving for a while the suggestive morphology of the Frog, it may be worth while for the palæontologist to reflect upon the empty spaces in the great vertebrate circle which are darkly but really revealed by what is seen in both the earliest and the latest stages of the Frog.

Territories vacant, but larger far than those now occupied by family after family, and order after order, have been suggested to me by my long attention to the growth of the skull and face in this Amphibian.

Empty spaces of almost indefinite extent seem, to my mind, to stretch themselves *below* the Myxinoid prototypes of the Batrachia, and *above and beyond* the Frogs and Toads, in the direction of the Mammalia.

This last space is wholly undefined, and no light has yet penetrated its deep abyss, in which lie buried the fundamental Mammalian types. The lowest Mammals known to us, the Platypus and the Echidna, may be fundamental to the Edentata; they are not, they cannot be, to the Marsupials, the Insectivora, and the Rodentia.

Between the Monotremes and the Batrachia we certainly have the Sauropsida—Reptiles, and Birds; but I am bold to say that no Sauropsidan lies in a direct line between, forms any part of a *phylum* which should connect together, the nobler Amphibian forms

and the lowest Mammal. On the Mammalian side of this empty space we must suppose a form which should be general to the whole class; I need not say that no such form is extant. The extraordinary and unlooked-for morphological elevation of the adult "Anuran," an elevation in very important structures attained by no Reptile or Bird, and which brings it almost into contact at certain points with the Mammalian margin, is very suggestive. Such a discovery sheds a certain but feeble light, useful though faint.

The fact that the higher Batrachia go on metamorphosing until several of their structures are so perfect as to require but the gentlest modification to make them fit for the Mammal, does not require one to suppose that the Toad and the Frog lie in the direct route from the Ichthyic to the Mammalian types. That such power of variation, such aptitude for transformation exists in these essential but metamorphic Fish, suggests the probability that some of the very earliest of the Amphibia, filial perhaps to forms far lower than the Lamprey, did not stop at the last metamorphic stage of an Anuran, but changed still further, and thus laid the foundation of the higher classes.

The formation of the *amnion* and the *allantois* in the early stage of an embryo may have been a *sudden variation*; when once developed, however, the essentials were present for the development of a *Reptile* ("Sauropsidan") as distinct from a mere Amphibian.

We are all looking for further traces of the *phylum* which shall complete the connexion between the cold-blooded, scaly types of "Sauropsida" and the feathered, warm-blooded Birds; even should this never be attained to, yet no one will doubt that it has existed.

An Amphibian, full of latent power of change, need not have taken in its metamorphosis merely the path that leads to the Reptile and the Bird; for the least deflection at first may have sufficed to bring about all the differences which now, in this late *human* period, we see between the Mammal and the Bird. These warm-blooded groups are huge culminating branches of the tree of Vertebrate life; yet it is not a wild fancy to suppose that they may once have existed together in the same common trunk.

So much for the vacant space *above* the Myxinoids; the *lower* is much larger and even more pathless.

The lowest existing Fish but one is the "Myxinoid" (Lamprey, Hag, *Bdellostoma*); between it and the lowest known Vertebrate, the Lancelet (*Amphioxus*), there is a gap the extent of which has never been imagined; and yet even the Lancelet itself is not necessarily the actual boundary form.

I have shown in my comparisons that the larval Lamprey (*Ammocœtes*) is only a little lower than my third stage of the Frog, whilst my fourth stage answers very closely to the adult Lamprey.

Let us imagine three families of extinct Fishes *below* the Lamprey:—first, a group arrested *as to type* at the Ammocœtine stage; secondly, a group which may be morpho- logically represented by my second stage of the Batrachian embryo; and thirdly, a group no higher than my first stage.

These three "Families" may have abounded in genera and species, and have been as

perfectly in harmony with their surroundings as the highly specialized and noble Ganoid Fishes. How far these groups would tend to fill up the space between the *Amphioxus* and the simplest of their species, I need not say. Every anatomist will at once see that a creature no higher in type than the unhatched embryo of the Frog is yet an untold distance in advance of the Lancelet, which yet is only the *known lowest* of the great Vertebrate subkingdom.

My next subject will be the Salmon, a subtypical "Teleostean;" after that I hope to work out one of the lowest of the placental Mammalia, namely the Guinea-pig.

The present paper has thrown some light upon the obscurer early stages of my last subject, the Fowl; this I have spoken of in the comparisons which have been made of the various types.

Meantime, if any one desires to earn the lasting gratitude of morphologists, let him work out the development of a "Myxinoid," a Lamprey, or, still better, the *Bdellostoma* (see MÜLLER's "Myxinoids," pls. 1, 2, 3, 6, 7, 8). In the last type, especially, the labial cartilages, the facial cartilages and branchial "basket," the axial structures, and the sense-capsules—these might all receive the most beautiful and invaluable elucidations if the early stages were known of a creature so low in the scale and yet at the same time so intensely specialized and modified from its primordial condition. It is impossible for us not to search after the types that arose above the Lancelet; and although they are most probably *nearly all extinct*, yet a clear comprehension of the stages of a *Bdellostoma* would give us pictures, diagrammatic indeed, but essentially true representatives of whole groups of lost "Families" of the simpler types of Fish.

This would be a new joy to the zoologist; but to the morphologist it would be as a lamp, giving the light of a new life to his science; and then would he be willing to break up his last idol, the mere creation of a fanciful transcendentalism, reasoning henceforth about the actual forms presented to him by Nature herself.

Explanation of Abbreviations.

al.n. alinasal.
al.s. aliseptal.
al.s. alisphenoid.
ar. articular.
ar.c. articular condyle.
a.sc. anterior semicircular canal.
au. auditory sac (periotic mass).
a.t. annulus tympanicus.
atl. atlas.
b.br. basibranchial.
b.h. basihyal.
b.t. basitemporal processes.
br. branchial arch.
ct. cutaneous system.
c.a.t. cartilaginous annulus tympanicus.
c.br. cerato-branchial.
cl. 1. 1st cleft.
c.p. claspers.
C 1. 1st cerebral vesicle.
C 1^a. rhinencephalon.
C 1^b. prosencephalon.
C 1^c. deutencephalon.
C 2. mesencephalon, or middle cerebral vesicle.
C 3. 3rd cerebral vesicle.
cr. cranium.
c.tr. cornu trabeculæ.
d. dentary.
d.e. derm.
e. eyeball.
e.br. epibranchial.
e.br. external branchiæ.
e.n. external nostril.
ep. epiotic.
e.pa. ethmo-palatal.
e.st. extrastapedial.
eth. ethmoid.
f. frontal
f.f. fenestral fossa.
f.m. facial muscle.
f.m.x. facial plate of maxillary.
f.n. fronto-nasal process.
fo. fontanelle.
f.r. fenestra rotunda.
f.s.o. fenestra ovalis.
f.w. facial wall.

g.l.t. gelatinous tissue.
h.br. hypobranchial.
h.c. hyaline cartilage.
h.h. hypohyal.
h.m. hyo-mandibular.
h.sc. horizontal semicircular canal.
hy. hyoid.
hy.h. hypohyal.
i.br. internal branchiæ.
i.f.st. infrastapedial.
i.h.m. infra-hyomandibular.
i.n. internal nostril.
inf. infundibulum.
it.st. interstapedial.
iv. investing mass.
iv.c. investing-mass connective.
i.v.p. inner valvular process.
l. labyrinth.
l.d.g. lower dentigerous plate.
l.l. lower labial.
lp. lips.
l.t. labial teeth.
lx. larynx.
m. mouth.
m.c. membranous cranium.
m.d. digastric muscle.
mk. Meckel's cartilage.
m.mk. mento-meckelian.
m.ob. medulla oblongata.
m.pg. metapterygoid.
m.pg.c. metapterygoid connective.
m.s. medulla spinalis.
mx. maxillary.
n. nasal.
nc. notochord.
no. notch.
n.p.x. nasal process of premaxillary.
o.c. occipital condyles.
ol. olfactory sac, or tube.
o.l. C 2. optic lobes.
op. opisthotic.
op.m. operculum.
op.o. opercular opening.
op.p. opercular process.
or.p. orbitar process.

o.s. orbito-sphenoid.
ot. otoliths, or otoconia.
p. parietal.
pa. palatal.
pa.s. parasphenoid.
p.b.c.f. posterior basicranial fontanelle.
p.br. pharyngo-branchial.
p.c.d. pericardium.
p.e. perpendicular ethmoid.
p.fo. posterior fontanelle.
p.g. pterygoid.
pg.p. pterygo-palatine.
phx. pharynx.
p.mx. premaxillary.
p.op. præoperculum.
p.p.p. pterygo-palatal process.
p.p.x. palatal process of premaxillary.
pr.f. prefrontal.
pr.hm. prehyomandibular.
pr.m. promontory.
pro. prootic.
pr.pa. prepalatal.
p.s. presphenoid.
p.sc. posterior semicircular canal.
pt.o. pterotic.
pt.hm. posthyomandibular.
p.t.o. postorbital.
pt.pa. postpalatal.
py. pituitary space.
q. quadrate.
r. rostrum of parasphenoid.
s.c. simple cartilage.
s.hm. suprahyomandibular.
s.mx. septo-maxillary.
s.n. septum nasi.
sn.c. snout-cartilage.
s.n.l. subnasal lamina.
s.o.f. subocular fenestra.
sp. suspensorium.
s.st. suprastapedial.
st. stapedial.
s.t. supratemporal.
st.f. stapedial fossa.
st.h. stylo-hyal.
su.s. supra-phenoidal.
sy. symplectic.
t. temporal bone.
tg. tongue.
th. thyro-hyal.
th.v. theca vertebralis.
t.m. temporal muscle.
tr. trabecula cranii.
t.c. trabecular commissure.
t.s.l. transeptal lamina.
t.ty. tegmen tympani.
u.d.g. upper dentigerous plate.
u.l. upper labial.
vb. vestibule.
v.e.n. valvular process of external nostril.
v.i.n. valvular process of internal nostril.
y. yelk.
z.m.x. zygomatic process of maxillary.
1. olfactory nerve.
2. optic nerve.
5. trigeminal nerve.
7^a. portio-dura nerve.
7^b. portio-mollis nerve.
8^a. glosso-pharyngeal nerve.
8^b. vagus nerve.

Description of the Plates*.

PLATE III.

First Stage.—Embryo of Frog 2–3 lines long; two days before and two days after hatching.

Fig. 1. Front view of head before hatching. × 20 diameters.

Fig. 2. Side view of the same, with ear-sac exposed. × 20 diameters.

Fig. 3. Same as last, with facial arches, as well as ear-sacs, exposed. × 20 diameters.

Fig. 4. Vertical section of head of the same. × 20 diameters.

Fig. 5. Horizontal section of the same, immediately below the cranial cavity. × 20 diameters.

Fig. 6. A similar section taken a little lower down. × 20 diameters.

Fig. 7. Another similar section, still lower down, and showing the floor of the mouth. × 20 diameters.

Fig. 8. A transversely vertical section of fore part of face, seen from behind. × 20 diameters.

Fig. 9. A section, like fig. 5, of a more developed embryo (3 *lines long*). × 15 diameters.

Second Stage.—Embryos 4 lines long, four or five days after hatching.

Fig. 10. Side view of head. × 20 diameters.

Fig. 11. Under view of the same. × 20 diameters.

Fig. 12. Vertical section of the same. × 20 diameters.

Fig. 13. Fore part of horizontal section of the same, from below, and showing the palate and inner nares. × 20 diameters.

PLATE IV.

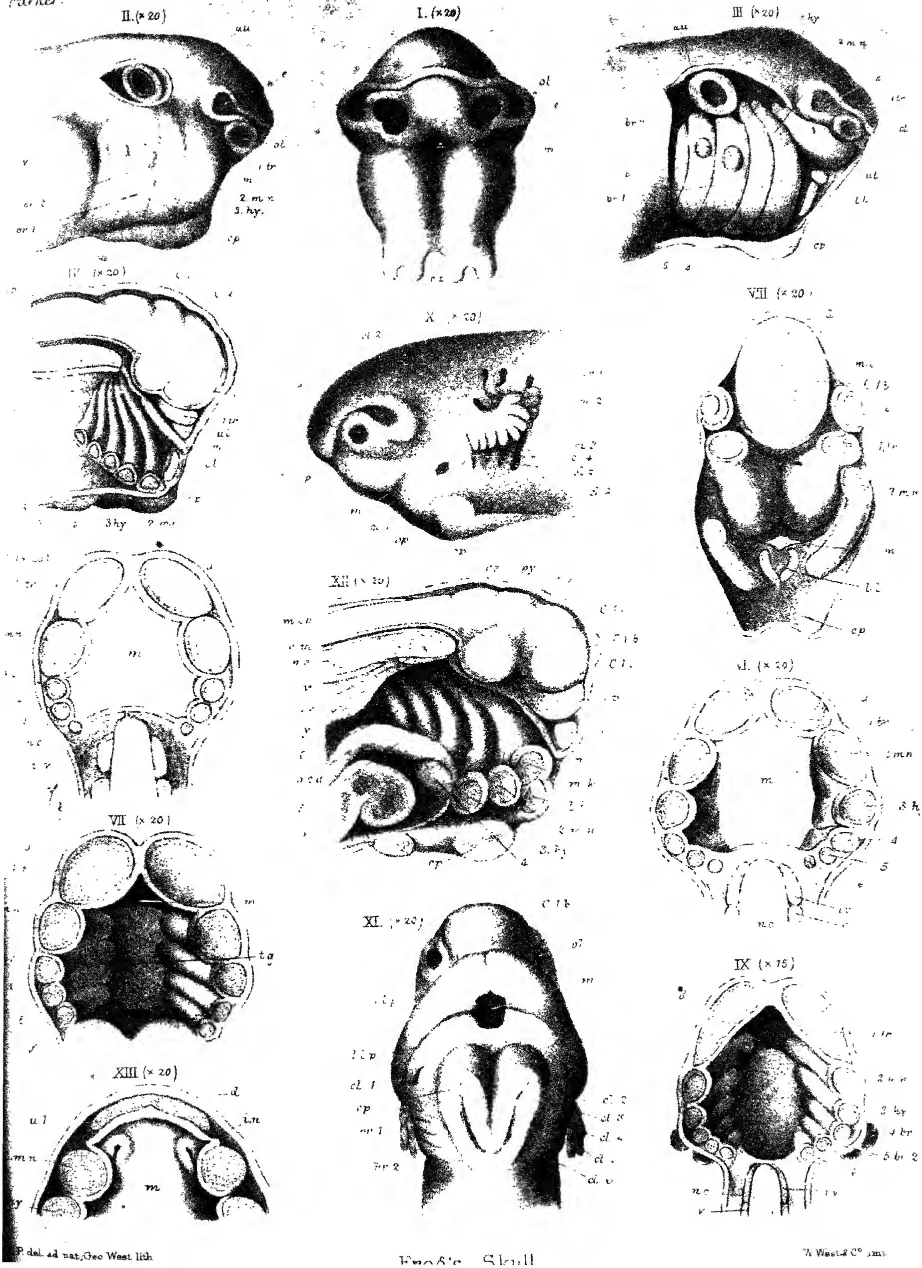
Second Stage (continued).

Fig. 1. Side view of head with sensory organs and facial arches exposed. × 20 diameters.

Fig. 2. Horizontal section of head through eyeballs, ear-sacs, brain, and notochord. × 20 diameters.

Fig. 3. Another horizontal section, dipping a little backwards, so as to expose the yelk-sac, below the notochord. × 20 diameters.

* The hyaline cartilage, whether totally unossified or slightly affected by "endostosis," is coloured *lilac*, the endostosis being shown by a coarser marking than in that which is wholly unossified. The more solid bones, including all true "ectostoses," are coloured ochre-yellow, whilst the "parostoses" and fibrous membranes are not tinted.



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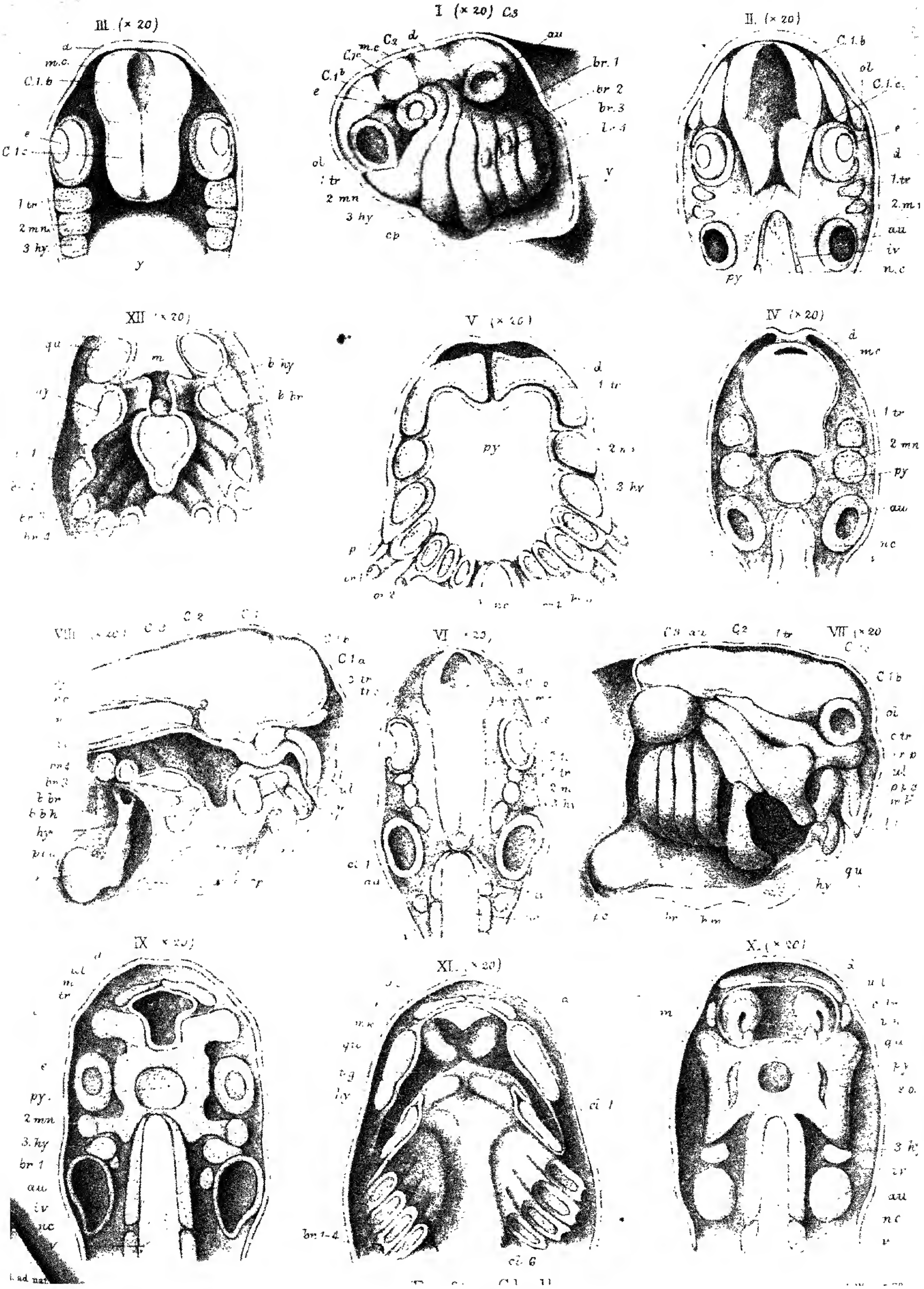
Frog's Skull

W. West & Co imp.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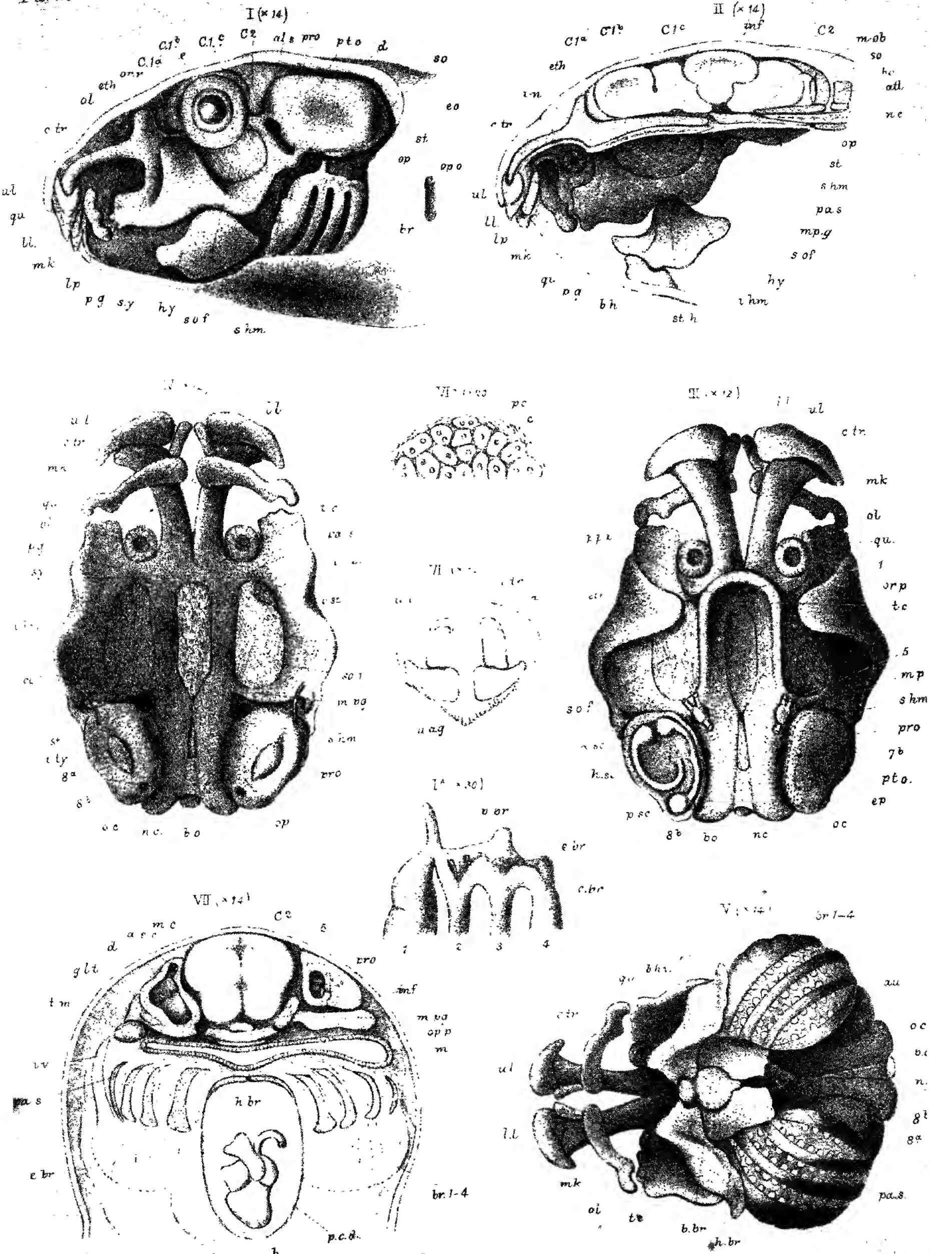


Frog's Skull

W West & Cᵒ imp.

Fig. 4. A similar section, in which the dip is forwards, so as to be below the eye and through the notochord and pituitary body and ear-sacs. × 20 diameters.

Fig. 5. A horizontal section below the brain and through all the facial arches and the end of the notochord: the pituitary body is indicated by dotted lines. × 20 diameters.

Fig. 6. Horizontal section of head, through eyes, ear-sacs, brain, and notochord. × 20 diameters.

Third Stage.—Young Tadpoles 5 lines long.

Fig. 7. Lateral view of head, with eyes removed, and nasal and auditory sacs and facial cartilages exposed. × 20 diameters.

Fig. 8. Vertical section of head, showing brain, notochord, oral cavity, and heart. × 20 diameters.

Fig. 9. *A dissected* horizontal section, showing basis cranii from above; with eyeballs, ear-sacs, upper labial cartilage, and part of oral cavity. × 20 diameters.

Fig. 10. A similar *dissected* section seen from below, and showing the openings of the nasal sacs into the fore part of the palate immediately in front of the pterygo-palatine connectives. × 20 diameters.

Fig. 11. Horizontal section through the facial arches and upper lip of a somewhat more advanced Tadpole, showing the floor of the mouth and its opening. × 20 diameters.

Fig. 12. A similar section lower down; here the mucous membrane has been removed, and with it the tongue and lips. × 20 diameters.

PLATE V.

Fourth Stage.—Tadpoles 1 inch long.

Fig. 1. Side view of head and part of trunk of Tadpole with first appearance of hind limbs, as far as to the left (or azygous) opercular opening. The skin has been removed, and the parts so dissected as to display the cerebral masses and sense-capsules, skull, face, and branchial arches. The hyoid and branchial arches are somewhat drawn downwards for display. × 14 diameters.

Fig. 1ᵃ. Upper part of branchial arches. × 30 diameters.

Fig. 2. A vertical section of the same as fig. 1, showing the right side from within. × 15 diameters.

Fig. 3. Upper view of skull and face of Tadpole somewhat more advanced, so dissected as to display the fundus of the cranium and the cavity of the ear-capsule. × 12 diameters.

Fig. 4. The same as fig. 3, but shown from below. × 12 diameters.

Fig. 5. Lower view of skull, face, and branchial arches of younger Tadpole (same as figs. 1 & 2): the outer "branchiæ" have been removed, and the dentate elevations from which the inner filaments grow are displayed. × 14 diameters.

Fig. 6. No. 1 of a series of vertically transverse sections through the skull and face of the younger Tadpole: this has been taken in front of the mouth, and is seen from behind. × 15 diameters.

Fig. 6^a. Part of the same, being the thin edge of the upper labial cartilage. × 200 diameters.

Fig. 7. Section No. 7, made through the optic lobes (*o.l.*), foramina ovalia (5), and pericardium (*p.c.d.*). × 14 diameters.

PLATE VI.

Fourth Stage (continued), *and part of Fifth Stage.*

Fig. 1. Section No. 2, made through the fore part of the nasal sac (*n*), the mouth, the ends of MECKEL's cartilages (*m.k.*), and through the substance of the "inferior labials" (*l.l.*). × 14 diameters.

Fig. 2. Section No. 3, made through the nasal sacs, close behind the lower labials (*l.l.*), and in front of the quadrate condyle. × 14 diameters.

Fig. 3. Section No. 4, made through the olfactory lobes (C 1^a), the quadrate condyle (*q.*), the orbitar process (*or.p.*), and *behind* the Meckelian and lower labial cartilages. × 14 diameters.

Fig. 4. Section No. 5, made through the prosencephalon (hemispheres) (C 1^b), eyeballs (*e.*), the quadrate condyle (*q.*), and the hyoid arch (*hy*). × 14 diameters.

Fig. 4^a. Part of hyoid arch, seen from behind.

Fig. 5. Section No. 6, made through optic lobes (C 2), infundibulum (*inf.*), metapterygoid (*m.pg.*), and close in front of pericardium (*p.c.d.*) and auditory capsules (*au.*). × 14 diameters.

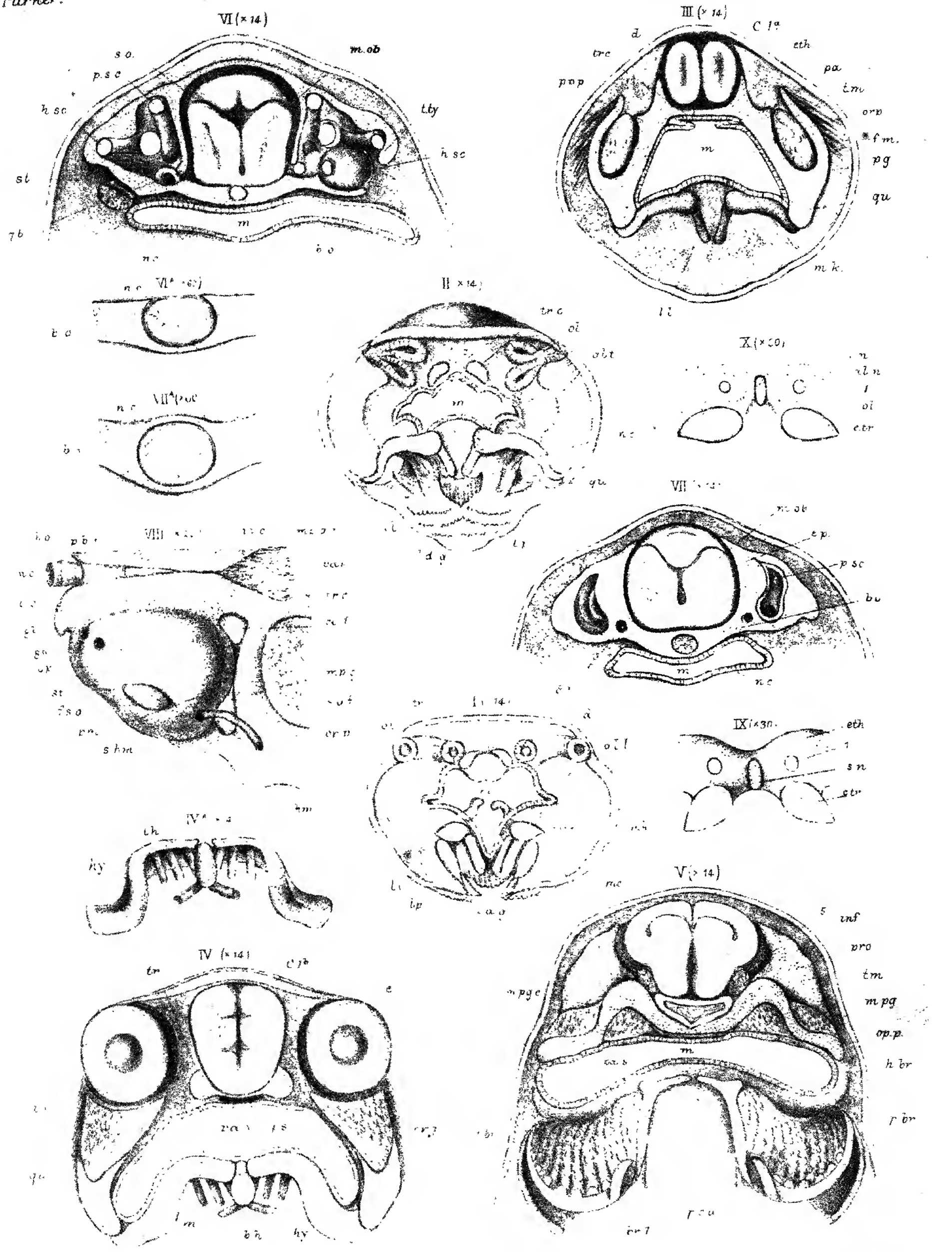
Fig. 6. Section No. 8, made through the medulla oblongata (*m.ob*), the middle of the auditory sac (*au.*), and behind the branchial arches. × 14 diameters.

Fig. 7. Section No. 9, made through the posterior part of auditory sac, in the occipital region, so as to cut through the epiotic eminences (*ep.*). × 14 diameters.

Fig. 8. Part of fig. 4, in Plate V., showing the metapterygoid (*m.pg.*), hyo-mandibular (*h.m.*), and auditory sac (*au.*) from below. × 20 diameters.

Fig. 9. A section of the skull in the 5th stage, showing the trabecular horns and the septum nasi, both in continuity with the front of the ethmoidal wall. × 30 diameters.

Fig. 10. Transversely vertical section of the skull in the 5th stage, a little in front of the ethmoidal wall, showing the trabecular horns distinct from the septum nasi. × 30 diameters.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Frog's Skull

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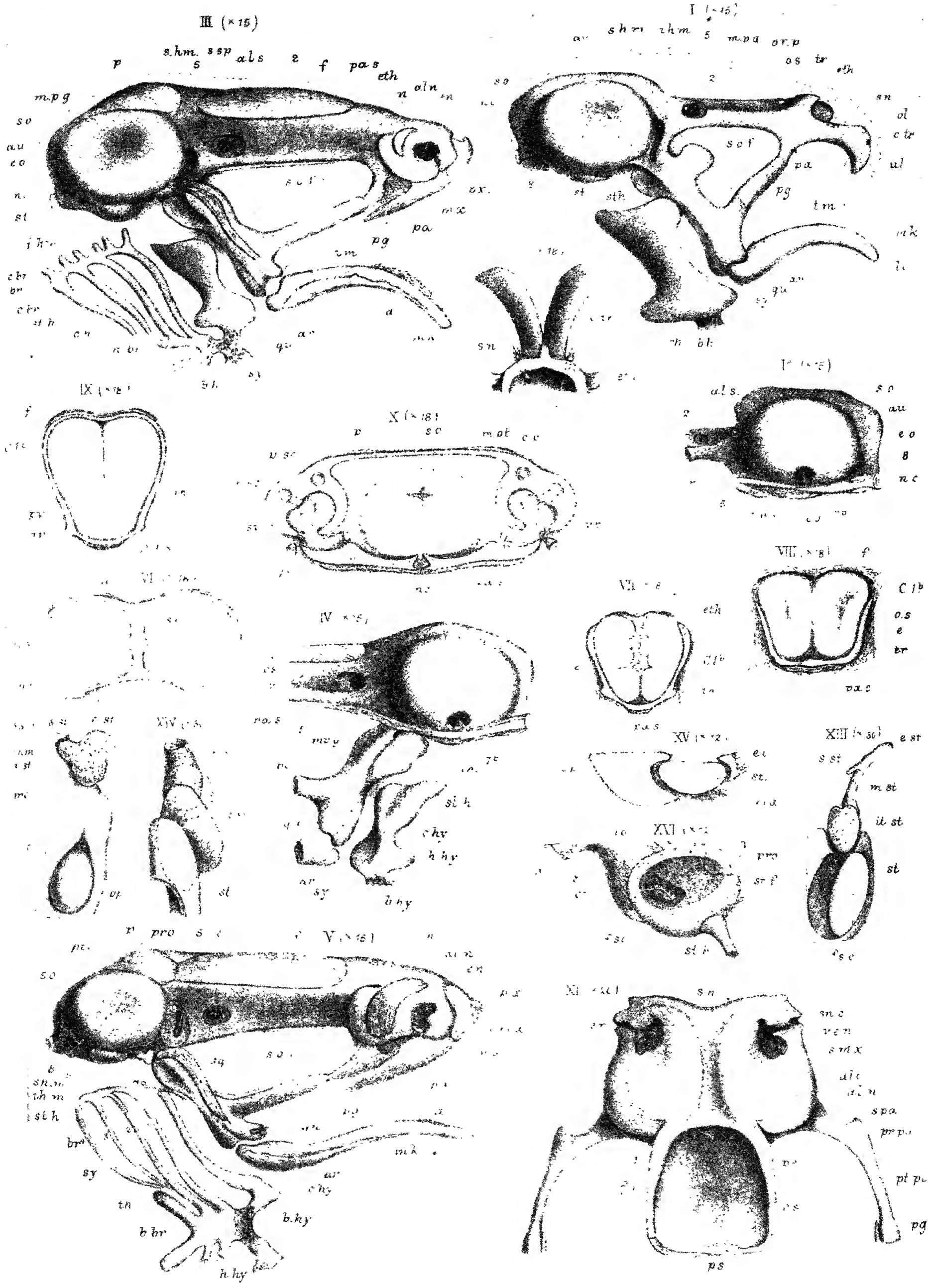


PLATE VII.

Fifth Stage (continued).

Fig. 1. Side view of skull, without the branchial arches, of a Tadpole further advanced, in which the hinder legs had increased in size and the tail had begun to shrink. The dotted lines carried from the foramen ovale (5) to MECKEL'S cartilage (*mk.*) show the direction of the temporal ("crotaphite") muscle (*t.m.*); the hyoid arch (*hy.*) is drawn downwards for display. × 15 diameters.

Fig. 1[a]. Part of the same, showing the auditory capsule (*au.*) from the inner side. × 15 diameters.

Fig. 2. Anterior part of cranium and cornua trabeculæ of the same, seen from above, and showing transverse ethmoidal wall, rudiment of septum nasi, and olfactory crura. × 18 diameters.

Sixth Stage.—Frogs with short tails.

Fig. 3 Side view of skull of a young Frog which had *moulted* the larval skin so as to expose the fore limbs, and in which the tail was reduced to half its former size. The facial arches are drawn downwards for display. × 15 diameters.

Fig. 4. Part of the same, seen from the inner side, the skull having been bisected. × 15 diameters.

Seventh Stage.—Frogs with tails absorbed.

Fig. 5. Side view of skull of a young Frog still further advanced, completely *curtailed*, with the facial arches drawn downwards for display. × 15 diameters.

Fig. 6. Section No. 1, made in a vertically transverse direction through the skull at this stage (7th): this shows the "aliseptal" (*al.s.*) and "subnasal" (*s.n.l.*) laminæ as outgrowths of the "septum nasi" (*s.n.*). × 18 diameters.

Fig. 7. Section No. 2, made through the hemispheres (C 1[b]) and through the skull-wall at the point where the ethmoid (*eth.*) passes into the orbito-sphenoid. × 18 diameters.

Fig. 8. Section No. 3, made through the hemispheres further backward, between the eyeballs (*e.*). × 18 diameters.

Fig. 9. Section No. 4, made through the fore part of the optic lobes (C 2) and the infundibulum (*inf.*), behind the great fontanelle; so that the skull is here completely roofed-in with cartilage. × 18 diameters.

Fig. 10. Section No. 5, made through the medulla oblongata (*m.ob.*) and through the middle of the auditory capsule. × 18 diameters.

Fig. 11. Upper view of anterior half of the same skull with the membrane bones removed. × 20 diameters.

Fig. 12. Part of fig 2, showing "suprahyomandibular" and stapedial cartilages. × 30 diameters.

Fig. 13. Eighth Stage.—Middle-ear chain, with fenestra ovalis and part of auditory capsule, the "stapes" partly displaced. × 30 diameters.

Fig. 14. Part of middle-ear chain (9th Stage) with medio-stapedial and stapedial cut through. × 30 diameters.

Fig. 15. Section through "stapedial plate," "fenestra ovalis," and exoccipital and opisthotic edges of the "fenestral fossa" (old Frog). × 12 diameters.

Fig. 16. Part of the skull of an adult Frog, with stapedial plate removed to show "fenestra ovalis" and "fenestral fossa." × 12 diameters.

PLATE VIII.

Seventh Stage (continued).

Fig. 1. Upper view of skull of young Frog with right mandible attached, and hyoid cornua cut through. × 12 diameters.

Fig. 2. The same, lower view. × 12 diameters.

Fig. 3. Part of fig. 1, suspensorium and auditory capsule. × 24 diameters.

Fig. 4. Part of fig. 2, ditto. × 24 diameters.

Fig. 5. Distal part of MECKEL's cartilage (*mk.*) with dentary plate (*d.*) still distinct. × 45 diameters.

Eighth Stage.—Frogs of 1st *early Summer.*

Fig. 6. Fore part of MECKEL's cartilage (ossified) with part of "dentary" and "articulare." × 45 diameters.

Ninth Stage.—Frogs of 1st autumn.

Fig. 7. Upper view of skull of a young Frog examined towards the end of the 1st summer; right mandible removed. × $7\frac{1}{2}$ diameters.

Fig. 7^a. Part of the same, showing right suspensorium and auditory region. × 15 diameters.

Fig. 8. The same skull, seen from below. × $7\frac{1}{2}$ diameters.

Fig. 8^a. Part of fig. 8, showing lower view of suspensorium and auditory region. × 15 diameters.

Fig. 9. Tenth Stage.—Side view of auditory region of adult Frog with semicircular canals exposed and parts of middle ear displayed. × $7\frac{1}{2}$ diameters.

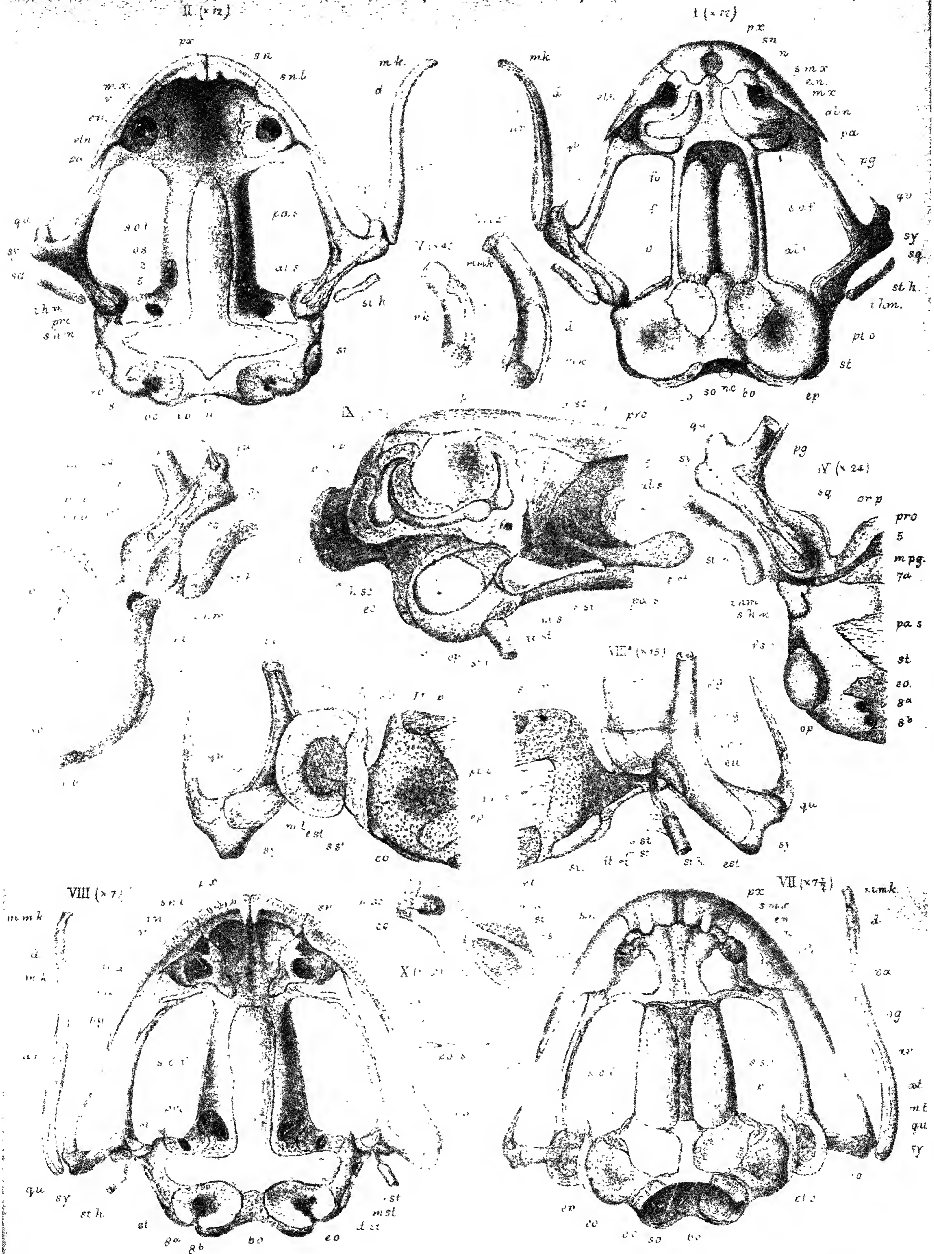
Fig. 10. Tenth Stage.—Section showing "fenestra ovalis" and "stapedial plate" from within. × 12 diameters.

PLATE IX.

Tenth Stage.—Old Frogs.

Fig. 1. Upper view of skull of adult Frog with all the bones *in situ*. × 4 diameters.

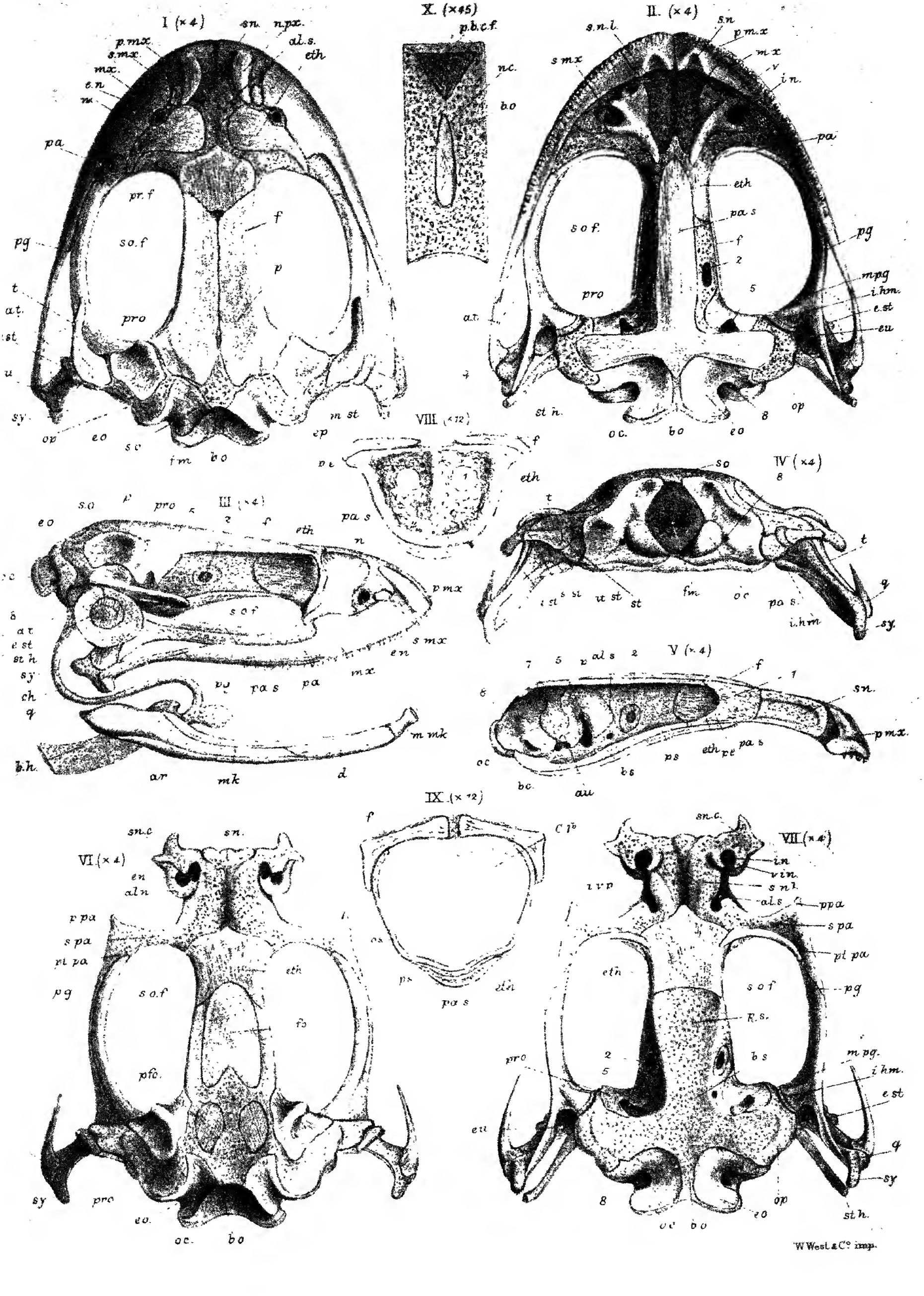
Fig. 2. Lower view of the same. × 4 diameters.



W.K.P. del. ad nat.

Frog's Skull

W. West & Co. imp.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II. (×4)
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VIII (×72)
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V (×4)
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W West & Co imp.

W West & C° imp

Fig. 3. Side view of the same. × 4 diameters.

Fig. 4. End view of the same. × 4 diameters.

Fig. 5. Sectional view of the same. × 4 diameters.

Fig. 6. Upper view of skull of adult Frog from which the parosteal bones have been removed. × 4 diameters.

Fig. 7. Lower view of the same. × 4 diameters.

Fig. 8. Section through the "girdle-bone" (ethmoid), seen from behind, and showing the gentle projection answering to the "crista galli" (the end of the meso-ethmoid), and the olfactory foramina piercing the ethmoid in the middle of the rhinencephalic fossæ. × 12 diameters.

Fig. 9. Section through the hemispheres at the anterior border of the orbito-sphenoidal walls. × 12 diameters.

Fig. 10. Posterior part of floor of the skull in the 7th Stage, seen from above, and showing "posterior basicranial fontanelle" and notochord (*n.c.*). × 45 diameters.

PLATE X.

Various Stages.

Fig. 1. Hyoid apparatus of young Frog in the 8th Stage. Magnified 12 diameters.

Fig. 2. Lower view of hyoid cornua and basihyobranchial plate, with ossified thyro-hyals (adult Frog). × 4 diameters.

Fig. 3. Section (more than half) through fore part of nasal capsule of adult Frog. 12 diameters.

Fig. 4. Section through right nostril, showing how the nasal canal is lined by the "septo-maxillary," a lobe of which is seen to project downwards, appearing in the inner nares. 12 diameters.

Fig. 5. Inner nostril of right side, seen from below (part of fig. 7 in Plate IX.); the descending spur of the septo-maxillary is seen enclosed in a fold of membrane. × 12 diameters.

Fig. 6. Part of fig. 3, further back—posterior face of the same section. × 12 diameters.

Fig. 7. Section (more than half) through hinder part of nasal capsules of the same. 12 diameters.

Fig. 8. Section (front view of left side) through the "foramen ovale" and anterior part of auditory capsule. × 12 diameters.

Fig. 9. Section (front view of right side) through the middle of the auditory capsule, and passing through fenestral fossa and stylo-hyal on the outside, and through the "meatus internus" on the inner. × 12 diameters.

Fig. 10. Section (front view of right side) through the supraoccipital and epiotic regions, the stapedial plate being severed at its hinder extremity. × 12 diameters.

Figs. 11–20. Diagrams showing the metamorphoses undergone by the first three facial arches, and their relation to the ear-capsule.

IX. *On the Fossil Mammals of Australia.*—Part IV. *Dentition and Mandible of* Thylacoleo carnifex, *with remarks on the arguments for its Herbivority. By Professor* Owen, *F.R.S. &c.*

Received September 27,—Read November 17, 1870.

Contents.

§ 1. In former Papers on the Fossil Mammals of Australia (*Thylacoleo*, Parts I. & II.) I inferred, from the size and position of the socket of the anterior tooth, from the structure of the root of the tooth therein implanted, and, above all, from the characters of the associated and completely preserved teeth, that such front tooth must have been laniariform, *i. e.* subcompressed and pointed, adapted for piercing, holding, and lacerating, like the canine of a Carnivore*.

To this the late laborious and experienced palæontologist, Dr. Falconer, has objected that, in referring to my paper, he finds "that the body of the tooth, of which the shape and direction are adduced as terms of comparison, together with the fore part of the symphysis, is wanting" †.

* Philosophical Transactions, 1859, p. 318; ib. 1866, pp. 79, 80.

† Quarterly Journal of the Geological Society, June 1862, vol. xviii. p. 353; also 'Palæontological Memoirs

To my statement, "that there is a socket close to the symphysis of the lower jaw of *Thylacoleo*, which indicates that the canine may have terminated the dental series there, and afforded an additional feature of resemblance to the *Plagiaulax*"*, Dr. FALCONER remarks:—"In all this, it will be seen, the argument is within the domain of conjecture; the tooth oscillates between canine and incisor; and not merely so, but the principles which are followed as guides in this walk of investigation are set aside, to give place to the illusory indications of mutilated external form..... If palæontological investigations were conducted in this manner there would be no limit to conjecture; the landmarks we profess to follow would be disregarded, and disorder would face us everywhere. But, happily, science furnishes unerring principles, which provide the corrective. I need hardly add that the argument drawn from *Thylacoleo* has, in my view, no bearing on the incisors of *Plagiaulax*, and gives no support to the carnivorous inference"†.

This rebuke, being doubtless kindly meant and penned in the interests of palæontology, I have hitherto borne in silence, hoping that less fragmentary fossils of *Thylacoleo* would ultimately reach me; and sustained, I must own, by a confident belief that they would confirm the inferences drawn from the position of the alveolus, suggesting the alleged feature of resemblance of *Thylacoleo* to *Plagiaulax*.

Nevertheless, the portion of mandible figured in Plates XI. & XIII. of the Phil. Trans. for 1859 being represented by a plaster cast, and the figures 5 & 6 in Plate IV. of the Phil. Trans. for 1866 being from photographs, I could not feel surprised that arguments in favour of the herbivorous nature and affinities of both *Thylacoleo* and *Plagiaulax* should have met with acceptance and support from some Anatomists, Naturalists, and Palæontologists‡.

I have again been favoured, through the kind offices of Sir DANIEL COOPER, Bart., with a collection of fossils obtained by his friend, Mr. ST. JEAN, of Gowrie, from the freshwater deposits of that locality, in Darling Downs, Queensland, Australia, which collection included the alveolar portion and certain teeth of the right upper jaw (Plate XI. figs. 1–5), and the major part of the left ramus of the lower jaw with certain teeth (Plate XII. figs. 1–5) of a full-grown *Thylacoleo carnifex*.

The teeth in the upper jaw are:—the anterior incisor with the terminal half of the crown broken away (i_1), the carnassial (p_4), and three antecedent small and simple obtusely conical teeth ($p_{1, 2, 3}$).

and Notes,' by the late HUGH FALCONER, F.R.S. &c., 8vo, 1868, vol. ii. p. 437. [In future references I shall use the numbers X. and XI. to signify the above volumes.]

* OWEN'S 'Palæontology,' 8vo, 2nd ed. (1861) p. 432.

† X. p. 354; XI. p. 438.

‡ *E. g.* Mr. BOYD DAWKINS, F.R.S., in Quarterly Journal of the Geological Society of London, vol. xx. 1864, p. 412: Mr. GERARD KREFFT, "On the Dentition of *Thylacoleo carnifex*, OWEN," Annals and Magazine of Natural History, 3rd series, vol. xviii. 1866, p. 148; Professor W. H. FLOWER, F.R.S., "On the Affinities and probable Habits of the extinct Australian Marsupial, *Thylacoleo carnifex*, OWEN," in Quarterly Journal of the Geological Society of London, March 1868, vol. xxiv. p. 307. [This volume and paper I shall refer to as No. XII.]

The teeth in the lower jaw are the root and base of the crown of the incisor (i), and the entire carnassial (p_4).

I was thus still driven, as far as these specimens went, to an inferential conclusion as to the form of the crown of the anterior incisor, both above and below. But, since preparing for the Royal Society a description of the specimens, I have been favoured by photographs and fossils of both these teeth nearly complete, and also with a plaster cast of the entire lower incisor, now in the Museum of Natural History at Sydney, New South Wales, through the kindness of the Trustees of that Museum and of their able Curator, Mr. GERARD KREFFT, Corr. M.Z.S.

The teeth transmitted and the subjects of the photographs were obtained from the Breccia-cave in Wellington Valley*, in the course of recent assiduous researches conducted by ALEX. M. THOMSON, D.Sc., Reader on Geology, Sydney University, and by Mr. KREFFT, in 1869, aided by the liberal grant of £200 voted by the Local Parliament of New South Wales in favourable response to the Memorial which I addressed to the Colonial Secretary, February 23rd, 1867†.

Whatever interpretation may ultimately be accepted in palæontology of the habits and affinities of *Thylacoleo*, additional and valuable materials for such interpretation have thus been added to the subjects of former descriptions: an account of these additions, with their bearing on the arguments that have been opposed to my conclusions, I have now the honour to submit to the Royal Society.

§ 2. *Upper Jaw and Maxillary Teeth.*—The specimen of this part of the skull (Plate XI.) includes almost the entire premaxillary (figs. 1–5, $_{22}$), with its alveolar (a, a'), nasal (n), and palatal (p) portions.

The alveolar portion contains the socket (a) of the anterior large laniariform incisor (i_1), that of a much smaller incisor (i_2) opening close to the first, and, after an interval of two lines, the front half of the socket (c) of a small canine (fig. 9), the division of which socket is made, or rather indicated, by the premaxillo-maxillary suture (s, s'): this third socket is rather larger than the second, and is more outwardly placed.

The nasal portion of the premaxillary forms anteriorly, above the deep socket of the first incisor, a thick obtuse margin (fig. 4, $_{22}$), convex transversely, concave vertically and also laterally toward the nasal cavity (ib. n); it becomes much thinner above the socket, then regains thickness at its upper part, where the plate arches inward to join the nasal bone. A ridge (r) for the attachment of the inferior "turbinal" divides the fore part of the nasal chamber into an upper (n) and a lower (n') passage.

The palatal process (figs. 2 & 3, $p_{22'}$) is thick and short; it projects forward about four lines in advance of the first large alveolus (fig. 1, p‡), is grooved above, lengthwise, where it forms that part of the floor of the nostril, n'; and it is also grooved or chan-

* Discovered by Colonel Sir THOMAS MITCHELL, C.B., F.G.S., and described in his work, 'Three Expeditions into the Interior of Eastern Australia,' 8vo, vol. ii. 1838.

† "On the Fossil Mammals of Australia.—Part III.," Philosophical Transactions, 1870, p. 569.

‡ As shown in the subject of the Memoir, Philosophical Transactions, 1866, Plate II.

nelled longitudinally at its under part, which channel (figs. 2 & 3, 22′) gains breadth and depth as it passes backward; but it is broken away after contributing an inch to the median palatal suture (ib. *p*).

The facial plate of the premaxillary repeats the characters of that figured in Plate II. Philosophical Transactions, 1866, and the suture (*s*, *s′*) with the maxillary has the same crenate character and course.

The outlet of the socket of the first incisor is $9\frac{1}{2}$ lines in fore-and-aft diameter, 6 lines in transverse diameter; the outer wall is outwardly convex, the inner one straight. The depth of the socket is $1\frac{1}{2}$ inch; it contracts to the closed end. The outlet of the second socket (fig. 3, i_2) is circular and small, $4\frac{1}{2}$ lines in diameter; it is also shallow, rapidly contracting to the closed end.

The outlet of the third socket (figs. 2, 3, *c*) is larger, deeper, and elliptical, $4\frac{1}{2}$ lines in long (fore-and-aft) diameter, 4 lines in transverse diameter; it is separated by a diastema of two lines and more from the second, and its hind wall is formed by the maxillary (21), the proportion being the same as that which the maxillary contributes to the premaxillary for the lodgment of the canine in *Thylacinus*. Besides this contribution to the third socket, the portion of maxillary of *Thylacoleo* here preserved shows three sockets of small tubercular premolars (ib. $p_{1, 2, 3}$) and the major part of that of the great carnassial tooth (ib. p_4).

A portion of the outer alveolar plate (figs. 1 & 5, 21) is preserved, and also a portion of the palatal plate (figs. 2 & 3, 21), showing its concavity near the carnassial.

The socket succeeding the third (*c*) is on the inner side of the hind or maxillary part of that socket, showing that the tooth it contained (figs. 2 & 3, p_1) held the same relative position to the third tooth (ib. *c*) as does the anterior premolar to the canine in *Lutra*; thus adding another to the extremely few instances simulating, in *Mammalia*, the double row seen in certain lower Reptiles and Fishes. The outlet of this socket is subcircular, 4 lines by $3\frac{1}{2}$ lines, and is 3 lines distant from the outer surface of the maxillary.

The next (fifth) socket (p_2) is nearer the outer border of the alveolar process, one line and a half behind the back part of the third socket; it is circular, three lines in diameter. It is immediately succeeded by a sixth socket (p_3) of similar size and shape, situated more outwardly as well as posteriorly, the alveolar wall curving from the premaxillo-maxillary suture outward and backward to the prominent fore part of the socket of the great carnassial (p_4).

This socket extends backward almost at a right angle with those of the three small antecedent teeth (fig. 3); its length from before backward is 2 inches 1 line; its greatest breadth near the fore part is 7 lines.

No part of the socket of the small tubercular molar shown in Plates XI. & XIV. fig. 1 of the Philosophical Transactions, 1859, is preserved in the present portion of the upper jaw; but this satisfactorily demonstrates the rest of the dentition of its side of that jaw, as respects size, kind, and number of teeth, and thus supplies what was less perfectly

shown in the subject of Plate III., Philosophical Transactions, 1866. (I may add that photographs, and specimens of this tooth (m_1) from the breccia-cave, illustrate the constancy of character in the solitary spelæan example of the true or tubercular molar series from the upper jaw of *Thylacoleo.*)

Of the first incisor (i_1), nearly one inch projects from the outer wall of the socket in the subject of fig. 1, Plate XI.; the inner wall (fig. 2, ib.) extends two lines lower down the tooth. The dimensions of the outlet of the socket give those of the corresponding part of the tooth, which very closely fits and adheres to the socket. The anterior border of the exposed part of the incisor shows a moderate curve convex forward; the posterior border, three lines below the socket, shows, after a slight basal convexity, the beginning of a curve concave backward. The exposed base of the tooth retains for four or five lines below the socket a coating of cement beneath which the enamel emerges. This is thicker toward the back than at the fore part of the crown, but nowhere exceeds half a line. Much of it is broken away from the base of the crown here preserved; and at the outer and back part of the base of the crown the enamel presents a free rounded edge, for two lines vertically, as if it were there interrupted. The dentine is extremely dense; the diameters of the broken part of the crown, which I take to be about halfway from the pointed end of the crown, are 7 lines by $5\frac{1}{2}$ lines; the dentine here presents, in transverse section, a narrow oval form, broader before than behind, and more convex outwardly than on the inner side.

Of the second incisor (i_2) one can infer from its socket that it had a root about 5 lines in length, tapering to an obtuse point, and a crown measuring 4 lines in diameter at its base.

The third tooth which has been displaced from the somewhat larger socket opening upon the premaxillo-maxillary suture, and which makes a slight prominence on the outside of the alveolar tract, at a short distance from the second, I conclude to have been a canine (c); the fang, or implanted part, has been 9 lines in length, slightly curved, tapering to the end.

The tooth remaining in the socket (Plate XI. figs. 2 & 3, p_1) on the inner side of the hind part of the canine (c) has the summit of the enamelled crown broken away; the diameters of the base of the crown are 4 lines and 3 lines. The root is firmly fixed in the socket: I regard this as the first premolar (p_1). Its internal position, its implantation in the maxillary at some distance from the suture with the premaxillary, and its continuation of the oblique line of the succeeding premolars, weigh with me against regarding it as a canine, according to the hypothesis of the tooth (c, Plate XI. figs. 1–3) being a third incisor, as in some hypothetical restorations referred to in the sequel.

The second premolar (p_2) *in situ* in the specimen (Plate XI. figs. 1, 2, 3) is somewhat smaller than p_1, with a very short enamelled crown, forming a low ridge extending from the outer side to nearly the inner side, and there meeting and blending with a second low ridge at right angles, close to the inner border of the crown. The enamel is limited to forming the low-ridged cap or summit of the tooth; the rest of the tooth

projecting from the socket is covered by cement. The length of the cement-clad root is given in the figure of, I believe, the homologous tooth in Plate XI. fig. 14.

The third premolar (p_3), also preserved in the specimen (Plate XI. figs. 1–3), is rather larger in size, has a similar extremely low and slightly prominent crown, with the same ridge running from the outer to the inner side, crossed by the shorter ridge at right angles near the inner side of the crown, to which the longer ridge extends, leaving the shorter ridge chiefly conspicuous behind it. The vertical extent of the cement-covered and enamelled part of the second and third premolars projecting beyond their sockets does not exceed 3 lines. This specimen resolves the doubt expressed with regard to their empty sockets in the specimen figured in Plate III. of the 'Philosophical Transactions' for 1866, p. 78*, and demonstrates that each socket contains its own small simply implanted tooth, and was not a division of a socket lodging a larger two-fanged premolar.

Beyond the third premolar the fore part of the crown of the maximized carnassial † (Plate XI. figs. 1 & 2, p_4) extends downward 10 lines. The shape, structure, vertical grooving, and dimensions of this tooth agree with those in the specimens described in the previous Memoirs.

The trenchant margin of the upper carnassial is worn, as usual, obliquely from without upward and inward, the cutting-edge of the enamel being external (Plate XI. fig. 2, p_4). This edge does not run straight, but sinks to form a low angle at the end of a well-marked external vertical groove (ib. o), marking off rather more than one-third of the hind part of the crown, which answers to the similarly but better defined hind lobe of the feline upper carnassial. The smoothly worn surface is thus divided into two parts, the anterior one being broadest anteriorly at the thickest part of the tooth, while the posterior gains breadth as it recedes toward the hind end of the crown. But the indications of resemblance to the feline carnassial, especially to that of *Machairodus* (Plate XI. figs. 15, 16), do not end here. The inner surface of the crown, about one-fourth of the way from the fore to the hind margin, projects and terminates in a ridge (r, figs. 2 & 3, Plate XI.), which expands to the base of the crown, representing the more developed ridge or vertical swelling of that part of the carnassial in *Machairodus* (fig. 15, v), from the broadening base (v') of which the tubercle of the upper carnassial, wanting in *Machairodus* as in *Thylacoleo*, is developed in *Felis*. An opposite vertical ridge on the outer side of the crown in *Thylacoleo* (fig. 1, p_4, u) represents the most prominent part of the middle lobe of the carnassial in Felines (fig. 16, u), whence the outer surface bends inward to the angle or groove dividing that lobe from the hind one. The outer surface in *Thylacoleo* bends in the same direction to the corresponding angle or groove (o, figs. 1 & 3), then curves outward to the hind end of the crown. This is very low and subobtuse, as is the corresponding end of the carnassial in Felines. But the fore part of the crown, in

* "At the fore part of the carnassial socket the alveolar border is excavated by either a similar socket for a two-rooted tooth, or by two contiguous sockets for two small single-rooted teeth."

† No evidence has yet been had that this or the antecedent permanent teeth had displaced deciduous predecessors; the adopted symbol p is to be taken with this reservation.

Thylacoleo, rises with a backward inclination to the highest, or vertically longest, part of the crown, from which a well-marked ridge traverses or forms the anterior margin of the crown (figs. 2 & 3, *z*). The anterior root is longer but narrower, antero-posteriorly, than the posterior one, as in the upper carnassial of Felines.

In *Hypsiprymnus minor* (Plate XI. figs. 17, 18) the premolar has a straighter edge, not bilobed; the outer side of the crown is indented with the four or more parallel grooves and ridges, at the apical half; the inner side is uniformly and obliquely worn, in degree according to age.

§ 3. *Mandible and Mandibular Teeth.*—The portion of lower jaw (Plate XII. figs. 1–5) from the deposit at Gowrie includes 6 inches in longitudinal extent of the left ramus, viz. from the fore part of the symphysis (*s*) to the fore part of the strongly inflected angle (*a*). This latter character is acceptable as confirmatory of the marsupial nature of *Thylacoleo*, in a way more intelligible or convincing to some than the cranial and maxillary characters adduced in support of that induction in the original Memoir (Philosophical Transactions, 1859); although I am not aware that the marsupiality of *Thylacoleo* has been, by any objector, called in question.

The fossil is massive, heavy, much petrified; it retains the fang and base of the crown of the anterior and sole incisor (*i*), the entire carnassial (p_4), and the two fangs of the anterior molar (fig. 3, m_1).

The small and simple socket of the second molar is indicated (m_2); and two or three small and very shallow alveoli (Plate XII. figs. 2 & 3, p_3, p_2) intervene between the incisor-socket and the inner side of the anterior fourth part of the carnassial. From the condition of the upper small premolars it may be inferred that there were two or three similarly small functionless and speedily lost teeth between the carnassial and the laniariform incisor of the lower jaw, occupying the sockets (p_3, p_2, figs. 2 & 3).

Assuming these to be three in number, the first and second are on nearly the same transverse line, so close together that the broken thin partition (?) gives the appearance of a single socket.

The entire length of the alveolar tract is 3 inches; from the back part of the last socket to the hind fractured end of the present fossil is 3 inches. As the extent from the fore part of the upper carnassial to that of the glenoid cavity in the skull figured in Plate II. & III., Philosophical Transactions, 1866, measures 5 inches 10 lines, it may be inferred that such must have been nearly, if not quite, the extent of the mandible from the fore part of the lower carnassial to the fore part of the condyle; consequently the entire length of the mandible would not be less than 7 inches. We may reckon that 1 inch, at least, is wanting from the broken hind part of the specimen figured (Plate XII.); and we may certainly infer that a greater proportion of the mandible was allotted to the joint and to the muscular forces working that instrument than to the dental weapons with which it was armed; concentrated as they here are, as in the fellest *Carnivora*, for fatal efficiency.

The symphysial contour (ib. figs. 1 & 2, *s*, *r*) rises from the lower border of the horizontal

ramus at an angle of 120°. The vertical diameter of the ramus anterior to the carnassial tooth is 1 inch 10 lines; it is the same anterior to the origin of the coronoid plate; and, save that the upper border is undulated by the alveolar opening, it runs parallel with the lower one. The outer wall swells out to lodge the anterior root of the carnassial, the vertical swelling subsiding at the lower fourth of the jaw. The dental canal has two small outlets anterior to the swelling. The outer wall becomes slightly concave lengthwise between the socket of the carnassial and the origin of the coronoid, which is broad and thick anteriorly (*c*), where it divides that concavity from the deeper one behind for the insertion of the large crotaphyte muscle (*f*, figs. 1 & 3, Plate XII.).

The specimen shows only the fractured base of the coronoid plate, the length of which in a straight line is 2 inches 6 lines; its direction is curved with the convexity inwards (fig. 3, *c*, *c'*): the fore part, formed by the buttress-like development of the outer wall of the ramus (*c*), is 7 lines in thickness; it rapidly decreases to $1\frac{1}{2}$ line, and returns to $2\frac{1}{2}$ lines in thickness at the hinder part (*c'*). The osseous tissue at the fore part of the coronoid is compact and dense. Toward the hind part is exposed the dental canal (fig. 2, *d*), broken across where it was traversing the base of the coronoid: the canal here is narrow tranversely. A narrow longitudinal groove between the base of the inflected part of the "ascending ramus" and the part of the dental canal (*d*) exposed by the fracture is continued as a shallow impression with a slight curve downward and forward, and then straight for a little way, becoming obliterated below the vertical parallel of the last molar (m_2). This is the only indication interpretable as a "mylohyoid groove."

The course of the fracture at the base of the coronoid from its thick fore part is backward and downward. The lower border of the ramus forms a thick ridge at the lower end of the symphysis, and subsides into a rounded or convex tract, gaining breadth as it recedes, and becoming flattened as it expands by the increased production of the angle of the jaw (Plate XII. fig. 4, *a*, *a'*). The crotaphyte fossa (ib. fig. 1, *f*) is not continued forward into the substance of the horizontal ramus, as in Potoroos and *Kangaroos*.

The symphysis (ib. fig. 2, *r*, *s*) is subtriangular, the lower and longer side being rather convex, the upper side almost straight; the base, which is turned backward and downward, is bilobed, the upper lobe, with the convex contour, being the longest. The length of the symphysis is 2 inches; the basal depth is 1 inch 5 lines. The upper part of the symphysis forms a slightly concave tract or platform, 9 lines in breadth at the fore and inner part of the carnassial, which increases as it recedes, sloping downward and backward (ib. fig. 3, *s*). It is bounded externally by the sockets of the incisor (*i*) and of the premolar teeth (p_{1-4}) in continuous series. There is no true diastema between the laniary and the carnassial; the three closely aggregated empty sockets of probably as many single-rooted, small, soon shed, functionless premolars occupy the intervening tract and something more, viz. by encroaching on the inner side of the fore part of the socket of the carnassial (Plate XII. figs. 2 & 3, p_2, p_3). Behind the symphysis the inner wall of the ramus (fig. 2) is moderately convex vertically, concave in a less degree lengthwise at the lower

part, where the curve becomes deepened posteriorly by the inbending of the angle (*a*). This part gains in thickness as it extends inward; the inner surface of the part preserved in the specimen described is vertical, and in that direction measures 6 lines (fig. 2, *a*). The fractured end (fig. 5, *a*) shows the three-sided character of this part of the angle, the upper and under surfaces converging to the thin horizontal plate (ib. *b*) connecting the angle with the part supporting the coronoid and condyle (ib. *c'*). The fore part or beginning of the neck of the condyle may be indicated by the smooth tract (fig. 3, *e*), which would then define the hind border of the coronoid process; or this narrow tract may indicate a minute transverse perforation of the ascending ramus. On the first notion the preserved fore-and-aft extent of the part relating to the support of the condyle (*e*, *c'*) is 10 lines.

One may hope ere long to receive a specimen with the whole of the rising branch of the mandible complete, showing both the shape and position of the condyle. Seeing that in *Bettongia* (p. 250, fig. 18), *Hypsiprymnus* (p. 250, fig. 17), *Phascolarctos* (p. 233, fig. 6), and all the marsupial vegetable feeders with a high-placed condyle there is a corresponding course of the base of the coronoid from before upward and backward, whilst in *Thylacinus* (p. 235, fig. 11) and *Sarcophilus* (p. 235, fig. 12), with a low-placed condyle, the base of the coronoid runs straight backward, I take ground for inferring a similar or relatively lower position of condyle from the slope of the base of the coronoid from before downward and backward, as indicated in the present jaw of *Thylacoleo*, and deem it not improbable that it may have resembled in both respects the *Plagiaulax*; thus exemplifying in the form of the mandible, correlatively with the dentition, the higher degree of carnivority in these extinct marsupial and diprotodont genera.

§ 4. *Photographs and Cave-specimens of Maxillary Teeth.*—Since finishing the description and figures of the foregoing specimens of maxillary and mandibular structure and teeth of the *Thylacoleo*, I have been favoured by receiving (May 20th, 1870) from the Colonial Secretary's Office, Sydney, New South Wales, a series of Photographs of Fossil Remains, and some duplicate specimens, obtained by Dr. A. M. Thomson and Gerard Krefft, Esq., from Limestone Caves in Wellington Valley, under the circumstances detailed in my paper on *Diprotodon*, p. 569, Philosophical Transactions, 1870.

I have subsequently been favoured by the Trustees of the Museum of Natural History, Sydney, and the able Curator, Mr. Gerard Krefft, with Photographs, some of them duplicates of the above, others of fossils since acquired from the same breccia-caves. The Trustees of the Sydney Museum have also transmitted to the British Museum duplicate specimens of these cave-fossils.

From this rich series of photographic illustrations and specimens I select for description and figures the following, which supplement and almost complete our knowledge of the permanent or fully developed dentition of *Thylacoleo carnifex*.

The tooth in "Photograph No. 28" (Plate XI. figs. 6, 7) is the anterior incisor, left side, upper jaw. It has its crown a little worn at the point; it is plainly "canine" in function as in shape. The enamelled part of the crown which projects beyond the

cement measures 1 inch 2 lines along the anterior curve. This is convex lengthwise, angular transversely, being traversed at the fore part by a low ridge (*r*, fig. 7, Plate XI.); the posterior border beyond a slight basal convexity is feebly concave lengthwise, rather flattened across, but chiefly bounded by a longitudinal ridge of enamel near the outer side: this ridge is feebly notched; the thinner enamel is continued from it, obliquely to the inner side of the crown, where the thicker enamel, of less longitudinal extent than the outer enamel, also developes a trenchant posterior ridge. The entire length of the tooth following the curve is 2 inches 3 lines: the fore-and-aft breadth of the base of the crown is 8 lines; the transverse breadth 5 lines: it accords, therefore, closely with the anterior incisor preserved in the portion of jaw above described (Plate XI. figs. 1, 2, 3, i_1). The photograph (No. 28 *b*) of the outer side of the answerable tooth shows a greater extent of preserved fang, though not quite entire at the end; the serration or notching of the long outer hind trenchant ridge or edge of the enamel is better marked than in the specimen. Together they concur in demonstrating the effective laniary character of the foremost tooth of the upper jaw of *Thylacoleo*.

A tooth in the photograph No. 28 *c* of upper teeth of *Thylacoleo* accords with the indications, as to form of fang and basal breadth of crown, afforded by the alveolus symbolized in figs. 2 & 3, Plate XI., as of the second incisor (i_2). I have therefore added a copy of it, fig. 9, in that Plate. It shows a root tapering to an obtuse point, 5 lines in length, and a crown 4 lines in diameter at its base, short, subconical, and obtuse, and may well be i_2 with a rather longer root from a less aged individual *Thylacoleo*: the crown of this tooth must project close behind the base of the crown of the front incisor.

The tooth (Plate XI. fig. 10) from the photograph No. 28 *d*, with a fang 10 lines in length, fitting by its fore-and-aft breadth such a socket as that marked *c* in figs. 1, 2, 3, Plate XI., answers to the indications there given. Another subject of the same photograph (fig. 11) is a more perfect canine of the opposite side, its bend of fang being contrary to that indicated by the right upper socket (*c*) in Plate XI.

Accordingly, I conclude the canine, *c* (Plate XI. figs. 9 & 10), to have a small subobtuse subcompressed crown, with a convex front outline from before backward, where it meets the hind shorter border of the crown at an angle which seems to form the rather blunt point of the tooth. The length of the enamelled crown is 4 lines, taken in the tooth's axis along the middle of the crown; the fore-and-aft breadth of the crown is 6 lines; the fang is an inch in length and rather curved, contracting to an obtuse point.

Amongst the duplicate cave-teeth are the crowns with portions of the fang of two such canines; the best-preserved fang is similarly curved (Plate XI. fig. 12), with a subtrihedral transverse section; but this form is more strongly marked in the short enamelled crown, the outer and inner sides meeting, anteriorly, at a trenchant border, strongly curved to the apex, which bends beyond the hind side (ib. fig. 12, *b*); this is flattened transversely, and is feebly concave lengthwise. The enamel has much less vertical extent here than along the anterior trenchant convex side of the crown. The outer side is slightly convex, and marked in one specimen by a longitudinal linear groove (ib. fig. 10); in the

other by two grooves (ib. fig. 11), recalling those in the upper canines of *Felis*; the inner side (fig. 12, *a*) in both teeth shows two longitudinal grooves, and a ridge of enamel behind the hindmost groove.

Of the three succeeding small teeth which I have assigned to the premolar series (*ante*, p. 214), I recognize, in the cave-specimens, by the similarity of their very short crown and straight root, those answering to p_2 and p_3 in the upper jaw (Plate XI. figs. 13 & 14); but there is a photograph of a larger tooth, though less than the canine, which in size at the base of the crown corresponds with the p_1 in place (id. ib.) on the inner and hinder side of the canine. This tooth has a low conical crown, 3 lines long by 4½ in basal antero-posterior breadth. The premolar (fig. 14, p_2) has a root 10 lines in length, curved near the end to which it contracts.

The two smaller succeeding premolars in place in the unique jaw (Plate XI. figs. 1–3) I have not thought proper to displace; the photographs, which plainly show the same very short extent of enamelled crown, give to an example of p_2 a straight fang of 7 lines in length, and to one of p_3 a similar fang 8 lines in length; both taper to an obtuse point. Amongst the duplicate teeth transmitted is a p_2 with half an inch of the solid straight fang, and the crown of a p_3 corresponding with that in the upper jaw (Plate XI. fig. 3).

The photographs include three specimens of the great carnassial (p_4) with an enamelled crown 2 inches in fore-and-aft basal extent, 9 lines in greatest vertical extent. The subject of one figure shows the two roots; the foremost of which is 1 inch 6 lines long and 1 inch in fore-and-aft breadth, where it becomes free; the hind root or division is 1 inch in length and about the same in fore-and-aft extent; its greatest transverse thickness is 6 lines, and it contracts to an obtuse hinder border. Both roots are shown to be strongly marked, as in the tooth *in situ* (Plate XI. figs. 1 & 2), by fine subwavy longitudinal striæ near their extremities, adding to the closeness of attachment to the alveolar periosteum. The characters of this huge carnassial in the fossil specimen are so closely repeated as to render figures of these photographs unnecessary.

A side view and a view of the grinding-surface of the small tubercular molar are given in the photograph No. 7: a similar specimen I have worked out of the breccia (Plate XI. fig. 3, m_1). This tooth closely resembles that shown on the inner side of the hind end of the great carnassial in Plate XI. Phil. Trans. 1859, and in Plate III. ib. 1866.

It is evident that the five small teeth between the upper laniary (i_1) and the carnassial (p_4, figs. 1–3, Plate XI.) can have had but insignificant functional relations. They could not be opposed to mandibular teeth, if even their homotypes had been present or retained in the lower jaw. But of these there seem to have been but two, or at most three, developed, of very small size, on the inner side of the fore part of the lower carnassial; and I have seen no specimens of mandible in which they are retained.

§ 5. *Photographs and Cave-specimens of the Mandible and Mandibular Teeth.*—The following are the most instructive photographs of portions of the mandible with

teeth of *Thylacoleo* from the Breccia-cave of Wellington Valley, in the series above referred to.

No. 10 gives two views of a portion of the right ramus (the outside view is given in Plate XIII. fig. 1). It is similar to the fossil from Queensland above described (Plate XII.), but more mutilated at the back part. The chief value of the specimen photographed is the retained incisor ($\ddot{\imath}$), from which only the apex of the crown is wanting, by an oblique fracture from above and behind downward and forward. In a photograph of a more mutilated mandible (ib. fig. 2), the inner wall of the alveolus of the incisor is broken away as far as the vertical line dropped from the fore part of the carnassial (p_4). The outer wall remains a few lines in advance of this in the subject of figure 1, but sufficient of the cement-covered root of the tooth is exposed to show a commencing contraction toward its implanted end. The incisor is directed upward at an angle of 130° with the long axis of the ramus, and the crown shows a curvature with the convexity forward and downward as in the lower laniaries of *Thylacinus*; the hind border is not straight or convex like the answerable upper border in the same tooth of *Bettongia* and *Hypsiprymnus*, but is serrato-trenchant and slightly concave lengthwise. A photographic view giving the transverse breadth or thickness of the incisor would have been instructive; but the portion of the tooth retained in the mandibular ramus figured in *Plate XII. fig. 1, $\ddot{\imath}$, a,* shows the more essential distinction from the long procumbent lower incisors of the herbivorous Marsupials in the degree of lateral compression of the crown and its proportion to the antero-posterior breadth, which in the laniary of *Thylacoleo* is intermediate between that in *Machairodus* and *Felis*.

The two anterior outlets of the dental canal are present, and in the same position in the cave-fossil (Plate XIII. fig. 1, *o*) as in the Queensland specimen (Plate XII. fig. 1, *o*). The postero-inferior emargination of the symphysial surface is repeated on the inner surface of the ramus of the subject of fig. 1, Plate XIII., as in Plate XII. fig. 2, *r*.

All the characters of the carnassial tooth (p_4) in the Queensland specimen are closely repeated; the crown is abraded in the same direction and to the same extent.

The crown of the first molar (m_1) is preserved in both the cave-specimens photographed, showing its raised, anterior, subtrenchant lobe, and its small low hind tubercular talon. On the outer side of this tooth is shown the subvertical surface formed by attrition against the hind part of the upper carnassial. The proportions of the anterior and posterior roots of m_1 are indicated in the photograph of the inner side of the subject of fig. 1, Plate XIII. The socket of the minute p_3 (ib.) plainly appears on the inner side of that for the anterior root of p_4 in the same photograph; but the shallower and larger ones of p_2 and p_1 have left no impression—were probably obliterated in the fossil. There can be no doubt as to the specific identity of the Wellington Valley cave-fossils with those of *Thylacoleo carnifex* from Melbourne (Lake Colungoolac) and from Queensland (Gowrie Creek).

The second fossil of *Thylacoleo* from the breccia-cave, the subject of the photograph

No. 29 of the series, is a smaller portion of the fore part of a right ramus, with the entire incisor, the carnassial, and first molar *in situ*. The whole length of the base of the incisor is exposed, and the obtuse termination of the closed and contracted end of the root (Plate XIII. fig. 2). The fractured state of the bone also shows portions of the fore and hind roots of the carnassial (p_4), the latter apparently the larger, contrary to that in the lower carnassial of Felines, which is not the homologous tooth, although with a similar adaptive modification of crown. The length of the incisor is 3 inches 3 lines, that of the enamelled crown appears to be about 1 inch 8 lines; the antero-posterior breadth of its base is 9 lines. The position, direction, and curvature of the incisor in this specimen accord with those in the photograph copied in fig. 1, Plate XIII., and with the restoration based on the direction of the empty socket in the subject of Plate IV. fig. 6, Philosophical Transactions, 1866. The vertical extent of the fore part of the carnassial (p_4) is 1 inch 9 lines, that of the enamelled crown being $7\frac{1}{2}$ lines.

All the evidences yielded by the specimen (figs. 1–3, Plate XII.), by the casts (Plate XI. fig. 3, Phil. Trans. 1859), and by the photographs (Plate XIII. figs. 1 & 2, p_4) concur in showing the closer resemblance of this sectorial tooth to the carnassial of the large placental *Carnivores* (Plate XII. figs. 9 & 12) than to the sectorial premolar in Rat-Kangaroos (ib. figs. 8 & 10). The crown of the tooth (fig. 11) is bent lengthwise, with the convexity outward, the concavity inward; and this is chiefly at the hinder half of the tooth (fig. 3, p_4). The fore part of the crown is the thickest, and that by the prominence of the inner surface at the anterior fourth, which makes a low obtuse ridge (*r*, fig. 11, Plate XII.) divided by a depression or channel from the anterior ridge (*a*) or border of the crown, which represents the prebasal ridge (*a*) in the carnassial of the *Hyæna* (fig. 12). The broader part of the trenchant surface (*b*, fig. 11) is anterior, as in *Hyæna* (*b*, fig. 12). The trenchant margin does not extend in a straight line, but is subconcave, though less so and more continuously than in *Hyæna*. The effect of these curves of the cutting part of the blades in *Thylacoleo*, as in *Felis* and *Hyæna*, is to make them meet at successive parts in the act of cutting, not by simultaneous opposition of the entire cutting-edges of the opposed blades. The vertical undulation of the enamel is finer, less marked, in the lower than in the upper carnassials, and is confined to the basal part of the inner surface, not to the apical half of the crown as in *Hypsiprymnus* (fig. 10).

In the cast of a specimen of a right mandibular ramus with the carnassial less worn than in the specimen Plate XII. figs. 1–3, the abraded surface is interrupted midway, indicating a bilobed character of the unworn margin, as in the lower carnassial of Felines; the abraded surface in the cast expands from the unworn part of the dividing notch forward toward the anterior end of the tooth and backward to the posterior end (Plate XII. fig. 11). The subject of figure 6, Plate XII., is a specimen worked out of the breccia transmitted by the Trustees of the Australian Museum, and shows the proportions of the two roots of p_4, lower jaw.

The Photograph No. 7 includes views of five examples of the large laniariform lower incisors, both outer and inner surfaces of the most entire specimen being given.

No. 1 shows the outer side of a left lower incisor wanting only the tip of the crown. The closed contracted end of the root is truncate. The length is 3 inches 4 lines, the greatest breadth from before backward 9 lines.

No. 3 in the photograph is of the inner side of a similarly entire right incisor (Plate XIII. fig. 4). The ridge (*d*) defining the inner side from the narrow posterior facet of the crown is clearly given in this photograph, which appears to be the incisor removed from the socket of the subject of fig. 1, Plate XIII., the same mutilation of the summit of the crown being shown. The implanted end of the root contracts in the same degree. and shows the same truncation, as in the subject of figs. 5 & 6, Plate XIII.

Photograph No. 43 gives, somewhat reduced, the inner side of the fore part of the right ramus, showing the symphysial surface, the carnassial, and the first molar. The extent and shape of the symphysis, as in Plate XII. fig. 2, are here repeated with the same vertical extent and lower contour of the fore part of the mandible. Photograph No. 37 is of the outer side of the same specimen, on the same scale, showing the trenchant part of the crown of the first molar (m_1) as in *Plagiaulax*. A view of the carnassial *in situ*, in a small fragment of the left ramus, showing the oblique external smooth wear of the trenchant tooth, is also given in photograph No. 43.

These evidences are acceptable as testifying to the constancy of the characters of the lower jaw and dentition in Thylacoleo carnifex.

§ 6. *Cave-specimens and Cast of Inferior Incisor.*—I have been favoured by Mr. KREFFT with a cast of the entire inferior incisor of *Thylacoleo*, from the breccia-cave in Wellington Valley; and since penning my notes on this cast and the photographs, an entire lower incisor and portions of others have come to hand in the series of cave-specimens worked out of the masses of breccia transmitted from the Wellington caves.

The incisor (Plate XIII. figs. 5, 6, 7) is long, subcompressed, subrecurved; the crown is pointed, trenchant anteriorly. The entire tooth is about equally divided into crown (fig. 5, *a'*, *b*) and fang (ib. *f*); but the enamelled part (*e*), when the root-cement is scraped away, is longer than at first appears; for the cement encroaches upon the enamelled crown in angular prolongations from the root, and further on the inner (fig. 6, *c*) than on the outer (ib. fig. 5) or hinder part (fig. 7) of the tooth. The crown becomes three-sided a little below the apex (*e*); the outer side (fig. 5, *a*) is broadest, and is transversely convex, the posterior border forming that of the crown. The inner side (fig. 6, *a*, *d*), of less breadth, is flat, but is divided by a longitudinal ridge (*d*) into two facets, the hinder one being the narrowest and inclining transversely to the hinder border (*n*). Toward the base of the crown the hind surface (*h*) becomes feebly concave between these marginal posterior ridges.

Thus the perforating part of this tooth is strengthened by four longitudinal enamel

ridges, in which the serrate or finely undulated or wrinkled character is more or less manifest, especially on the trenchant anterior border, defining the outer (fig. 5) from the inner (figs. 4 & 6, *i*) surfaces of the crown, also on the supplementary ridge (*d*) on the inner facet (*i*).

As the fore-and-aft breadth of the crown increases from the apex downward, the stronger convexity of the antero-external part is limited to the fore part of that facet, the hinder part of the facet becoming less convex or almost flattened, and at the base of the crown even feebly concave, where the cement (fig. 5, *c*) encroaches on the enamel. A similar concavity marks the outer part of the base of the laniariform incisor of *Plagiaulax*, Falconer, X. fig. 1, p. 366; XI. pl. 33. fig. 1, *a*. The anterior subserrate ridge is the longest, the postexternal ridge subsides a little sooner in approaching the base, the postinternal ridge is next in length, and the supplementary inner ridge is the shortest. The enamel-case of the crown is entire, but is thickest upon the more convex anterior part of the antero-external facet, and where it forms the ridges (see the section, Plate XIII. fig. 8).

The posterior facet at the apical part of the crown meets at a right angle the inner side, but lower down it slopes from the postinternal ridge, backward as well as outward, to meet the outer facet at *b*. This gives a more trenchant character to the subserrate ridge or border (*d*) between these surfaces.

The effective cutting power of the postinternal ridge, where the angle between the posterior and internal surfaces of this three-sided bayonet-like tooth becomes a little open, is enhanced by the prominence of the ridge, supplemental strength being given to the piercer by the added postinternal ridge. The cement-clad root (Plate XIII. figs. 5, 6 & 7, *c*, *f*) gradually contracts to its subtruncate closed extremity.

In the specimens of lower laniary above described, as in the cast and photographs of that formidable tooth of *Thylacoleo*, evidence is given of its conforming in its limited or temporary growth, as in its shape, proportions, and structure, with the canine of the Felines, but with superadded modifications strengthening and perfecting it for its work as a piercer, holder, and lacerator.

§ 7. *Guide to inferring function from form of Teeth.*—Thus, through the cooperation of a liberal and enlightened Legislature and Administration, and of esteemed friends and fellow-labourers in Sydney, New South Wales, ample evidence has been got of both upper and lower laniary incisors, as well as of the rest of the dentition of *Thylacoleo carnifex*.

It is with pleasure, though without surprise, that I have been enabled to confirm the inferences expressed in my former papers, on such elements of that dentition as I then only knew "in part."

Whether the "principles which are followed as guides in this walk of investigation were" therein "set aside, to give place to the illusory indications of mutilated external form"*, I cannot determine, because Dr. Falconer does not define the principles to which

* X. p. 354; XI. p. 437.

he refers. It is certain that the indications of the mutilated parts of my original specimens have not proved "illusory."

But such indications were not the sole grounds of my conclusions; I was also guided by a principle. It is that laid down by CUVIER in the van of his immortal work of Restoration of the extinct Mammals of the Paris Basin.

The aberrations of some contemporary labourers in this field show that it will bear repetition:—"La première chose à faire dans l'étude d'un animal fossile est de reconnoître la forme de ses dents molaires; on détermine par-là s'il est carnivore ou herbivore"*.

These test-teeth were fortunately entire in the upper jaw of the skull of *Thylacoleo*†, and in the cast of the lower jaw originally described‡.

The major part of the molar series was represented by one large and most efficient carnassial, followed by a single small tubercular above, which was opposed to a semi-tubercular molar and a second more minute tubercular tooth below.

Here was no molar machinery for the mastication of vegetable food, but a maximized modification of the teeth for the division of fleshy fibre, and so much of the tubercular form added for the final crush or squeeze of gristle or other tough part escaping the shears, as exists in the most carnivorous of the placental mammals.

§ 8. *Location of Laniaries.*—From these facts, with faith in the Cuvierian principle, I inferred a concomitancy of laniary teeth at the fore part of the jaw "to pierce, retain, and kill"§ the prey, whether such laniaries held the relative position to a suture technically determining them to be "incisors" or "canines." It is now determined that these laniaries are, as was inferred in my Second Paper‖, incisors. A co-adapted pair at the fore part of the lower jaw were opposed to a slightly separated pair in the upper jaw.

To this demonstration applies the following objection against the inference as to function:—

"Throughout the *Mammalia*, where teeth perform the functions of canines to 'pierce, retain, and kill,' they are held well apart through the interposition of a line of incisors—the end being obvious: the points of penetration are doubled, the grasp is strengthened by widening the base, and the dilacerating and killing powers are multiplied"¶.

To this I reply that, were a pair of bayonets cemented side by side and the forces of two brawny arms concentrated on the thrust, their perforating and lethal power would be increased. I fail to see how such "collateral arrangement in the axis" of the piercing force "would place them at a disadvantage to the end to be attained"**.

Dr. FALCONER admits that "a Rat when seized can inflict a smart wound on the hand"††. I can add experience of loss of young poultry showing by the wounds on their legs how they had been brought down from the perch, and by wounds on the neck how they had

* Recherches sur les Ossemens Fossiles, 4to, tom. iii. (1822) p. 1.

† Philosophical Transactions, 1859, Plate XI. figs. 1 & 2.

‡ Id. ib. fig. 3.

§ OWEN'S 'Palæontology,' 8vo, 1860, p. 320.

‖ Philosophical Transactions, 1866, p. 80.

¶ X. p. 352; XI. p. 435.

** Id. ib.

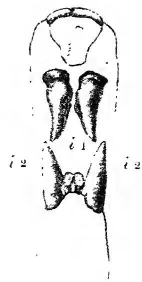
†† Id. ib.

been killed, and then more or less devoured by the nocturnal murine omnivorous Rodent, thus demonstrated to have acted in this carnivorous fashion by virtue of the pair of scalpriform incisors "arranged collaterally in the axis," &c.

But there are several genera and species of CUVIER's "Carnassiers" in which incisors having the size, form, and office of laniaries* are not "held well apart through the interposition of a line of 'other' incisors"†.

The European Otter (*Lutra*) shows, indeed, this divaricate arrangement, but an African Otter (*Potamogale*, Cut, fig. 1) does not; a co-adapted pair of laniaries (i_1) at the fore part of the upper jaw were opposed to a slightly separated pair in the under jaw (i_2).

Fig. 1.



Laniary incisors, front view, *Potamogale velox* twice nat. size‡.

In the *Insectivora*, as in the *Marsupialia*, there are two types of the teeth which are developed and shaped "to pierce, retain, and kill," in other words, two local conditions of "laniaries." In some, *Gymnura*§, *Centetes*‖, *e. g.*, the laniaries answer to the 'canines' of *Carnivora*, and are separated by interposed 'incisors' in both upper and lower jaws, as they are in *Sarcophilus* and *Thylacinus*; in other *Insectivora* the laniaries are approximated, and are formed by 'incisors'; as, *e. g.* in *Solenodon*¶, *Erinaceus***, *Scalops*, *Urotrichus*, and other *Soricidæ* generally, in which a juxtaposed pair at the fore part of the mandible†† oppose a corresponding pair at the fore part of the upper jaw. These incisors usurp the functions of the canines in *Gymnura*, *Talpa*, &c. The transference of the laniary form and function from the canines to the incisors, the development of these latter into the dental instruments "modified to pierce, retain, and kill," is the rule, or is found in the majority of *Insectivora*. In the Japanese Mole-shrew (*Urotrichus talpoïdes*)‡‡ "the incisor is long, conical, and pointed;" it is grooved on the inner side: "the lower canine is small, its office being transferred to the incisor"§§. This large laniary tooth may be

* "Technical canines vary as much in shape, proportion, and function as do technical 'incisors:' are sometimes, indeed, implanted by two roots instead of one." See 'Odontography,' pl. 110. fig. 3 (Mole).

† X. p. 352; XI. p. 435.

‡ For the subject of this Cut I am indebted to the author of the instructive Memoir on *Potamogale*, Zool. Trans. vol. vi. p. 1, Professor ALLMAN, F.R.S., of the University of Edinburgh, where the unique skeleton of the *Potamogale* is preserved.

§ OWEN's 'Odontography,' pl. 111. fig. 4, *a*, *b*.

‖ Ib. pl. 110. fig. 6.

¶ 'Odontography,' pl. 111. fig. 1 (the front view (*b*) may be compared with that of *Thylacoleo* in XII. p. 312, fig. 2).

** Ib. pl. 110. fig. 5.

†† *Sorex*.—"In the lower jaw there is, as is known, one very elongated pointed incisor on each side." "The canine is a small conical tooth, the smallest of the lower jaw."—MIVART, "On the Osteology of Insectivora." Journal of Anatomy and Physiology, vol. ii. p. 11.

‡‡ Catalogue of Bones in the British Museum, 8vo. p. 109.

§§ MIVART, *ut suprà*.

the homologue of the laniariform i_2 in *Potamogale*; but the pair of developed incisors in *Urotrichus* are what Dr. FALCONER would have described as "approximated and placed collaterally, as in the placental Rodents"*; *i. e.* they are in contact, side by side (Cut, fig. 2). In the large African Ferine (Carnassier, CUV.), with the habits and food of the Otter, discovered and called *Potamogale* by DU CHAILLU, the first incisor of the upper jaw (fig. 3, i_1) "closely resembles a large projecting canine"†. In the lower jaw "the first is very small. The second incisor is high, conical, curved; it is sharp-pointed, and resembles a canine"‡. "The condyle is borne on a distinct neck"§. The low position of the condyle (*b*), associated with these laniariform incisors in *Potamogale* (fig. 3), is not alluded to either by ALLMAN (*l. c.* p. 11) or by MIVART (*l. c.* p. 127).

Fig. 2.

Front view of laniary incisors, magn., *Urotrichus talpoides*, TEMMINCK.

It is important in association with the position and juxtaposition of the laniaries in this large fish-hunting Ferine, with regard to the moot question of the carnivority of *Thylacoleo* and *Plagiaulax*.

Fig. 3.

Side view of upper laniary and mandibular dentition. *Potamogale velox*, nat. size (*ex* ALLMAN, *loc. cit.*).

In *Myogale*, as in *Solenodon*, the first upper incisor "is much larger and more vertically extended than in any other tooth in the skull. Its crown is triangular, it is in contact with its fellow of the opposite side, and predominates more than does any other form yet reviewed, except, perhaps, that of *Sorex*. It is opposed mainly to the second incisor of the lower jaw, the first being much smaller...... This second incisor is exceedingly large, pointed and conical"‖. "The canine is a small obtusely pointed tooth¶," as in *Thylacoleo*. In *Solenodon*, *Potamogale*, and *Myogale* the laniariform incisors of the lower jaw, to which those in *Urotrichus* and *Sorex* may be homologous, are divided from each other, at their base, by a very small pair of anterior incisors; but the extent of separation is slight, and cannot affect in any appreciable degree their piercing power.

Indeed, to concentrate is to give force, to divide is to weaken. If it be not carried too far, divarication may aid in the secondary work of holding the pierced prey: but for the primary lethal operation, the base for the grasp of the biting muscles is as broad in *Thylacoleo* as in *Felis*. An estimate of the concentrated force of these enormous muscles

* X. p. 352; XI. p. 435.

† ALLMAN, "On the Characters and Affinities of *Potamogale*," Trans. Zool. Soc. vol. vi. (1866) p. 6. I am indebted to the accomplished author of the above-cited interesting Monograph for the drawing which is the subject of the Cut, figure 1.

‡ Ib. p. 7.

§ Ib. p. 11. (Compare this "pedunculate" character with that in *Plagiaulax*, figure 10.)

‖ MIVART, *op. cit.* Journal of Anatomy and Physiology, vol. ii. p. 124.

¶ Id. ib. p. 125. "*Scalops*. The first incisor is very large, and is opposed to one nearly as long in the lower jaw, where the canine is wanting, or represented by a most minute and rudimentary tooth."—Ib.

upon the pair of lower laniaries approximated as one piercing, lacerating organ, with the superlative degree of carnassiality of the premolar, suggested the expression of the pouched Lion having been "one of the fellest and most destructive of predatory beasts"*.

The Curator of the Museum of the Royal College of Surgeons, and now Hunterian Professor, adopts the argument from divarication of the laniaries in the *Carnivora* known to Dr. FALCONER, and salves the exceptions by affirming "the *modus operandi* of the Hedgehog in snapping up and devouring a beetle is totally different from that of a Cat in seizing and killing a Rat or a Rabbit"†. And one may conclude that the *Thylacoleo*, from the nearer resemblance of its laniaries and of the jaw working them to those of the Cat, would show, also, some difference from the Hedgehog in the snapping or seizing of its prey. But Professor FLOWER, in a question of such importance to Physiology as the reconstruction of *Thylacoleo*, should have defined the 'total difference' between the mode of application by the Hedgehog of its 'approximate' laniaries and that of the application of the Cat or Stoat of their 'divaricate' ones in the killing of a young Rabbit; for the Hedgehog invades the burrows of the prolific rodent to devour the offspring; it is by no means exclusively insectivorous.

Was the well-armed mandible, with its low and advantageous joint for a strong grip, applied by *Potamogale* in piercing, holding, and killing its fish in so different a fashion from that of the like mandible in *Lutra*, as to lend any countenance to the assumption that the juxtaposed long terminal incisors of the lissencephalous Otter were put to the service of an herbivore—to the same service as they are in the Koala, *e. g.*? Yet, if Professor FLOWER's argument and diagrams‡ mean any thing, they mean this!

The *Thylacoleo's* approximate incisors§ are relatively as long, as sharp, as laniariform as are those of *Potamogale*; and if we turn to the teeth (Plate XIV. p 2–4, m 1, 2), which tell us truly the use to which such incisors were put, they speak directly and plainly that it was for capturing and killing a higher prey than fishes.

§ 9. *Comparison of the Teeth of Thylacoleo with those of Phascolarctos.*—The light thrown by the large carnassial and small tubercular teeth on the application and function of the laniaries of *Thylacoleo* is sought to be obscured by conjectural figures of the structure of those laniaries and of the jaw that works them.

Fig. 4.

Front view of mandible and teeth (*Thylacoleo*), as restored, one-third nat. size, by Professor FLOWER (XII. p. 312, fig. 2).

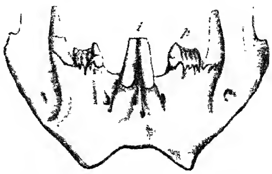
In fig. 2 (XII. p. 312), entitled "*Thylacoleo carnifex* restored," Professor FLOWER represents the incisors with truncate summits, like those of an herbivorous marsupial. This restoration is reproduced in Cut, fig. 4. The carnassial of *Thylacoleo* (ib. p) has features too broad and pronounced to be misunderstood. The herbivorous Marsupial selected

* Philosophical Transactions, 1859, p. 319. † XII. p. 318. ‡ XII. p. 317, & pp. 312, 313, figs. 2, 4, 5.

§ These teeth are represented too broad in proportion to their length, or too short in proportion to their breadth, in XII. fig. 2.

by Professor FLOWER for comparison (in XII. p. 313, fig. 4), copied in Cut, fig. 5, appears to have a similar carnassial (*p*); but this appearance is due to the foreshortening of the series of the grinding-teeth of the Koala.

Fig. 5.



Front view of mandible and teeth (*Phascolarctos*), three-fourths nat. size (after Professor FLOWER, XII. p. 313, fig. 4).

My business here is simply to set forth the facts which guide to a right conclusion, and to put them as correctly as I am able. The incisors of *Thylacoleo* are neither truncate nor flattened by attrition at their ends; their character, from nature, is given, of the natural size, in the front view of the mandible (Plate XIII. fig. 3). They may be blunted by use, or the point may be broken off, as in figures 1, 4, Plate XIII., from the photograph No. 10. The laniaries of an old Lion usually show the same effects of usage. Professor FLOWER gives a front view of the incisors of *Phascolarctos*, and a side view of the incisors of *Hypsiprymnus*; but a view of the working surface, from which the best idea can be formed of the use to which such incisors, in the two Marsupial herbivores, are put, is not given. I have supplied this omission in the upper figure of Cut 6, *i*, where the working surface of the lower incisor of the phytophagous diprotodont Marsupial may be compared with that of the zoophagous one (Plate XIII. fig. 7).

Returning to CUVIER's test of the diet of an extinct animal, which test gives the use of the long anterior teeth, whether canines or incisors, of such animal, I may recall attention to the single, small,—one may truly say, viewing the enormous carnassial against which it abuts—minute tubercular in the upper jaw of *Thylacoleo* (Plate XI. fig. 3, m_1). Then, as regards the lower jaw (Plate XIII. fig. 1), the molar (m_1) following the carnassial (p_4) has the anterior half of the crown compressed transversely, the sides converging to a trenchant margin: this approximation to the form of its homologue in Felines, from the close and extensive abutment of the tooth against the upper carnassial, forms a continuation of the shear-blade structure, and gives the lower blade an extent equal to that of the larger carnassial above. The tubercular part of m_1 below forms a mere basal talon to the carnassial part of that tooth, whilst m_2 is a truly minute tubercular, and, seemingly, soon lost.

The demonstrated structure of the laniaries of *Thylacoleo* is in harmony with the zoophagous work which the molar teeth are plainly designed to transact.

Now, being solely desirous to test CUVIER's principle in reference to the approximate pair of long incisors of *Phascolarctos*, I subjoin what is essential to such test, and what Professor FLOWER omits, viz. a side view of the dentition of the Koala, reduced one-half, together with a view of the grinding-surface of the molar teeth, natural size (Cut, fig. 6), corresponding with those of the *Thylacoleo* shown in Plate XI. fig. 3, Plate XII. fig. 3.

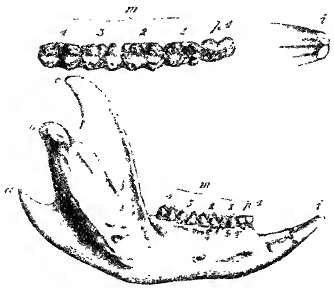
The tooth (p_4), probably homologous with the carnassial of *Thylacoleo*, and that which most resembles, or rather least differs from, it in the shape of the crown, occupies less than one-eighth of the dental series in *Phascolarctos*, in *Thylacoleo* it occupies nearly

one-half; in *Phascolarctos* it forms one-fifth of the molar series, in *Thylacoleo* it forms two-thirds of that series; these proportions are masked in the foreshortened view, fig. 5.

The premolar in the upper jaw of the Koala (p_4, fig. 6) presents a flat surface, with a breadth of crown two-thirds of the length, the surface being slightly raised by a border of thick enamel at the periphery, and by a plicate island of enamel in the middle. It is a true pounder of vegetable substances, with the structure of a type-grinder of an herbivore. Such contrast in proportions, structure, form, and function with p_4 in *Thylacoleo**, as is presented by the tooth (p_4) of *Phascolarctos*, would not be surmised by an uninstructed comparer of the restored *Thylacoleo* (fig. 4, copied from XII. fig. 2) with the corresponding view of its alleged herbivorous analogue and ally (fig. 5, copied from ib. fig. 4).

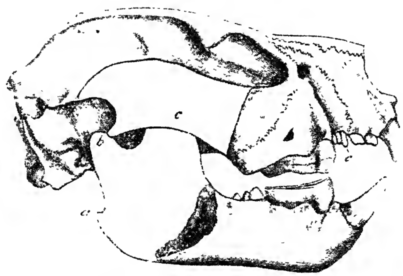
§ 10. *Mandibular Characters of Carnivorous and Herbivorous Marsupials.*—A high-placed condyle is associated with the rotatory movements of the jaw in herbivorous Marsupials (XII. fig. 3) as in herbivorous Placentals. Professor FLOWER's restoration

Fig. 6.



Mandible of Koala (*Phascolarctos fuscus*), one-half the natural size, and grinding-surface of teeth, natural size, from nature.

Fig. 7.

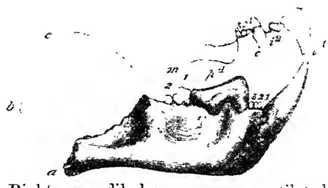


Skull of "*Thylacoleo carnifex*, restored," one-fourth natural size (after FLOWER, XII. p. 312, fig. 1).

(fig. 7, copied from XII. p. 312, fig. 1) gives a similar position to the mandibular condyle (b) in *Thylacoleo*, and the angle (a) is there indicated after the pattern of the lower jaw of the Koala (fig. 6, a).

The first fossil mandible of *Thylacoleo* which permits a deduction to be made of the relative position of the parts in question (Plate XII. figs. 1 & 2, and Cut, fig. 8), demonstrates the fallacy of the restoration in fig. 7, and shows a structure harmonizing with powerful vertical movements of the mandible, not with the horizontal grinding required for the comminution, and mixing with abundant saliva, of vegetable matters.

Fig. 8.



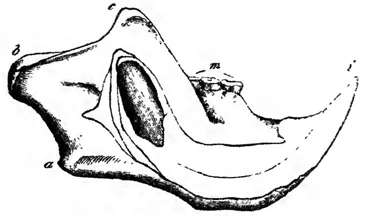
Right mandibular ramus, mutilated behind, from nature, one-fourth natural size (*Thylacoleo*).

* Figured in Plate XI. figs. 1 & 2, Philosophical Transactions, 1859.

The fact is patent; but the inference has been called in question.

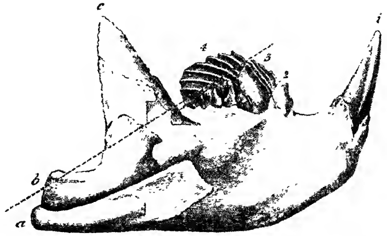
Dr. FALCONER writes, in June 1862, "Next, as regards the depressed position of the condyle—below the level of the grinding-teeth. The author of 'Palæontology' states that it is 'a character unknown among any herbivorous or mixed-feeding animal.' I again refer my reader to the figure (pl. 34. fig. 13) of the lower jaw of the Aye-aye"*. This figure is essentially the same as that in my Memoir read before the Zoological Society, January 14th, 1862 (fig. 9). Prior to that date the depressed position of the condyle *to* the level of the grinding-teeth (fig. 9, *b*) was a character unknown in any herbivorous or mixed-feeding animal; it is still so unknown as depressed *below* that level, such as we see it in *Plagiaulax* (fig. 10). Dr. FALCONER, proceeding with his evi-

Fig. 9.



Mandible of *Chiromys Madagascariensis*, natural size (Zool. Trans. vol. v. pl. 20. fig. 9, Memoir of January 14, 1862).

Fig. 10.



Mandible of "*Plagiaulax Becklesii*, magnified four diameters" (after FALCONER, XI. pl. 34. fig. 1).

dence of the herbivority of the gnawing Lemur and his comparison of its mandible with that of *Plagiaulax*, admits that "the condyle looks still more depressed in *Plagiaulax Becklesii*; but this is, in part, owing to the inflected margin of the angle being broken off in the fossil, while it is entire and salient in the recent form, thus elevating the condyle above the lower plane of the ramus, and leading to an appearance of a greater amount of difference than exists in nature"†.

Here we are at issue on a matter of fact. I affirm that the condyle (*b*, fig. 10) in *Plagiaulax* would not be so elevated above the lower plane of the ramus if the angle (*a*, fig. 10) were entire and salient, because the saliency has an inward, not a downward, direction. The correspondence in this respect, as well as in the low position of condyle, of *Plagiaulax* with the carnivorous Thylacoleo, Dasyures, and Thylacines, and its difference from the herbivorous marsupials (fig. 6) and larvivorous Lemur (fig. 9) is as clearly demonstrated in the Purbeck fossils, as if the inflected angle had received no fracture whatever.

For other differences of mandibular structure and configuration between *Plagiaulax* and *Chiromys*, invalidating Dr. FALCONER'S argument in favour of a common herbivorous or rodent nature associated with a common depressed position of condyle, I refer to figs. 9 & 10.

In reference to the subject of the latter illustration, Dr. FALCONER has remarked, "Pro-

* X. p. 361; XI. p. 445.

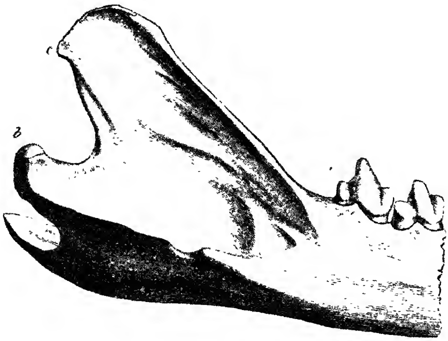
† X. p. 361; XI. p. 446.

fessor OWEN meets the argument in my paper by the assertion that the condyle of *Plagiaulax* is 'pedunculate, as in the predaceous marsupials.' If so, I invite him to adduce the instance, bearing in mind that the question here is one of degree*".

The virtual acceptance of this "invitation" had been given years before in the instance of the Phascolothere†, which, like *Plagiaulax*, is an extinct marsupial carnivore from an oolitic deposit.

The required structure is shown in the Cut (fig. 11, *b*) of the articular extremity of

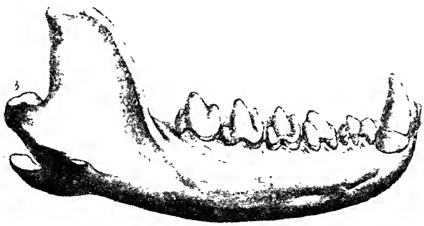
Fig. 11.



Articular end of mandible, *Thylacinus cynocephalus*: two-thirds nat. size.

the mandible in *Thylacinus cynocephalus*‡. In nearly the same degree is the condyle pedunculate in *Potamogale* (fig. 3), and in a somewhat less degree in *Sarcophilus* (fig. 12, *b*).

Fig. 12.



Mandible of *Sarcophilus ursinus* (after nature).

I may assume that the extent of condylar prominence and position in the instances cited will be admitted to have justified, as, indeed, they formed the basis of, the assertion of the community of the "pedunculated condyle," such as it is shown in figs. 10 & 11 at *b*.

* X. p. 362; XI. p. 447.

† Transactions of the Geological Society of London, 2nd series, vol. vi. (1839) p. 58, pl. 6.

‡ See 'Descriptive Catalogue of the Osteological Series contained in the Museum of the Royal College of Surgeons of England,' 4to, 1853, p. 347, nos. 1903–1908.

It will be observed that, in the Thylacine—the most carnivorous of modern Marsupials—the depth of the notch between the condyle (*b*) and coronoid (*c*), or what Dr. FALCONER indicates by the term "pedunculate," is relatively greater than in *Sarcophilus*. The two recede in this respect progressively further from the Koala and the Aye-aye. If *Plagiaulax* had shown less degree of "pedunculation" and a higher position of the condyle than in *Thylacinus* or *Sarcophilus*, and had, in these respects, approached nearer to *Chiromys* or *Phascolarctos* in mandibular modifications, an inference of corresponding approximation in diet, or herbivorous application of jaw, would have been legitimate. I claim the same quality for my conclusion, that as the carnivorous characters of the lower jaw are maximized in *Plagiaulax* (fig. 10) with strong indications of the same structure in *Thylacoleo* (fig. 8), so the carnivority of both genera is the more plainly demonstrated.

It is not, however, a solitary character, but an association of characters, which establishes this conclusion.

Position of condyle relates to the force with which the mandible is worked, shape and pedunculation of the condyle to the direction of the working force.

The flattened or less convex articular surface favours the rotatory movements; the more convex, especially transversely extended and pedunculate or subpedunculate, condyle indicates the ginglymoid articulation with greater extent of divarication or wider gape, and more habitual movements of the jaw in one plane, or limited more or less thereto. The rotatory grinding movements of the mandible are commonly associated with a high position of the condyle and vegetable diet; the vertical biting movements are commonly associated with a low position of the condyle and animal diet. But the advantage of a long lever afforded by a lofty coronoid process (figs. 10, 11, 12, *c*) and low-placed condyle (ib. *b*) may correlate with powerful biting and gnawing actions, as in the working of the maximized scalpriform teeth of *Chiromys* (fig. 9, *i*).

Here, however, the coronoid (*c*) is comparatively low: the condyle (ib. *b*) is narrow; its convex curve is chiefly longitudinal, or from above downward; there is no constriction or neck; the supporting part of the articular surface is continued directly upward and forward to the coronoid (*c*), and almost as directly downward and forward to the angle (*a*)*.

In similarly placed condyles for biting, piercing, lacerating, and killing actions of the mandible, and where such condyles are associated, as is the rule, with laniariform not with scalpriform anterior teeth, the condyle is more prominent; the part of the ascending ramus supporting the condyle curves toward the coronoid process, in a course at first more or less deeply concave, then vertical or recurved; and a similar well-marked concavity divides the condyle from the angle of the jaw, save in the most decidedly zoophagous of the *Marsupialia* (*Sarcophilus*, *Thylacinus*, *Plagiaulax*), in the latter of which the convex condyle forms, as it were, the upper and back part of the angle itself.

The condyle in *Plagiaulax* (fig. 10, *b*) projects a little below the horizontal level

* OWEN, 'Monograph on the Aye-aye,' 4to, 1863, p. 20. pl. 8. figs. 7, 9 (also 'Transactions of the Zoological Society,' vol. v. pt. 2. pl. 20. figs. 7, 9).

of the alveolar series; in *Chiromys* and *Carnivora* it is on that level. But if the mandibular condyle in *Plagiaulax* agreed in all characters with that of the rodent Lemur, this would not show *Plagiaulax* to be a vegetable feeder. The direct testimony of the insectivorous or rather larvivorous habits of the Aye-aye is too strong and too sure to be done away with by the enforced food on which a captive individual may have been compelled artificially to subsist.

However, for the instruction of any physiologist or palæontologist who may still deem the position of the condyle in *Chiromys* to throw light upon the food and nature of *Plagiaulax* and *Thylacoleo*, it may be stated that in every secondary mandibular character *Plagiaulax* differs from *Chiromys*, and resembles *Sarcophilus*, *Thylacinus*, and *Phascolotherium* *. The supporting part of the condyle sinks below the transversely extended upper part of the convex articular surface, before curving forward and upward to the coronoid, leaving an entering notch between that process and the coronoid which, in the type specimen of *Plagiaulax Becklesii* (fig. 20 F, p. 258), closely corresponds in form with that in *Thylacinus* and *Phascolotherium*.

The fractured line of the angle of the jaw is not beneath the neck of the condyle, but on the inner side of the inferior border of the rising ramus passing to the lower end of the condyle. That part of the angle which has been broken off did not extend, as Dr. FALCONER states, below the condyle as in the Aye-aye, but to the inner side thereof, as in *Sarcophilus*, *Thylacinus*, and *Phascolotherium*†.

Whoever may have watched a living Thylacine or Ursine Dasyure must have been struck with the width of its gape. The extent of such motion of the mandible is due to the freedom of the joint (figs. 11, 12, *b*) and its distance from the moving lever (*c*). The like or even greater relative backward position of the condyle must have equally or more favoured "the power of separating the jaws in front essential to a predaceous animal having laniary teeth," like those of the *Thylacoleo* and *Plagiaulax* (fig. 10), "constructed to pierce, retain, and kill"‡. And we have direct proof in the sessile condition of the condyle in the Aye-aye that the power of separating the jaws was more restricted in that carnivorous and rodent Lemur.

§ 11. *Testimony as to the native food of the Aye-aye.*—The advantage to the forcible action of the jaw by the backward position of the condyle is recognizable, whether the fore teeth of the jaw be fashioned for "biting," *i. e.* piercing as a dagger and becoming infixed in a prey, or for "eroding" hard wood, as a gouge or chisel.

Modifications of the mandible might be expected to be associated with the different actions and applications of the fore teeth, aided or advantaged by the carrying back the condyle and lengthening the lever of the biting powers.

Prior to 1861 such backwardly placed as well as low-placed condyle was not known

* British Fossil Mammals, 8vo, 1846, p. 65.

† It is this "broad" part of the condyle which gives it the "ovate or pyriform outline" (XI. p. 445). In *Thylacinus* and *Sarcophilus* a part of the articular surface also extends down from the back of the condyle.

‡ XI. p. 447.

in any real or alleged herbivorous or mixed-feeding animal. The anatomy of *Chiromys* added, in that year, the interesting and instructive exception (admitting the Aye-aye to be a mixed feeder). If it had been contended that the lower-placed condyle shown in *Plagiaulax*, and deducible in *Thylacoleo*, was absolute, independently of other characters and considerations, in demonstrating the carnivorous nature of these marsupials, the discovery of the structure of the mandible of the Aye-aye would have placed a seeming objection and a feasible argument in the hands of an advocate of the non-carnivorous character of *Thylacoleo* and *Plagiaulax*. If it were proved that the Aye-aye is a vegetable feeder, not to say herbivorous, the same advantage would be his who had interpreted the fossil remains of *Thylacoleo* and *Plagiaulax*, notwithstanding the low-placed condyle, as those of vegetarians, having their nearest affinities "to the marsupial herbivores, such as *Halmaturus*, *Hypsiprymnus*, and *Phascolarctus*" *.

But the only testimony we have at present of the natural food of *Chiromys* shows it to be "carnivorous" in the sense of subsisting on the flesh or insect-tissues of wood-boring larvæ; all the peculiarities of its structure are physiologically or teleologically intelligible only on this basis. HUNTER, it is true, made his captive Sea-gull subsist wholly on grain †, and induced a Kite to eat and thrive on bread alone ‡.

Save for loyalty to truth in the abstract one might be willing to accept the evidence adduced by Dr. FALCONER § of the food given to captive Aye-ayes as proof of its being naturally a vegetable feeder; but I believe the position of the mandibular condyles to be related to the powerful working of the pair of incisors. Such work is not needed for dividing the stems of rice or the stalks of dates or bananas. Nor are the Aye-aye's conditions of condyle present in *Hypsiprymnus* or in any other vegetable feeder. No one can admit the Aye-aye to be a strict vegetarian who gives credit to the subjoined testimony:—

"It so happened that the thick sticks I now put into his cage were bored in all directions by a large and destructive grub, called here the *Moutouk*. Just at sunset the Aye-aye crept from under his blanket, yawned, stretched, and betook himself to his tree, where his movements are lively and graceful, though by no means so quick as those of a Squirrel. Presently he came to one of the worm-eaten branches, which he began to examine most attentively; and bending forward his ears, and applying his nose close to the bark, he rapidly tapped the surface with the curious second digit, as a Woodpecker taps a tree, though with much less noise, from time to time inserting the end of the slender finger into the worm-holes as a surgeon would a probe. At length he came to a part of the branch which evidently gave out an interesting sound, for he began to tear it with his strong teeth. He rapidly stripped off the bark, cut into the wood, and

* X. p. 352; XI. p. 435.

† HOME, 'Lectures on Comparative Anatomy,' 4to, vol. i. p. 271. OWEN, 'Catalogue of the Physiological Series, Museum of the Royal College of Surgeons,' 2nd ed. 8vo, p. 151, prep. no. 523.

‡ HUNTER, 'Animal Economy,' OWEN's Ed. 8vo, 1837, p. 112.

§ X. p. 364; XI. p. 449.

exposed the nest of a grub, which he daintily picked out of its bed with the slender tapping finger, and conveyed the luscious morsel to his mouth. I watched these proceedings with intense interest, and was much struck with the marvellous adaptation of the creature to its habits, shown by his acute hearing, which enables him aptly to distinguish the different tones emitted from the wood by his gentle tapping; his evidently acute sense of smell, aiding him in his search; his secure footsteps on the slender branches, to which he firmly clung by his quadrumanous members; his strong rodent teeth, enabling him to tear through the wood; and lastly, by the curious slender finger, unlike that of any other animal, and which he used alternately as a pleximeter, a probe, and a scoop"*.

SONNERAT, besides specifying the compulsory food on which his captive Aye-aye perished in two short months, not being able longer to sustain life thereon, describes the long slender naked middle digit:—"il s'en sert pour tirer des trous des arbres les vers qui sont sa nourriture"†. I understand this to mean that larvæ—"vers"—are its natural or staple food. The affirmation may have been made from SONNERAT's observations on *Chiromys* in a state of nature, or on the reports of natives of Madagascar, or on both authorities. It is a better testimony of its natural "nourriture" than the compulsory diet of confinement, and ought to be quoted in a consideration of the present important question.

For to what condition is Comparative Anatomy reduced if we reject the testimony which Dr. FALCONER does not cite, and admit, upon the testimony he does cite, that *Chiromys* is a vegetable feeder! Were the scalpriform teeth enabled, through the low position of a terminal condyle, to gouge out the hard woody fibre for food in order that the animal might masticate such fibre? Only upon this hypothesis could *Chiromys* be cited as an exception to the correlation of such position of mandibular joint with animal diet. But xylophagous habits involve complex ever-growing molars, like those of the Voles, the Beavers, and Capybaras. A reference to the molar teeth of the Aye-aye at once indicates its true diet, and the part played by the lower jaw and its chisels in obtaining it. Observation of the living animal in its native woods vindicates the Cuvierian principle, and gives the rational explanation of both dental and maxillary machinery. Instead of being an exception. the low condyle enters into the rule of its association with the getting of food of an animal nature.

Now let us return to the application of the Aye-aye's mandibular structure to the explanation of that in *Thylacoleo* and *Plagiaulax*. "The large front teeth in *Chiromys* are curved in segments of circles, the working surface is elongate, in breadth equalling that of the base of the tooth, with a front convex enamelled border, forming the obtuse apex of the gouging surface"‡.

* Letter from Dr. SANDWITH, quoted in "OWEN on the Aye-aye," Trans. Zool. Soc. vol. v. pt. 2. 1863, p. 37.

† Voyage aux Indes Orientales, &c., Paris, 4to, 1782, p. 122.

‡ OWEN, 'On the Aye-aye,' 4to, 1863, p. 25.

With what molars are these scalpriform teeth associated? Few, small, tubercular. Adapted for squeezing the soft animal nutriment out of the tegumentary covering of a caterpillar, not adapted for trituratory mastication of such vegetable food as calls for the more complex and massive molars of the Kangaroos, Potoroos, and Koalas. With what kind of teeth is the low-placed and backwardly placed condyle of *Thylacoleo* and *Plagiaulax* associated? and what may be the diet indicated by such association? For the response to these questions the palæontologist, guided by the Cuvierian principle, refers to the great carnassial and the small tubercular molar teeth.

§ 12. *Comparison of the Mandibular Condyle in Thylacoleo, Plagiaulax*, and *Rodentia.*—In placental Rodents the mandibular condyle is longitudinally horizontal, transversely convex; its long diameter is from before backward; it represents the section of a cylinder. The glenoid cavity of *Thylacoleo* shows that its condyle has been convex from before backward or longitudinally, and with its long diameter transverse; not limiting the jaw, as in Rodents, to horizontal movements chiefly to and fro, but adapting the jaw to hinge-like vertical motion, needed for the due action of the terminal laniaries and the large carnassials.

The mandibular condyle in the more ancient and smaller Marsupials with a closely analogous dentition is demonstrably similar to that which is here inferentially ascribed to the condyle of *Thylacoleo*. It is in *Plagiaulax* convex longitudinally, or from before backward, and that in so great a degree that the most prominent part of the convexity looks backward. "Its long diameter is disposed subvertically, and the outline is ovate or pyriform, the broad end being uppermost"*. This broad end is the transversely extended part of the convex condyle†. Dr. FALCONER, nevertheless, affirms that the form of condyle presented by *Plagiaulax* is "common in the placental Rodents"‡; yet is constrained to add, "with the difference, however, that in the latter the condyle having to work backwards and forwards in a groove, its articular surface is disposed longitudinally"§.

But this difference precludes an ascription of community of form of condyle between *Plagiaulax* and *Rodentia*; and in so far as the difference is such as to enable the condyle in *Plagiaulax* to work the jaw upward and downward, or vertically, it lends itself to those actions which the jaw has to perform "among the *Carnivora*."

The kind and degree of difference which the mandibular condyle presents in *Chiromys* and in *Plagiaulax*, already pointed out, in like manner illustrates its application in the latter to predatory actions, and is consequently and concomitantly associated with a difference of form of the entire mandible: that part in *Plagiaulax* conforms with the lower jaw in *Sarcophilus* and *Thylacoleo* in as marked a degree as it differs from the mandible in *Chiromys*, in placental Rodents, in *Phascolarctos*, and in *Hypsyprimnus*.

* X. p. 360; XI. p. 445.

† OWEN, Monograph on British Mesozoic *Mammalia*, pl. 4. fig. 10, A, *c* & B.

‡ X. p. 360; XI. p. 445.

§ Id. ib.

§ 13. *Comparison of Incisors of Diprotodont Paucidentata with those of Chiromys and Rodentia.*—*Thylacoleo* and *Plagiaulax*, it is affirmed, "agree with *Chiromys* in the collateral position and upward direction of their strong incisors"*. Doubtless; but they differ in the character of the terminal surface indicative of the kind of work to which those incisors were respectively put in *Thylacoleo* and *Chiromys*. Admitting the Aye-aye to be "as rare and aberrant among existing Mammalia"† as are *Thylacoleo* and *Plagiaulax* among fossil Marsupials, yet the Aye-aye shows on its lower front teeth a long smooth sloping surface, the result of the scraping, cutting, chisel-like action of the opposed scalpriform incisors‡.

Thylacoleo and *Plagiaulax* preserve the pointed termination of the lower incisors, or if they be blunted or broken, they show no signs of habitual attrition. *Chiromys* combines a compressed form with peculiar fore-and-aft breadth of the incisor, which has its thick enamel limited to the front border and to a contiguous portion of the sides, but coextensive in length with the deeply implanted tooth. The lower incisors of *Thylacoleo* and *Plagiaulax* have the proportion of transverse to fore-and-aft breadth, and the continuous sheath of enamel (Plate XIII. fig. 8) limited to the exposed crown, which are characteristic of the laniaries in *Potamogale* and *Felis*: consequently the crown or exposed part of the long and large incisor of *Thylacoleo* and *Plagiaulax* is that alone which is curved, and the division into crown and root is recognizable. The entire scalpriform incisor of *Chiromys*, like that of true Rodents, is curved in the segment of a circle§ and the tooth (fig. 9, *i*) preserves its diameters of depth and breadth from the end of the worn, sloping, eroding surface of the crown (*i*) to the base of the implanted part, and this part is much longer, with a concomitantly longer socket, than in *Thylacoleo* and *Plagiaulax*. The above-defined broad and striking differences between the lower incisors of *Chiromys* and those of *Plagiaulax* and *Thylacoleo* militate strongly against the conclusion of *Plagiaulax* and *Thylacoleo* being Marsupial forms of Rodent, or "Rodent types of Marsupial"‖, and are decisive against the alleged "clear evidence of their phytophagous and rodent plan of construction"¶.

Dr. FALCONER pursues his argumentation as follows:—"Let us now consider OWEN's inference as to the function of these teeth. It is expressed thus: 'The large front tooth is formed to pierce, retain, and kill; the succeeding teeth are like the blades of shears, adapted to cut and divide soft substances like flesh,' &c. Professor OWEN has elsewhere described the premolar of *Hypsiprymnus* as trenchant**, and I have shown above that the tooth is essentially alike in *Plagiaulax*." (as Professor FLOWER contends that

* X. p. 364; XI. p. 449. † Id. ib.

‡ OWEN, 'On the Aye-aye,' &c., pl. 20. fig. 3.

§ "The incisors are long, large, much compressed, regularly curved in segments of equal circles, the upper pair describing one-fourth, the lower pair one-half of such circle."—OWEN, *op. cit.* p. 55.

‖ X. p. 349; XI. p. 431. ¶ X. p. 353; XI. p. 436.

** Odontography, vol. i. p. 389.

it likewise is in *Thylacoleo* *). "If, therefore," proceeds Dr. FALCONER, "the function is to be deduced with such facile certainty from the mere form, the premolar of *Hypsiprymnus* ought also to be carnivorous. But we know that the genus is so strictly herbivorous that the family to which it belongs has been regarded as representing in the *Marsupialia* the Ruminants of the placental Mammals. With this fact before us, is it likely that the premolars of *Plagiaulax* and *Thylacoleo* 'were applied to cut and divide flesh'?"†

To this I reply; consider the difference of the molar teeth following the trenchant one in *Plagiaulax* and *Hypsiprymnus* respectively, and the true solution will be given‡.

§ 14. *Interrupted and continuous applications of Teeth.*—The deduction of the carnivorous nature of the extinct Marsupials in question was drawn not from shape merely, but from correlation of teeth. The "arbitrariness," if such quality be predicable of the conclusion, is applicable to the guiding principle, not to the palæontologist confiding therein. If reference to confirmatory and collateral facts was omitted in the Papers of 1859 and 1869, it was under the belief of their being the common property of the interpreters of fossil remains.

In the adaptively modified dentition of the class *Mammalia* the differently shaped teeth are put, some to occasional and interrupted, others to continuous use. The incisors of the child biting a piece of bread and butter exemplify the first functional character; the molars which pound the piece bitten off, the second. These incisors are trenchant or sharp-edged like the premolar of a Bettong or the carnassial of a Cat; their action in the human subject leaves a clean semicircular border of the bitten slice. The teeth, of whatever kind, incisors, premolars, or molars, which have the continuous work show its effects by an abraded surface. Those which have the occasional and interrupted work show little or no indication of such. The laniaries of *Carnivora*, when they have pierced and lacerated the prey, have done their work; the gnawing off of the pieces of flesh adapted for the imperfect tubercular mastication and for deglutition is effected by continuous action of the sectorial blades, the mouth being turned sideways to the food, as may be seen in the Felines and other *Carnivora* which have the largest and best-shaped carnassial teeth. Such action and application of these teeth are exemplified by the clean-worn, smooth, oblique surface sloping in the opposed blades down their opposite and opposed sides.

The same functional character, relating, that is to say, to "time" or "prolongation of use," is indicated by the incisors. Those in *Thylacoleo* and *Plagiaulax* show no evidence

* XII. p. 310. "The great cutting premolar of *Thylacoleo* bears no real comparison with the carnassial tooth of the Carnivora, but with the compressed premolar of the *Hypsiprymni*."—Ib. p. 316.

† X. p. 356; XI. p. 440.

‡ Dr. FALCONER, indeed, does strengthen his attack by calling in the argument from correlation; but he selects a different tooth from that of the Cuvierian principle:—"When this conclusion as to the herbivority of *Plagiaulax*" and *Thylacoleo* "from similar trenchant characters of premolars with those of *Hypsiprymnus* is coupled with the obviously phytophagous type of the incisors, the conviction will be confirmed."—X. p. 357; XI. p. 440.

of mutual continuous attrition. In Rodents, whether placental or marsupial, the oblique surface of wear or use in both upper and lower pairs of the large front teeth has suggested the comparison with the chisel, and the term "scalpriform." Such incisors have a trenchant margin as the human incisor has; but the superadded sloping surface of attrition in the Rodent indicates the continuous as distinguished from the occasional application of such front teeth. The Lemurine Aye-aye presents the same character as a guide to the inference of function of incisors, and at once exemplifies the difference of such function and that of the homologous pairs of pointed unworn teeth in *Thylacoleo* and *Plagiaulax*.

It is proper, in pursuing comparisons for the purpose of arriving at truth, that, besides the front view of the incisors of the Koala*, we should contrast their working surface (fig. 6, *i*) with that in the corresponding teeth of *Thylacoleo*. A comparison of Cut, figure 5, *i* with figure 3, Plate XIII., will show that the one has the continuous or frequent action, the other the intermittent and occasional. It is evident that the six incisors of the upper jaw, as well as the lower pair, in the Koala, work much and continuously in cropping and gnawing off the vegetable food which the large, numerous and complex grinders (fig. 6, *m*) pound to pulp for the bolus of deglutition.

A minor but sufficiently conspicuous degree of attrition characterizes the narrower upper and the lower procumbent incisors of the Bettongs and Rat-Kangaroos.

In the *Bettongia penicillata*, with such worn incisors and with all the molars in place and showing habitual use, the trenchant premolar retains its vertical groovings to the cutting-edge of both the outer and inner sides. They have been used to divide the grass-blade or the leaf-stalk, or other tough part or fibre of the vegetable food; but the more important and continuous work of mastication has had grinders in number, size, massiveness, and complexity of horizontal area fitted to perform it. Old age is attended with seeming exceptions to this rule in both human incisors and hypsiprymnal premolars, which then show the wear or work of a life.

Independently of the correlative guide, the worn surfaces of the *Thylacoleo's* carnassials show, like those of the Lion's, and like the scalpriform incisors of the Rodents, that their work and office were of the continuous kind; which, with their shape and position in the jaw, was for flesh-cutting, not for wood-cutting, or leaf-cutting or grass-cutting; for the succeeding few and small tuberculars could do nothing to the purpose with slices of such vegetable substances.

How far this deduction of function from mere form may be "facile" or "arbitrary" it is not for me to say; but it by no means authorizes any one to infer, because the correlation of the premolars of *Thylacoleo* and *Plagiaulax* with few and small tuberculars and large laniaries favours their carnassiality, that "the premolars of *Hypsiprymnus* ought also to be carnivorous"†. All that the mere form of that tooth shows is, that it cuts. What manner of substances were so cut can be inferred from the asso-

* Exclusively given by Professor FLOWER in his advocacy of the herbivority of *Thylacoleo*, XII. p. 313, fig. 4.

† X. p. 357; XI. p. 440; also XII. p. 318.

ciated teeth, more especially those defined by CUVIER as the fittest to yield the required information.

§ 15. *Work of Molars in Herbivora.*—Vegetable substances need for their assimilation not only dividing but crushing and reduction to pulp by commingling of salivary secretions during the grinding process. Hence large salivary glands are associated with numerous broad-crowned grinders. Palæontology is not left in so helpless a condition as it is made to appear in the following passage:—"There is no reason to suppose that the large trenchant premolars [of *Thylacoleo*] were not as well adapted for chopping up succulent roots and vegetables as for 'dividing the nutritive fibres' of animal prey"*. But my task has been to show, not only for what they were adapted, but what they did "chop up." "It may have been," proceeds the writer, "some kind of root or bulb; it may have been fruit"*. And so it may, according to the conditions of life and organization imagined by Professor FLOWER, but not according to those of the Creation open to our observations and comparisons. No known herbivorous Mammal is limited to teeth for slicing or "chopping up" vegetable food.

There is no difference, indeed, between X. and XII. on the main question at issue between them and me; but they are at variance between themselves on one point. Dr. FALCONER was unable to resist the proofs of carnivority from the demonstrated molar dentition of *Thylacoleo*; but, having committed himself to a different interpretation of the like dentition in *Plagiaulax*, he defended his position with an ingenuity which excited in the author of XII. and others the sentiments expressed by the epithets "masterly," "amply demonstrated," &c.

Professor FLOWER, however, with the unmistakable evidences of essential conformity between the dentition of *Plagiaulax* and *Thylacoleo*, consistently applied himself to show that *Thylacoleo* was as good an herbivore as *Plagiaulax*. He says, "Dr. FALCONER, in his anxiety to show that *Plagiaulax* could not have been carnivorous, has endeavoured to separate it as much as possible from *Thylacoleo*, laying great emphasis on all the points of divergence that could be found between them. He was evidently under the impression that the latter had been proved to be a carnivorous Marsupial, without staying to inquire into the arguments on which the assumption rested"†.

§ 16. *Family relations of Thylacoleo and Plagiaulax in the Marsupial Order.*—Of the existing groups of pouched *Herbivora* Professor FLOWER, in his paper on the Affinities of the extinct Australian Marsupial, which is "branded with such a direful title as *Thylacoleo carnifex*"‡, inclines to select the *Macropodidæ* as the one to which that Marsupial belonged; and, therein, more especially the *Hypsiprymni* or Rat-Kangaroos, in which he "sees at once in the great cutting premolar a miniature of that of *Thylacoleo*"§.

In a "Postscript" he derives encouragement of his views from "some remarks 'On the Dentition of *Thylacoleo carnifex*, Ow.,' by Mr. GERARD KREFFT, the able Curator of the Australian Museum, Sydney, in the Ann. & Mag. Nat. Hist. vol. xviii. ser. 3, p. 148,

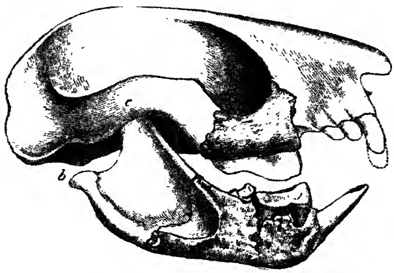
* XII. p. 318. † Ib. p. 308. ‡ Ib. p. 314. § Ib. p. 310.

1866;" accompanied by "a conjectural restoration of the then unknown anterior part of the skull and incisor teeth," which, Professor FLOWER proceeds to assert, "subsequent discoveries have in great measure confirmed"*.

I may here remark that, as my "Description of an almost entire Skull of the *Thylacoleo carnifex*," was "Received June 8,—Read June 15, 1865" (Phil. Trans. 1866, p. 73), the anterior part of the skull and incisor teeth were not unknown in September 1866, nor at the date of Mr. KREFFT's paper, May 24, 1866. The degree of confirmation which the restoration of the skull, according to the herbivorous hypothesis, has subsequently received, may be estimated by the comparison of fig. 7, p. 233 and fig. 13, with Plates XI., XII. & XIII., and more especially with Plate XIV. of the present paper.

Mr. KREFFT in this communication, and in its conjectural illustration (fig. 13), inclines to refer *Thylacoleo* to the *Carpophaga*†, deeming it "not much more carnivorous than the Phalangers of the present time‡".

Fig. 13.



Restoration of the skull and teeth of *Thylacoleo*, by Mr. KREFFT, on the herbivorous hypothesis. (Ann. & Mag. Nat. Hist. 1866, vol. xviii. pl. xi.)

But in the "List of the Fossils from the Caves of Wellington Valley," appended to the 'Report to the Trustees of the Australian Museum regarding the examination of those Caves,' Mr. KREFFT writes:—"5. Teeth and bones belonging to the gigantic Kangaroo-Rat named *Thylacoleo carnifex* by Professor OWEN."

Of the same opinion I infer to be Mr. BOYD DAWKINS, F.R.S., from the following passage in his instructive paper "On the Rhætic Beds and White Lias of Western and Central Somerset:"—"The presence of the *Macropoda* (Van der H.) (=*Poëphaga*, OWEN) is proved by the discovery of the Kangaroo-Rat allies,—viz. in the Purbeck beds, of the *Plagiaulax*, the true affinities of which have been so amply demonstrated by Dr. FALCONER§; in the Rhætic bone-bed, of the *Microlestes* of Frome and Diegerloch, closely allied, according to Professor OWEN, to *Plagiaulax* (Palæont. p. 303); and, lastly, in the strata below the bone-bed, by the discovery of the *Hypsiprymnopsis Rhæticus* of the Watchet shore"‖.

To the evidence and question of the affinity of *Thylacoleo* and *Plagiaulax* to existing groups or families of the Marsupialia I next address myself.

* XII. p. 319.

† OWEN, "Classification of the Marsupialia," Trans. Zool. Soc. ii. p. 322.

‡ Ann. & Mag. Nat. Hist. 1866, xviii. p. 149.

§ Quart. Journ. Geol. Soc. vol. xiii. p. 261, vol. xviii. p. 348.

‖ Id. ib. vol. xx. 1864, p. 412. But see the examination of the grounds of the determination of this rhætic fossil as the tooth of a Potoroo, in my "Monograph on Mesozoic Mammals," in the volume for 1870 of the Palæontographical Society, pp. 8–10.

The pouched *Mammalia* show two taxonomic modifications of the anterior mandibular teeth: in one, several pairs of incisors intervene between the canines; in the other, one pair of incisors of large size are present and no canines. The first condition characterizes the "polyprotodont section," the second the "diprotodont section"*. The existing representatives of the latter group of *Marsupialia* are confined to the Australasian area; some of the former group are American.

In both sections there are modifications of dentition, of digestive organs, and limb-structures, which in an interesting degree run parallel with each other,—the arboreal diprotodont Phalangers and Petaurists, *e. g.*, with the Opossums and Phascogales, and the saltatory *Bandicoots* and *Chœropus* with the Potoroos and Kangaroos; while the gradatory carnivorous Polyprotodonts have no known existing diprotodont correlatives.

But my knowledge of mammalian organization does not authorize me to assert that the diprotodont type of *Marsupialia* could not be so modified as to subserve carnivorous habits. I recognize no sufficient ground for the confidence that predatory dentition must be associated with three or more incisors antecedent to the canine, or "by the interposition of a line of incisors" between the two canines of either the upper or lower jaw.

Dr. FALCONER, in reference to the known Marsupial genera, asserts:—"In all the carnivorous genera and species, fossil or recent, of which the dentition has been accurately determined, there are three or more incisors, followed by a canine, on each side of the jaw, above and below; and the empirically observed result is consistent with a rational interpretation of the arrangement, in reference to their food and the means of procuring it. On the other hand, in all the existing strictly phytophagous genera, there is only a solitary incisor (being that next the axis) on either side of the lower jaw, and no canine"†. I shall presently inquire how far this alleged generalization applies to known existing species, premising that it can only be affirmed as bearing on the interpretation of the fossil remains of *Thylacoleo* and *Plagiaulax*, by demonstrating the inaccuracy of my determinations of the dentition of those extinct genera, and by resting on the foregone assumption that no Marsupial genus can or could be carnivorous unless it had the canine or caniniform tooth preceded by three or more incisors, and that "a solitary incisor," however shaped and associated with other teeth, must make a "strictly phytophagous Marsupial."

My endeavours, and whatever success may have attended them, in the interpretation of animal structures, have depended mainly on careful avoidance of antecedent assumptions of the extent of secondary modifications with which a dentition primarily fashioned for animal food might be associated. I leave my mind open, for example, to deduce consequences from observing the modifications of size, shape, and direction of the "solitary incisor on either side of the lower jaw," and the form, size, and number of the premolars, and more especially of the true molars associated therewith. To think or reason otherwise would be simply to argue in a circle, as thus:—"All carnassial Marsupial genera have incisors as well as a canine; *ergo*, no Marsupial genus with a laniari-

* OWEN's 'Anatomy of Vertebrates,' vol. iii. p. 293.

† X. p. 351; XI. p. 434.

form tooth not preceded by incisors can be carnivorous. All phytophagous Marsupial genera have a pair of developed incisors, approximated and placed collaterally in the lower jaw as in placental Rodents; *ergo*, a Marsupial fossil mandible with such incisors must be of an herbivore." But it is affirmed:—"the incisors of *Plagiaulax* are framed, in regard to number, order of suppression, collateral position, and relation to the premolars, in exact correspondence with the type of the Marsupial herbivores, such as *Halmaturus*, *Hypsiprymnus*, and *Phascolarctos*, and wholly at variance with the Carnivorous type"*. And if considerations of the shape of the incisor, modification of its working end, angle of its projection, degree of curvature of the tooth, evidence of its temporary growth or otherwise be set aside, the same may be affirmed of the lower incisors of *Thylacoleo*.

Any one enjoying a sense of confidence in the impossibility of a modification of the diprotodont type of Marsupial dentition for carnivorous habits may well dispense not only with a consideration of all those characters of the teeth in question which truly point to their function, but also of the modifications of size, shape, and number of the molar teeth associated with such pair of lower laniariform incisors.

I am not cognizant of any facts subversive of the Cuvierian principle as to the teeth which should first be observed in an unknown fossil by the palæontologist in quest of the nature of its food, and I cleave to the belief of their primary importance as throwing light on the problem to be solved.

I have qualified Dr. FALCONER's generalizations even when restricted to existing Marsupials, as "alleged." Let me recall to the recollection of his followers some of the instances which invalidate the general averments adduced to show that *Thylacoleo* and *Plagiaulax* must be herbivorous because diprotodont.

The small insectivorous Marsupial *Tarsipes* combines with its two "well-developed, long, slender, and pointed lower incisors"†, minute molars unfitted, as in *Plagiaulax* and *Thylacoleo*, for mastication of vegetable food. Its simple alimentary canal, only exceeding the entire length of the animal by about one-half, is "destitute of cæcum"‡, as in the small Polyprotodont *Phascogale*§. "When intent upon catching flies it would sit quietly in one corner of the cage, eagerly watching their movements"‖.

According to the "fundamental principles which comparative anatomy supplied" to Dr. FALCONER "for his guidance" (but which principles he nowhere defines), *Tarsipes*, like *Thylacoleo* and *Plagiaulax*, having "a pair of developed incisors approximated and placed in the lower jaw collaterally," should have been "phytophagous."

Let us test the contrasted conditions of the generalization as to incisors by another appeal to living Nature. "The Root-feeding Dalgyte"¶, or Australian "Native Rabbit" (*Peragalea lagotis*), is a miscellaneous eater. The specimen in the Zoological Gardens

* X. p. 352; XI. p. 435.

† WATERHOUSE, 'Natural History of Mammalia' (*Marsupialia*), p. 342.

‡ *Op. cit.* p. 343.

§ OWEN, Art. "*Marsupialia*," Cyclopædia of Anatomy, vol. iii. p. 300, fig. 122 (*Phascogale flavipes*).

‖ GOULD, 'Mammals of Australia,' vol. i. (*Tarsipes rostratus*).

¶ GOULD, 'Mammals of Australia,' vol. i. Introduction, p. xvii.

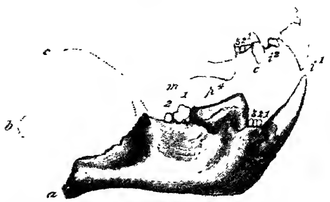
was dieted on bread and milk. "In its native grassy haunts its food consists of insects and their larvæ, and the roots of trees and plants"*, for the mastication of which its broad flat grinders† are well adapted. Nevertheless the canines proper are separated in the upper jaw by not fewer than ten incisors, and in the lower jaw by six incisors‡.

The cloven-footed *Chæropus*, equally polyprotodont, but with digital characters more closely resembling those of the Artiodactyle Ungulates than in any other marsupial genus, is not carnivorous. The condition of the molars associated with the "three or more incisors followed by a canine on each side of the jaw," clearly points to that fact. The accomplished naturalist and explorer of Australian haunts of animal life thus testifies of *Chæropus castanotis*:—"As its dentition would indicate, its food consists of insects and their larvæ, and of vegetable substances of some kind, probably the bark of trees and tuberous roots"§.

In fact the parallel and convergent modifications of all those structures which truly influence and indicate the food and habits of the animal have been noticed by all who have devoted the requisite attention to the Marsupial order. GOULD well remarks, "*Hypsiprymni* grub the ground for roots, and live somewhat after the manner of *Peramelides*, with which, however, they have no relationship" ‖; meaning within the ordinal limits—the one group being "diprotodont," the other "polyprotodont," with modifications of the two subordinal types bringing them to close similarity, if not identity, of locomotion, diet, and mode of obtaining food.

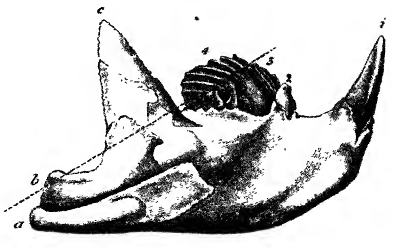
In the case of a fossil mandible of either genus the palæontologist, referring to the molar teeth, would be led to the like inference as to food and habits, although he would see in one a pair of large approximate incisors and no canines, in the other canines with small incisors interposed.

Fig. 14.



Mandible and teeth, *Thylacoleo*, reduced to one-fourth nat. size.

Fig. 15.



Mandible and teeth, *Plagiaulax*, magnified 4 diameters. (After FALCONER, Quarterly Journal of the Geological Society, vol. xiii. 1857, p. 280, fig. 14.)

Thylacoleo (fig. 14) and *Plagiaulax* (fig. 15) more closely resemble each other in

* GOULD, *tom. cit.* (*Peragalea lagotis*).

† Cyclopædia of Anatomy, vol. iii. (1841), Art. "*Marsupialia*," p. 274, fig. 96.

‡ Ib. Art. "*Marsupialia*," *ut suprà*.

§ GOULD, 'Mammals of Australia,' vol. i. (*Chæropus*).

‖ Id. ib. Introduction, p. xix.

their dentition and shape of mandible than they do any other family of diprotodont Marsupials*. From the characteristic reduction in size and number of the molar teeth I have associated them as members of a "paucidentate" family or section.

To which of the existing families of Diprotodonts is the paucidentate one most nearly allied? *Thylacoleo* best lends itself to the solution of this question, its maxillary as well as mandibular dentition being now, I may affirm, accurately determined. It is highly probable, from the close conformity of *Plagiaulax* to *Thylacoleo* in the peculiarly and extremely modified dentition of the lower jaw, that the maxillary teeth also resembled those of the larger diprotodont carnivore. Of this the dental formula is:—

$$i.\ \tfrac{2-2}{1-1},\ c.\ \tfrac{1-1}{0-0},\ p.\ \tfrac{4-4}{4-4},\ m.\ \tfrac{1-1}{2-2}:=30.$$

No existing Diprotodont offers a like formula. That of the *Poëphaga*† departs further than in most other diprotodont families, because there is no tooth interposed between the incisor and sectorial in the lower jaw, and in most Kangaroos not more than two are developed between the front incisor and sectorial in the upper jaw on each side, the two intervening teeth being both incisors—both anterior to the maxillo-premaxillary suture. *Hypsiprymnus* and *Bettongia* have a small canine in that suture, and two incisors between the larger front incisor and the sectorial in the upper jaw, but no teeth in that interspace in the lower jaw (figs. 17, 18). Of the more important true molar teeth (id. ib. m_{1-4}), the first three have "a quadrate form, presenting four equidistant blunt tubercles which are joined in pairs by transverse ridges, but with these ridges less elevated than the points of the tubercles; there is a slight trace of the band of the tooth" ('cingulum' of my 'Odontography') "on the front and back part of each molar as in *Macropus*. The hindermost" (fourth) "molar is generally small, almost round. Cases occur in which the last molar tooth is absent; and, what is more extraordinary, I have observed an extra tooth on each side of the upper jaw in a species of *Hypsiprymnus*"‡.

Thus in these mixed feeders, but with the vegetable diet predominating, the molar teeth adapted to such diet are never fewer and commonly more in number than in the most typical placental *Herbivora*. In relation, apparently, with the drier and tougher vegetable fibres of Australia, the premolar is trenchant and strengthened by vertical grooves and ridges. In one of the New Guinea Tree-Kangaroos (*Dendrolagus dorcocephalus*) this trenchant tooth (p, fig. 16) is proportionally larger than in the Australian Potoroos and Bettongs, but the light-giving teeth (the true molars) "are conformable with the *Macropus* type"§.

* Dr. Falconer asserts, "*Thylacoleo* and *Plagiaulax* may be regarded as being as wide apart among the Marsupials as the two former (*Machairodus* and *Moschus*) are among placental Mammals."—X. p. 358; XI. p. 442.

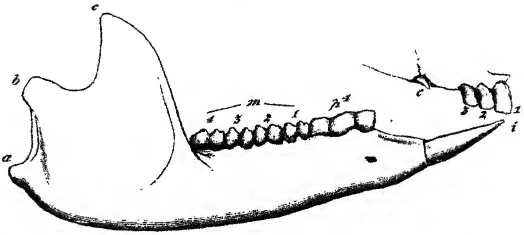
† I hold by this term, preferring it to the subsequently propounded one, *Macropoda*, of Van der Hoeven, because the latter is equally applicable in its descriptive sense to the long-legged, saltatory Polyprotodonts.

‡ Waterhouse, 'A Natural History of the Mammalia' (*Marsupialia*), 8vo, 1845, p. 194.

§ Ibid. p. 182, pl. 10. fig. 3. In my 'Odontography' I showed that the "maximum of development of the trenchant premolar was attained in the arboreal Potoroos of New Guinea (*Hypsiprymnus ursinus* and *Hyps. dorcocephalus*), in the latter of which its antero-posterior extent nearly equals that of the three succeeding molar

There is no greater contrast in the Diprotodont series than that presented by the molars in the *Poëphaga* and the *Paucidentata*—the Kangaroos and Potoroos on the one hand, the *Thylacoleo* and *Plagiaulax* on the other. A trenchant tooth may exist for other purposes than that of cutting vegetable matter, notwithstanding the stress laid by Messrs. FALCONER, FLOWER, and BOYD DAWKINS on the degrees of resemblance subsisting between the sectorials in the *Paucidentata* and *Poëphaga*. The differences which are pointed out in the present paper outweigh the resemblances in number and importance, irrespective of the characters given by the rest of the dentition.

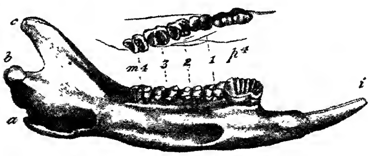
Fig. 16.



Macropus (Dendrolagus) dorcocephalus, mandible and teeth, nat. size.

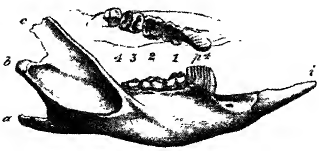
First, as to relative size. With all the additions of poëphagous species made to our Zoological Lists since 1840, I still find the *Macropus dorcocephalus** (fig. 16) to present the nearest approach to *Thylacoleo* in the relative magnitude of the trenchant premolar (ib. *p* 4). Including with that tooth the four succeeding molars, as the "molary series," the premolar constitutes nearly two fifths of that series: in *Thylacoleo* (fig. 14) the pre molar (p_4) constitutes seven-tenths of the molary series.

Fig. 17.



Hypsiprymnus minor, mandible and teeth, nat. size.

Fig. 18.



Bettongia penicillata, mandible and teeth, nat. size.

In some Potoroos, *Hypsiprymnus minor*, *H. Grayi*, e. g. (fig. 17), the premolar (p_4) a little exceeds in fore-and-aft extent the two succeeding molars (m_1 & $_2$), but in most it

teeth" (p. 389). Dr. FALCONER misquotes this as a "statement that in two Potoroos of New Guinea its antero-posterior extent nearly equals that of the three succeeding molars."—X. p. 358; XI. p. 442. But in *Dendrolagus ursinus*, *D. inustus*, and *D. Brunii* the proportion of the premolar does not exceed that of *Hypsiprymnus Grayi*.

* The many and small gradations which those additions have made known, in retained rudimental or functionless canines, in hairiness of muzzle, of tail and other parts, in shape of ears, in proportion of fore and hind

falls short; and in *Bettongia penicillata* (fig. 18) it is reduced to the extent of one and a half of the succeeding molars, or to one-fourth of the entire "molary series."

A comparison more closely bearing upon the use to which a sectorial premolar has been applied is that of the relation of its fore-and-aft length with the length of the "diastema" or interval between it and the incisor; for the jaws of marsupial *Herbivora* are commonly characterized by length, and those of *Carnivora* by shortness.

Long as the premolar is in *Dendrolagus dorcocephalus*, the slender jaw is prolonged to as great an extent before it gives exit to the procumbent incisor; this interval is never less, usually more, than the fore-and-aft length of the trenchant premolar in all *Poëphaga*. In *Thylacoleo* the interval between the fore border of the homologous lower premolar and the outlet of the incisor's alveolus is one-fourth the fore-and-aft extent of such premolar.

The contrast between *Thylacoleo* and herbivorous Diprotodonts, in the proportion of the trenchant premolar of the upper jaw to the succeeding molars (which in the great carnivore are reduced to one, Plate XI. fig. 3, m_1, as in *Felis*), is still more striking and decisive as to the use of such premolar than in the lower jaw. With the predominance of antero-posterior over vertical extent of crown in the trenchant border, and in the proportions of the two roots of the lower one, the resemblance of the premolar of *Thylacoleo* to that in any poëphagous or herbivorous Diprotodont ceases. It has not the parallel ridges and grooves which characterize the homologous tooth in the Potoroos (*Hypsiprymnus*, *Bettongia*, *Potorous*, &c.).

In the upper sectorial premolar of *Thylacoleo*, the two best marked ridges are the one defining the anterior border (Plate XI. figs. 1–3, *z*), and the one terminating the inner prominence of the swollen fore part of the tooth (ib. *v*) answering to the somewhat more developed ridge in the upper carnassial of *Machairodus* (ib. figs. 15, 16, *v*).

The slight outswelling of the base of this ridge (Plate XI. figs. 2, 15, *v'*) I regard as a rudimental homologue of the internal tubercle of the upper carnassial in *Felis*. Thus the carnassial in *Machairodus* (*Drepanodon*) offers an instructive intermediate modification of that tooth between *Felis* and *Thylacoleo*. I am the more impressed by the degree of resemblance through adaptive modification of the sectorial premolar in the carnivorous marsupial, seeing the differences that might be expected, as, indeed, some do exist, in homologous teeth, developed for the same office, in two such different routes of derivative modification as are exemplified by the Marsupial and Placental series of mammalian structures.

One vertical ridge on the outer and broader fore part of the crown (Plate XI. fig. 1, p_4, *u*) feebly represents the second lobe of the feline carnassial; it is divided by a shallow vertical depression from the part (ib. *z*) representing the anterior lobe of that tooth*.

limbs, in length or curvature of claws, &c., have afforded the "Gattungsmacherei" grounds for *Halmaturus*. *Lagorchestes*, *Heteropus*, *Petrogale*, *Osphranter*, *Dendrolagus*, *Hypsiprymnus*, *Bettongia*, *Potorous*, *Dorcopsis*, &c.

* This structure is better marked in an upper carnassial of *Thylacoleo* from the breccia-cave, of which photographs of the outer and inner sides were transmitted to me in the series above noted.

The rest of the outer surface is feebly undulated, and that more toward the base than the upper part of the crown. On the inner side, the basal undulation, through vertical risings and sinkings of the enamel, is more feebly marked*. The chief vertical indent on the outer side of the crown of the Thylacolean upper carnassial is near the posterior third (Plate XI. figs. 1, 3, *o*), and answers to the deeper vertical notch in *Machairodus* (ib. fig. 15, *o*) and *Felis* which defines the posterior lobe of their upper carnassial. It is interesting to note that this notch is less marked in *Machairodus* than in *Felis*, and also that the concavity of the outer side of the carnassial from before backward (*i. e.* from the outer ridge representing the second or middle lobe in *Machairodus* and the hind end of the crown) represents the more angular concavity due to the deeply vertical groove on the outer part of the carnassial in the above placental *Carnivora*.

Professor FLOWER (XII. p. 309) states that the resemblance of the great premolar of *Thylacoleo* to the carnassial of the true *Carnivora* is merely superficial; and he specifies among the differences, "especially the absence of any distinct inner lobe or tubercle" (in the upper molar) "supported by a third fang" (ib. p. 310). He was, probably, not cognizant of the example afforded by one of the extinct true *Carnivora* of the absence of the inner lobe or tubercle, or, rather, its reduction to a ridge, the lower swollen base of which (Plate XI. fig. 15, *v'*) may be compared to "a less developed homologue of the inner tubercle in the normal species of *Felis*"†.

Dr. FALCONER, indeed, repudiates this partial homology, and affirms "of the upper carnassial of his Sewalik *Machairodus*" that "neither the anterior lobe nor the middle one bears the slightest indication of bearing an internal tubercle" (XI. p. 456); and this further evidence of transitional structure between the Feline and Thylacoleonine carnassials will probably be acceptable to Professor FLOWER, though it is enunciated, as I think, in exaggerated terms.

The well-defined vertical ridges and intervening grooves on both outer and inner surfaces of the crown of the sectorial premolar of the Potoroos vary in number in different species, but are countable and pretty constant in such species, rising from four to eight or more; they are best marked on, and sometimes limited to, the apical half of the crown, the enamel at the base being smooth and even. The fore part of the Potoroo's sectorial is not broader (is usually narrower) than the hind part, and the cutting-edge runs straight or nearly so.

The transverse expansion of the fore part of the lower carnassial of *Thylacoleo*, representing the thicker anterior lobe of the carnassial of the Felines, the fore-and-aft convexity of the outer surface of the crown, and the concavity of the inner surface answering to that which defines the two lobes of the blade in Felines are better marked than

* Dr. FALCONER, quoting my original description of the carnassial in *Thylacoleo* as being "slightly grooved vertically on the inner side," correctly proceeds: "these indentations disappear about halfway up towards the edge, where the surface becomes reticulately rugose, being precisely the reverse of what occurs in the last premolar of *Hypsiprymnus*" (X. p. 356, XI. p. 440).

† OWEN, British Fossil Mammals, 4to, 1846, p. 178.

are those correspondences in the upper carnassial, and are unmistakable. A broader well-defined prominence on the fore part of the inner surface of the crown of the lower sectorial (Plate XII. fig. 11, *r*) leaves a part anterior to it (ib. *a*) representing the anterior basal talon, chiefly marked or extended upon the inner surface of the fore part of the crown in the lower carnassial of *Felis* and *Hyæna*. The indications of vertical elevations of enamel are more feeble in the lower than in the upper sectorial, and are chiefly seen at the basal part of the inner surface. The notch at the middle of the trenchant border in the less worn lower carnassial (Plate XII. fig. 11) clearly indicates divisions resembling, though more feebly marked, the anterior and posterior lobes of the homologous tooth in the placental *Carnivora* (ib. fig. 12).

The absence of the anterior transverse expansion, and the straight line described by the trenchant border of the lower sectorial of the Potoroos, is, at least, as strongly marked in the lower jaw (ib. figs. 8, 10, 13) as in the upper one (Plate XI. figs. 17, 18). In juxtaposing the specimens of the homologous teeth in *Thylacoleo* and any Potoroo for a true deduction of comparative similarity and difference, "one sees at once that the great cutting premolar of the *Hypsiprymni* or Rat-Kangaroos is" not "a miniature of that of *Thylacoleo*" *. And, if it were, the function of such sectorial could not be deduced from mere shape, but from the nature of the other teeth wherewith it is associated, and the modifications of the jaws by which such dentition was worked.

The student in reading of the "great cutting premolar of the Rat-Kangaroos" must bear in mind that the epithet is relative. Where such tooth is greatest in those vegetarians it is small in comparison with its homologue in *Thylacoleo*. The difference of shape, direction, term of growth, and of every character meaning function is still greater and more obvious in the incisors of the Diprotodonts compared than in the sectorials; and the degree and kind of difference shown by *Thylacoleo* testifies to the carnassiality of the main representative tooth of the molary series.

Against the association of that great carnivore with the *Poëphaga* "(= Macropoda, V. d. H.)," there are opposed not only the differences above demonstrated in the homologous sectorial teeth, but the absence of the third pair of upper incisors and the presence of premolars in advance of the sectorial one in both jaws of *Thylacoleo*. It will be admitted by candid readers of both my Papers on that genus that I have been reticent of conjecture or assumption; but I venture to say that when the limbs of *Thylacoleo* are restored they will not be "macropodal," not minimized at the fore part and maximized at the hind part of the body, for bipedal saltatory actions to bear it swiftly away from carnivorous pursuers, or to carry it far abroad from pasture to pasture and from scrub to scrub in quest of vegetable food, but that they will agree in the main with the limbs of *Leo*, *Thylacinus*, and *Sarcophilus*.

Pursuing the comparison of *Thylacoleo* with other *Diprotodonts*, we may at once dismiss the arboreal *Phascolarctidæ*, with a trenchant premolar (at least in the young Koalas), on the same grounds as those on which the *Poëphaga* are rejected from the association.

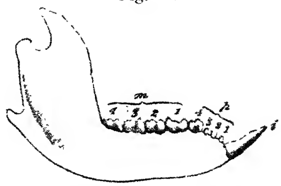
* XII. p. 310.

The superficial resemblance in the comparative views given in XII. pp. 312, 313, figs. 2 & 4, of the skulls of the Koala and of the *Thylacoleo* (as restored by Professor FLOWER), vanishes when they are turned from the front to the side view, as in figs. 6 & 8, p. 233.

The sectorial of *Phascolarctos* forms one fifth of the molary series, and mainly through "the greater relative size than in other vegetable-feeding Diprotodonts of the four following molar teeth"*. The incisor formula differs by excess, as the premolar formula does by defect, compared with the dentition of *Thylacoleo*, and this in the same way and degree as in Kangaroos and Potoroos.

We must pass to another family of Diprotodonts to find the two minute (I termed them "functionless") premolars† in advance of the last which retains its sectorial use and equality of length with the succeeding molar. In *Phalangista ursina*, *Ph. maculata*, and *Ph. chrysorrhoa* the functional premolar is preceded by two rudimental premolars as in *Thylacoleo*. In *Phalangista Cookii* (fig. 19), where the upper canine is minute and protrudes at the maxillo-premaxillary suture, two small premolars intervene between it and the homologue of the upper carnassial of *Thylacoleo*: the same degree of correspondence in numerical formula is represented by some Petaurists‡; but I have failed to find any species of "*Carpophaga*"§ in which three premolars appear between the functional one and the canine, or any species in which the upper incisors are reduced to two on each side. That a tendency to deviate by such reduction was amongst the inconstant characters of organization of diprotodont *Marsupialia* is exemplified by the Wombats, in which no incisors are developed behind the large upper anterior pair. *Thylacoleo* shows an interesting intermediate stage of the incisive formula, viz. $i.\,\frac{2-2}{1-1}$, between the $i.\,\frac{1-1}{1-1}$ of *Rhizophaga*‖ and the $i.\,\frac{3-3}{1-1}$ of all other existing families of Diprotodonts.

Fig. 19.



Mandible and teeth, *Phalangista Cookii*, nat. size.

Of all known *Marsupialia*, recent or fossil, *Plagiaulax*, so far as its dentition is accurately determined, is most closely allied to *Thylacoleo*. In the lower jaw the true molars are similarly reduced to two of small size and tubercular form. One cannot suppose that they were opposed by more tuberculate molars above; the analogy of *Thylacoleo* (Plate XIV.) would point to fewer. A character, indeed, of the first of the inferior molars of *Plagiaulax*, overlooked by Dr. FALCONER, would indicate that it worked sectorial-wise, like the fore part of the anterior lower molar of *Thylacoleo*, upon the back part of the blade of a large upper carnassial; I allude to the smooth vertical wall-like surface of the inner side of the outer half of the crown of *m* 1, in *Plagiaulax*¶.

* OWEN, "Classification of *Marsupialia*," Trans. Zool. Soc. (1839) p. 326. † *Loc. cit.* p. 323.

‡ E. g. *Petaurus* (*Belideus*) *flaviventer*, Cycl. of Anat. Art. *Marsupialia*, tom. cit. p. 264, f. 89.

§ OWEN, "Classification of *Marsupialia*," *ut suprà*, p. 322.

‖ Ibid. p. 329.

¶ OWEN, Monograph on Mesozoic Mammals, *tom. cit.* t. iv. figs. 9, 12.

Taking the same range of the molary series for comparison as in *Thylacoleo* and existing Diprotodonts, in reference to the character of size of the last trenchant premolar, the tooth equals in antero-posterior extent one-half of that series in *Plagiaulax*. But in this more ancient Diprotodont the premolars anterior to the last large one have not undergone the extreme degradation which they show in the tertiary fossil (*Thylacoleo*) and in some existing Diprotodonts of Australia. They are modified, in *Plagiaulax*, for sectorial function, and are so combined with the last and largest sectorial as to work with it as one instrument, obliquely ridged and notched at the convex cutting-margin, like a section of a circular saw. I have elsewhere* pointed out the advantage of this modification of carnassial in dividing the integuments and other tissues, tougher and drier than those in Mammals, of the lacertian members of the cold-blooded class which so abounded with the small carnivorous Marsupials in the same Mesozoic period and place.

If it be admitted that, so far as the lower jaw and its dentition show, *Plagiaulax* (figs. 10 & 15), with its two or three reduced anterior premolars, its suddenly enlarged hind premolar, its disproportionately small and few (two) tubercular molars, and its large laniariform upcurved incisor, comes nearest to *Thylacoleo* (figs. 8 & 14), it is plain, from the antecedent comparisons with existing Diprotodonts, that there are no grounds for inferring the *Macropoda* to have been derived from the *Paucidentata*, or these from Rat-kangaroos.

What we do learn from consideration of the fossils in question is, the fact of an additional and most interesting modification of the Diprotodont section of the Marsupial order or subclass, unknown before the discovery of these fossils. We further learn that such modification, which, from the extreme reduction of the true molar series, I have been led to take as the character of a "paucidentate" family of Marsupials, was already established at the Purbeck period; yet with modifications interestingly exemplifying the tendency to the "more generalized condition of structure" as compared with the newer tertiary extinct form.

§ 17. *Tendency from the general to the particular in the Dentition of the Paucidentate Marsupials.*—But I am here met by another objection. Dr. FALCONER, attacking the principle of the tendency to transition in organisms from generalized to specialized structures as they approach in geological position the present time, writes: "Among other arguments, they insist that the earliest Eocene Mammalia, both carnivorous and herbivorous, possessed, in most cases, the full complement of teeth; while forms characteristic of later times, such as the Felidæ and Ruminantia, are remarkable for special suppression of these organs. If the generalization were really of as wide an application as has been claimed for it, we ought to find evidence of closer adherence to the general archetypic model the further back we recede in time. But so far is *Plagiaulax*, at present the oldest well-ascertained herbivorous mammal yet discovered, from giving any countenance to the doctrine, that it actually presents the most specialized exception, so to speak, from the rule to be met with in the whole range of the *Marsupialia*.

* OWEN, Monograph on Mesozoic Mammals, *tom. cit.* t. iv. figs. 9, 12.

fossil or recent. It had the smallest number of true molars of any known genus in that subclass, six at least of the normal number of incisors being also suppressed" *.

But *Plagiaulax*, viewed as a member of the same predaceous group of *Marsupialia* as *Thylacoleo*, affords an interesting instance of adherence to the law above disputed. The extinct pouched carnivore of the tertiary period shows a single carnassial tooth on each side of the lower jaw; the extinct pouched carnivore of the oolitic period retained in one species three premolars of the carnassial type, in another species four (the normal or type number) on each side of the lower jaw. The parallel runs very close with that which the placental *Carnivora* show within the limits of tertiary time; as when we compare the miocene *Hyænodon* and its three lower carnassials with the modern *Hyæna*, where they are reduced to one, or when we compare the miocene *Amphycyon* with its three upper true tubercular molars with the modern *Ursus*, where they are reduced to two, or the modern *Felis*, where they are reduced to one. If, also, the oolitic *Phascolothere*, although it is known (to me) only by half its lower jaw and the teeth of that moiety or "ramus," be compared with the modern Opossum, represented by the same part, the more generalized type is conspicuous in the absence of the degree of differentiation of the individual teeth in the oolitic fossil jaw which characterizes the homologous teeth in *Didelphys*. The canine is marked by only a slight superiority of size from the antecedent teeth, which are of similar shape, and divided from each other by similar intervals, in *Phascolotherium*. In *Didelphys* the canine is marked by greater relative size and difference of shape from the close-set group of small incisors anterior to it. The seven molars in *Phascolotherium* show gradational differences of size, but none of shape; save some simplification of the two smallest, which are the first and the last of the series of seven teeth. In *Didelphys* the last four molars are abruptly and markedly differentiated from the three preceding ones, so that zoologists distinguish the four as "true molars" from the three which are their "false molars." *Phascolotherium* does not lend itself to this distinction†.

A still more generalized type of dentition is shown by the multiplication of slightly differentiated teeth in the genera *Amphitherium*, *Amphilestes*, *Spalacotherium*, *Peralestes*, *Stylodon*, &c., of the lower and upper oolites. One solitary form (*Myrmecobius*) alone remains at the antipodes with minute and slightly differentiated teeth, in number exceeding the type one in most modern Mammals, and recalling that in lower and wider vertebrate groups.

The two or three smaller but functional premolars in advance of the large lower carnassial in the mesozoic *Plagiaulax* are reduced to two more minute functionless and speedily lost premolars in the neozoic *Thylacoleo*.

* Quart. Journ. Geol. Soc. vol. xiii. p. 276; XI. p. 427.

† This well-known fact in comparative odontology is here repeated in reply to the question addressed by Professor Huxley to the London Geological Society: "in what circumstance is the *Phascolotherium* more embryonic, or of a more generalized type, than the modern Opossum?"—Quarterly Journal of the Society, vol. xviii. (1862) p. li.

This fact invalidates the averment of "the contradictory bearing of the dental system of *Plagiaulax* upon the assumption that the earliest Mammals had the full complement of teeth:" which averment Dr. FALCONER reiterates and "calls special attention to," in X. p. 365, XI. p. 451. For if, in place of assuming *Plagiaulax* to be the earliest mammal, and, as such, with the full complement of teeth, or "the oldest well-ascertained herbivorous mammal," it be viewed as no more than it is, viz. a geologically earlier form than *Thylacoleo* with a dentition similarly modified for carnivority, the degree of difference between the two members of the *Paucidentata* is affirmatory instead of contradictory, in relation to the rule in question, rightly stated.

§ 18. *Relation of Size to Carnivority.*—One other argument against the predatory way of life of the subject of the present Paper remains for notice, although its very suggestion implies a sense of the insecurity of the grounds on which the herbivorous habits and affinities of *Plagiaulax* and *Thylacoleo* have been advocated.

They are affirmed to have been animals too small, too feeble, to have preyed upon others, especially when much larger than themselves.

Whoever has witnessed the well-known zoological phenomenon of the pertinacious pursuit and fatal attack of a hare by the diminutive weasel would pause, however, before venturing on such grounds of objection.

Dr. FALCONER, selecting for his purpose the most diminutive of the species of *Plagiaulax*, affirms: "The entire length of the specimen, including the six molars and premolars, together with the procumbent incisor (according to the metrical line *e*), does not exceed ·4 of an inch, of which the six cheek-teeth united make only about two and a half lines (·25 inch). I ask any zoologist or comparative anatomist to look at it, and say whether the dental apparatus of this extremely minute creature is competent to perform the duties required of a predaceous carnivore. Magnitude in this case is an important ingredient, as it necessarily involves measure of force. Could *P. minor* have preyed on small Mammals and Lizards? Is it not more probable that this pigmy form was itself an object of prey in the Purbeck fauna?"*

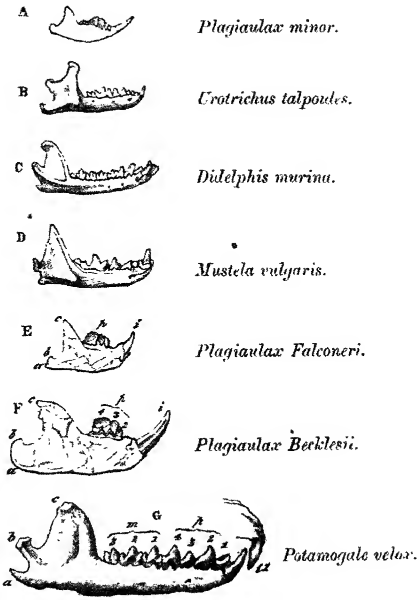
To this I reply, that I have now before me the original of fig. 15, *Plagiaulax minor* of the Quarterly Journal of the Geological Society of London for August 1857, xiii. p. 281, reproduced in the subsequent paper of Dr. FALCONER in Quarterly Journal &c. for June 1862 (X. p. 367), and copied in pl. 34. fig. 2 of the posthumous work (XI. p. 416).

The specimen (fig. 20, A) shows two molars and four premolars; the incisor is neither chisel-shaped nor procumbent, but rises with a slight curve to its pointed apex at an angle of 120°, with the line of the molar alveoli. The length of the dental series from the apex of the laniariform incisor to the hind part of the second molar is seven-sixteenths of an English inch, precisely the length of the dental series in *Urotrichus talpoïdes* (ib. B), a ferine mammal, 5 inches long from the snout to the tip of the tail, with a skull 1 inch in length, and an approximate pair of lower pointed incisors upcurved at the same angle as in *Plagiaulax minor*, but relatively less and shorter.

* X. p. 363; XI. p. 448.

Some Opossums, *e. g.* the murine and dorsigerous Philanders (fig. 20, C)*, have the mandible intermediate in size between that in *Plagiaulax minor* (A) and in *Plagiaulax Becklesii* (F); it is both shorter and weaker than in the latter species. A naturalist and good observer (Dr. CARTER BLAKE, F.G.S.) has expressed to me his surprise at witnessing, while in Central America, the disproportion of size between those mouse-like predaceous Marsupials, and the Lizards and Snakes on which they prey.

Fig. 20.



The above figures of the mandible and mandibular teeth are of the natural size.

I am not cognizant of any grounds afforded by zoology which forbid the supposition that a mammal of five inches in length, with the carnivorous type of dentition of *Plagiaulax*, may have been able to capture and kill the diminutive Lizards (*Saurillus*, *Macellodus*, *Nuthetes*†, &c.) abundantly associated with *Plagiaulax* in the Purbeck shales. Comparative anatomy suggests that the modifications of the dentition of *Plagiaulax minor*, as compared with the similarly sized Shrew (*Urotrichus*, fig. 20, B) and Opossum (*Philander murinus*, ib. C.), would give the Purbeck marsupial both the disposition and power to attack and prey upon animals of a larger size and higher organization than worms and insects. But the question of the carnivority of *Plagiaulax*, if weighed by

* In *D. dorsigera*, from tip of incisors to condyles, 10 lines; in *D. murina* $11\frac{1}{2}$ lines.

† OWEN, 'Quarterly Journal of the Geological Society,' 1854, vol. x. p. 420.

"magnitude as a measure of force," is not fully or fairly tested by the exclusive example of the most diminutive species.

In *Plagiaulax Falconeri* (Ow.*, fig. 20, E) the extent of the dental series, lower jaw, is six-sixteenths of an inch; in *Plagiaulax Becklesii*, Fr. (fig. 20, F) it is ten-sixteenths of an inch. The entire length of the mandible in this species, inclusive of the incisor, in a straight line, is 1 inch two-sixteenths; the depth of the ramus at the back part of the large carnassial is five-sixteenths of an inch.

In the Weasel (*Mustela vulgaris*, Cuv., fig. 20, D) the extent of the dental series, lower jaw, is eight-sixteenths of an inch; the depth of the ramus at the back part of the large carnassial is two-sixteenths of an inch.

With the greater relative depth and consequent strength of the jaw of *Plagiaulax* a greater size and strength of both laniary and carnassial teeth are concomitant. The condyle, which is on the level of the dental series in the Weasel, is below that level in *Plagiaulax*. Every modification of the small marsupial by which it departs from the little blood-thirsty Placental is in the direction of greater carnivority.

In *Phascogale penicillata* the extent of the dental series, lower jaw, is fourteen-sixteenths of an inch. It has four true molars in such lateral series, with relatively smaller laniaries and still smaller sectorial premolars than in *Plagiaulax*; the mandibular condyle is raised a little above the dental line; the carnivorous adaptation of both jaw and teeth is less marked than in the Purbeck marsupial. But what is the testimony in regard to the habits of the existing pouched carnivore no bigger than a rat?

Gould, who would be the last to repeat testimony to which zoology and comparative anatomy ran counter, writes "*Phascogale penicillata*, small as it is, comparatively, is charged with killing fowls and other birds"†.

I can bear personal testimony, and that to my own loss, of the attack and slaughter of nearly full-grown Shanghai Pullets by *Mus decumanus*. Comparative anatomy lends more aid to the credibility of the predatorial powers of the carnivorous marsupial than of the equally small rodent; but that both of them do attack and destroy animals more than twice their size and weight is a zoological fact.

Though magnitude may be, in one sense, a measure of force, it by no means necessarily implies the application of such force, and consequently is any thing but "an important ingredient" in the question of the carnivority of *Mus, Mustela, Phascogale*, and *Plagiaulax*.

But whatever bears on the interpretation of the singular dentition of the small "paucidentate" marsupial, logically applies to the larger one.

Mr. Krefft gives drawings of sections of the "lower incisor of *Thylacoleo, Nototherium*.

* Monograph of the Fossil Mammalia of the Mesozoic Formations, p. 84, plate iv. figs. 16, 16 A.

† "Mammals of Australia," fol. Introduction, p. xviii. Mr. Waterhouse remarks, "In the Phascogales, where the two foremost of the lower incisors are large, their increased development is, as it were, at the expense of the posterior incisors, which are very small, and the canine which follows them is but moderately developed."—Nat. History of the Mammalia, vol. i. (1845) p. 256.

Diprotodon, *Thylacinus*, and *Sarcophilus*," also of what he terms the "upper incisor and lower incisor of *Felis tigris*, ... showing the relative size of the teeth in these animals, and proving sufficiently that the *Thylacoleo* was far inferior in strength to a modern Tiger, and no match for ponderous Diprotodonts and Nototheriums"*.

If the carnassial tooth were selected instead of an incisor, it would show on the above basis that *Thylacoleo* was "far superior in strength and carnivority to the modern Tigers and Lions." But I would submit that the test of relative size of a single tooth, if even the answerable or homologous one were recognized by the tester, is not a decisive or sufficient one in the present question.

It is evident that Mr. KREFFT's figures 7 & 8 are sections of the canine, not the incisor, of the Tiger. But if that tooth in the *Hippopotamus* were exemplified by a similar section, it would be no element, or a very deceptive one, in concluding as to strength or carnivority. The canines of *Moschus* and other like instances will at once suggest themselves to the competent Comparative Anatomist.

To the assertion of the "gigantic herbivorous *Nototherium*" &c. being "many times as large as the *Thylacoleo*"†, I will oppose a few matters of fact and mensuration. The length of the skull of the largest species of *Nototherium* (*N. Mitchelli*) is 1 foot 6 inches ‡; that of the skull of *Thylacoleo carnifex* is 10 inches 8 lines: were the occipital ridge and spine entire in the specimen measured (Plate XIV.) it might be set down at 11 inches. It will be within the bounds of accuracy to say that the Notothere was twice as large as the *Thylacoleo*, not more. The skull of the *Diprotodon* is 3 feet in length; it is, however, large in proportion to the trunk and limbs; bulk for bulk, it was probably not much larger in comparison with the *Thylacoleo* than is the Giraffe in proportion to its destroyer the Lion. The disproportion between the Wolverene (*Gulo luscus*) and its prey the Reindeer must be greater than that which the dimensions of the known fossils of *Thylacoleo* and *Diprotodon* suggest. The length of a Lion's skull before me is 1 foot; that of the skull of a South-African Giraffe is 2 feet 2 inches. If we next compare, not a single tooth merely, but the whole lethal tooth-weapons of *Thylacoleo* and *Felis tigris*, we get the following results. The length from the fore part of the laniary to the hind part of the carnassial, upper jaw, is in *Felis tigris* 3 inches 7 lines; in *Felis spelæa* 4 inches; in *Thylacoleo carnifex* 4 inches 3 lines. In the lower jaw the proportions are reversed; but the difference affords no reasonable ground for inferring such inferiority of strength or destructive power as to support the inference that *Thylacoleo* was incapable of playing the same part in relation to the *Nototheres* and *Diprotodons* as the Lion now performs in relation to the Buffaloes and Giraffes.

* "On the Dentition of *Thylacoleo carnifex* (Ow.)," in Annals and Magazine of Natural History, Third Series, vol. xviii. 1866, p. 148.

† Professor FLOWER, F.R.S., however, adopts the argument from size, and rejects the hypothesis "that *Thylacoleo* was the destroyer of the gigantic herbivorous Marsupials (many times as large as itself) with which its remains are found associated, the Diprotodons and Nototheres."—XII. p. 318.

‡ OWEN, "On some Outline-drawings and Photographs of the Skull of *Nototherium*," Quarterly Journal of the Geological Society of London, vol. xv. p. 173, pl. vii. (1858).

The remains of the large extinct *Herbivora* of the Pleistocene period in Britain, which have been found in the limestone-caves of Weston-super-Mare, Torquay, Pickering, &c., are held to have been parts of animals which have fallen a prey to the contemporary *Carnivora*, now also extinct. The caves of the limestone-district of Wellington Valley, Australia, reveal phenomena of extinct animal life closely analogous. I infer that the fossils, always more fragmentary than those from the tranquil freshwater deposits, of the Diprotodons, Nototheres, large Kangaroos, and Wombats, surpassing in size any existing species, were remains of animals which had fallen a prey to contemporary *Carnivora*, and by them had been dragged into the cave.

Now, no predaceous species bearing such proportion to the *Diprotodon* and *Nototherium* as the spelæan Lion, Bear, and Hyæna bore to the Mammoths, Rhinoceros, Oxen, &c., has hitherto been detected in Australian bone-caves, save the *Thylacoleo carnifex*. To its associated fossils, the Thylacine or the Dasyure (*Sarcophilus*), the objection of defective strength and bulk might be specious; but it is inapplicable to the *Thylacoleo*.

§ 19. *Conclusion.*—In the main the descriptions or definitions of the characters of the fossil remains of *Thylacoleo* and *Plagiaulax* by my antagonists and myself are the same; and the chief difference herein is that I interpret the fractured surface of the angle of the jaw in a specimen of *Plagiaulax* as indicative of that part being bent inward immediately below the neck of the condyle as in *Sarcophilus* and *Thylacinus*, whilst Dr. FALCONER contends that the part broken away descended below the condyle as in the mandible of the Aye-aye. And so, with regard to *Thylacoleo*, I interpret the evidences of its fossil mandible as indicative of an agreement with that in existing Marsupial *Carnivora* in the form and proportions of the coronoid process and in the position of the transversely extended condyle. Messrs. KREFFT and FLOWER restore the mandible of *Thylacoleo*, in regard to these light-giving structures, according to the analogies of the carpophagous Phalangers and Koalas and the poëphagous Potoroos, assigning to the upper jaw the same incisive formula, for dissenting from which I have given reasons.

I cannot find better words to express my conviction of the state of the question as now analyzed and tested than those of the gifted and lamented Palæontologist, whose criticisms, as reproduced in his posthumous work, reiterated, as it were, from the grave, have overcome the reluctance which, till now, has kept me silent. In those words, therefore, I venture to remark, that, if my inferences and conclusions be favoured by acceptance, it will not imply that my opponents had "fallen into errors of observation and description"*, so much as it will expose "the fallacious train of reasoning which had led them astray"†.

Should *Thylacoleo* be permitted to rest, after the facts and inferences from the scanty fossil evidences at my command, in the section of diprotodont Marsupials, with *Plagiaulax*, amongst the predaceous feeders on flesh, and not with *Hypsiprymnus* amongst the harmless *Herbivora*, it will only be further proof of the worth and truth of the principle which CUVIER laid down as our guide in such dark routes in Palæontology.

* X. p. 350; XI. p. 433.

† Id. ib.

Already, since writing the above, evidence has reached me, in the last Packet from Sydney, which I shall probably be not the only one to hail as undesigned witness to what I deem the truth of the matter.

Amongst the fossils obtained by Professor THOMSON and Mr. KREFFT from the breccia-caves of Wellington Valley were several ungual phalanges, some of which, equalling or surpassing those of a Lion, were compressed, the vertical exceeding the transverse diameter, and being considerable in proportion to the length: these phalanges are curved and pointed, but the point is more or less blunted or broken, apparently after interment. They supported a claw, and in most there are traces more or less plainly discernible of a bony sheath which bound or strengthened the attachment of the base of the claw. These specimens, at present, I know only by photographs of the natural size.

Plate XIII. fig. 12 is of one of these ungual phalanges, 1 inch 9 lines in length, 1 inch 3 lines in basal depth. The articulation (*a*) occupies the upper half of the basal surface; it is concave and divided by a median vertical ridge, adapting it to the pair of convexities on the distal end of the penultimate phalanx. A strong tuberous process (*b*) for the insertion of the flexor tendon projects from the lower part of the basal half of the bone. A ridge (*c*) anterior to the joint may indicate the attachment of the sheath broken away.

Figure 14 gives an under view of this phalanx, showing the breadth of the apophysial part of the base, and the compressed character of the decurved claw-bearing part of the phalanx.

Figure 13 is a side view of a similar phalanx, 1 inch 8 lines in length, 1 inch in basal depth. The upper part of the articular surface (*a*) is more produced, or better preserved, than in figure 12; and the indication of the sheath (*c*, *c*) is more considerable and begins more in advance. The insertional tuberosity (*b*) also extends rather more forward.

In the next photograph (ib. fig. 11) the bony basal claw-sheath (*c*, *c*) is evidently preserved; its anterior margin is 1 inch 3 lines in advance of the hind part of the phalanx; but this, as well as the under surface of the back part, appears to be mutilated. One half or side of the sheath has been broken away, exposing the core of the claw (*d*), the pointed termination of which is better preserved than in the preceding specimens.

From these specimens may be inferred a spelæan animal with subcompressed decurved pointed claws, equalling or exceeding those of the Lion or Tiger in size, but supported by phalanges resembling those of *Thylacinus*, *Dasyurus*, and the Opossums in being non-retractile, or wanting the characteristic low position of the joint in the sheathed claw-bones of placental Felines, but resembling those phalanges, rather than the non-retractile ones of the Marsupials above mentioned, in the proportion of depth to length and breadth.

A claw may be adapted to pierce, retain, and lacerate (as, for example, the large sheathed one of *Myrmecophaga jubata*), and be used as a weapon against a mammal of equal or superior size only in defence (as when the great Anteater causes the death of its assailant the Jaguar by the tenacity of its grip). So, likewise, may the claws of the Megatherioids have been put to such occasional defensive uses against their probable


a

assailant the *Machairodus neogæus*, although, as in the Anteater, the habitual service of the claws may have related to insects or vegetable diet.

One is guided in a conjecture as to the uses of claws by the evidence afforded by the associated fossils of the animals which, if unguiculate, would have had claw-bones of the size of those under consideration.

No evidence of a Megatherioid or other Edentate animal has been had from any cave or fossiliferous deposit in Australia. The shape of the ungual phalanges in Kangaroos and Wombats is known. The ungual phalanges (Plate XIII. figs. 12–14) are too small for *Nototherium* and *Diprotodon*, if even one were to entertain the idea of those huge Marsupial *Herbivora* having had sheathed, compressed, decurved, pointed claws, like those which the plalanges in question plainly bore. These phalanges are as much too large for the *Thylacinus* and *Sarcophilus*. But there is no other associated Carnivore corresponding in size with that of the animal indicated by them, save the *Thylacoleo*.

It is open to any one to repeat, with respect to these phalanges, the remark which has been made on the fossil metacarpal of the carnivorous type from Australia, the size of which is such, as the articular surfaces (*a* in figs. 11, 12 & 13, Plate XIII.) show to have entered into the formation of the paw terminated by such claw-phalanges, viz. "That the metacarpal bone figured in Phil. Trans. 1859, Plate XIII. belonged to the same animal as the skull is only conjectural"*.

All that has been above advanced in searching out the nature of the ungual phalanges made known to me by photography is conjectural; but if a Palæontologist or Comparative Anatomist is willing to lend friendly aid in such difficult gropings after the things of the past, he should point out in what particulars he deems the grounds of the conjecture to be defective.

A great proportion of the fair edifice of Palæontology still rests upon a scaffolding of wise and well-founded "conjecture."

Description of the Plates.

PLATE XI.

Fig. 1. Portion of right upper jaw-bone (*maxilla*) and teeth, outer side view.

Fig. 2. Portion of right upper jaw-bone (*maxilla*) and teeth, inner side view.

Fig. 3. Portion of right upper jaw-bone (*maxilla*) and teeth, under view with working-surface of teeth: the relative size and position of the tubercular is shown at *m* 1.

Fig. 4. Portion of right upper jaw-bone (*maxilla*) and teeth, front view.

Fig. 5. Portion of right upper jaw-bone (*maxilla*) and teeth, hind view.

Fig. 6. Crown of a less worn upper laniary (*i* 1), outer side; from a breccia-cave.

* XII. p. 309.

Fig. 7. Upper laniary (i_1), front view; from a breccia-cave.
Fig. 8. Part of crown of upper laniary (i_1), inner side view; from a breccia-cave.
Fig. 9. Second incisor (i_2), outer side; from a breccia-cave.
Fig. 10. Right upper canine (c), outer side; from a breccia-cave.
Fig. 11. Left upper canine (c), outer side; from a breccia-cave.
Fig. 12. a, right upper canine (c), inner side: b, ib. hinder side; from a breccia-cave.
Fig. 13. First upper premolar, outer side; from a breccia-cave.
Fig. 14. Second upper premolar, side view; from a breccia-cave.

The foregoing figures are from *Thylacoleo carnifex*, nat. size.

Fig. 15. Second upper carnassial, *Machairodus*, inner side view, nat. size.
Fig. 16. Second upper carnassial, *Machairodus*, working-surface, nat. size.
Fig. 17. Right upper premolar, inner side view, nat. size, *Hypsiprymnus*.
Fig. 18. Right upper premolar, working-surface, nat. size, *Hypsiprymnus*.

PLATE XII.

Fig. 1. Left mandibular ramus, wanting the "rising branch," outer side view; ib. $\ddot{i}$, a section of fractured laniary.
Fig. 2. Left mandibular ramus, wanting the "rising branch," inner side view.
Fig. 3. Left mandibular ramus, wanting the "rising branch," upper view, with working-surface of carnassial.
Fig. 4. Under view of beginning or fore part of the inflected angle.
Fig. 5. Back view of fractured ditto.
Fig. 6. Lower carnassial tooth (p_4), outer side, with fangs exposed.

The foregoing figures are from *Thylacoleo carnifex*, nat. size.

Fig. 7. Lower carnassial, outer side, *Machairodus*, nat. size.
Fig. 8. Left lower premolar, outer side, *Hypsiprymnus*, nat. size.
Fig. 9. Left lower carnassial, inner side, *Machairodus*, nat. size.
Fig. 10. Left lower premolar, inner side, *Hypsiprymnus*, nat. size.
Fig. 11. Working-surface of lower carnassial, *Thylacoleo*, less worn than in fig. 3, nat. size.
Fig. 12. Working-surface of lower carnassial, *Hyæna*, nat. size.
Fig. 13. Working-surface of left lower premolar, *Hypsiprymnus*.

The line I indicates the total length of the mandible of *Thylacoleo* when entire.

PLATE XIII.

Fig. 1. Horizontal ramus of right mandible, with teeth, outer side view, nat. size; from a "photograph." (Original, a cave-specimen in the Museum of Natural History, Sydney, New South Wales.)


a

Fig. 2. Fore part of mandibular ramus, showing depth of implantation of laniary (i), and part of socket of carnassial (p_4); from a "photograph." (Id. ib.)

Fig. 3. Front view of mandible and teeth; the bone in outline: from original fossils.

Fig. 4. Right lower incisor, inner side view; from a "photograph." (Original, a cave-specimen in the Museum of Natural History, Sydney, New South Wales.)

Fig. 5. Right lower incisor, from a breccia-cave in Wellington Valley, outer side view.

Fig. 6. Right lower incisor, from a breccia-cave in Wellington Valley, inner side view.

Fig. 7. Right lower incisor, from a breccia-cave in Wellington Valley, back view.

Fig. 8. Right lower incisor, from a breccia-cave in Wellington Valley, transverse section, one-third from apex of crown.

Fig. 9. Right lower incisor, from a breccia-cave in Wellington Valley, transverse section of base of crown.

The subjects of the foregoing figures are from *Thylacoleo carnifex*, nat. size.

Fig. 10. Left mandibular ramus and teeth, *Bettongia*, nat. size.

Fig. 11. Inner side view of a sheathed ungual phalanx (claw-core exposed by the removal of part of bony sheath) of an unguiculate mammal.

Fig. 12. Outer side view of a similar ungual phalanx, with more of the bony sheath preserved.

Fig. 13. Side view of a similar ungual phalanx; sheath mutilated.

Fig. 14. Under view of the same phalanx.

(From "photographs:" the originals in the Museum of Natural History, Sydney, New South Wales, were obtained from the same breccia as the subjects of figs. 1 and 2, and are from a large carnivore, probably of the same species.)

PLATE XIV.

Section of the skull of *Thylacoleo carnifex* as at present known, showing the cerebral cavity and the entire dentition from nature, nat. size. Owing to the crowding of the abortive premolars ($p_{1, 2, 3}$) to the inner side of the functional one (p_4), a complete view of the dental system cannot be had from the outer side of the jaws: if illustrated from that point of view it must be more or less diagrammatically; this Plate is, therefore, added, to be contrasted, as to number and relative size and position of the teeth, with the restorations which have been published in support of the herbivorous hypothesis of *Thylacoleo*. I have refrained from completing anatomically the articular part and ascending ramus of the mandible: one cannot doubt but that, ere long, an entire mandibular ramus of *Thylacoleo carnifex* will be obtained. The section of the cerebral cavity confirms the ascription of the extinct carnivore to the Lyencephalous subclass. The cerebellum, as in *Dasyurus*, rises wholly behind the

cerebrum, with apparently the interposition of part of the mesencephalon. The prosencephalon (*pr*) is very small when its proportion to the entire skull is compared with that in the Lion or Tiger. The rhinencephalon (*rh*), projecting and conspicuous anterior to the cerebrum, is characteristically large. A "sella" (*h*) is plainly indicated at the part of the basisphenoid where that cavity is "indicated only by the internal orifices of the entocarotid canals" in *Thylacinus* *. The chiasmal fossa is shown at *o*: the precondylar foramen at *p.c.*

The symbols, letters, and numerals are explained in the text.

List of Woodcuts.

* Descriptive Catalogue of the Osteological Series, &c., 4to, p. 349.

X. *On the Mathematical Theory of Stream-lines, especially those with four Foci and upwards.* *By* William John Macquorn Rankine, *C.E., LL.D., F.R.SS. Lond. & Edin., &c.*

Received January 1,—Read February 10, 1870.

Introduction.

§ 1. *Object and Occasion of this Investigation.*—*A Stream-line* is the line that is traced by a particle in a steady current of fluid. Each individual stream-line preserves its figure and position unchanged, and marks the track of a filament or continuous series of particles that follow each other. The motions in different parts of a steady current may be represented to the eye and to the mind by means of a group of stream-lines; for the direction of motion of a particle at a given point is that of a tangent to the stream-line which traverses that point; and when the fluid is of constant density, as is sensibly the case with liquids, the comparative velocities at different points are indicated by the comparative closeness of the stream-lines to each other. Even when the fluid is gaseous, the comparative *mass-velocities* are indicated by the closeness of the stream-lines—the term *mass-velocity* meaning the mass which traverses a unit of area in a unit of time. Gaseous fluids, however, will not be considered in the present paper.

Stream-lines are important in connexion with naval architecture; for the curves which the particles of water describe relatively to a ship, in moving past her, are stream-lines; and if the figure of a ship is such that the particles of water glide smoothly over her skin, that figure is a *stream-line surface**, being a surface which contains an indefinite number of stream-lines. The stream-lines of a current gliding past a circular cylinder in a direction transverse to its axis, and also those of a current gliding past a sphere, have long been known.

In a paper entitled "On Plane Water-lines in two Dimensions," read to the Royal Society in 1863, and published in the Philosophical Transactions, I have given a detailed

* Note added December 1870.—This limitation is necessary in speaking of the figures of ships; for although every surface is a possible stream-line surface, the surface of a ship is not even approximately an actual stream-line surface unless it is such that she does not drag along with her a mass of eddies of such volume and shape as to cause the actual tracks of the particles of water to differ materially in form from those which would be described in the absence of eddies. The surfaces which fulfil this condition are what are called by shipbuilders "*fair*" surfaces; and their forms have in a great many cases been determined by practical experience. In order to determine, at all events approximately, the actions of such surfaces on the water, it is necessary to be able to construct them by geometrical rules based on the principles of the motion of fluids; and the methods described in this paper afford the means of doing so.—W. J. M. R.

investigation of the mathematical properties of a very extensive class of stream-lines, representing the motions of particles of water in layers of uniform thickness. Those stream-lines closely resemble the water-lines, riband-lines, and other longitudinal sections of ships of a great variety of forms and proportions; and there is scarcely any known figure of a fair longitudinal line on a ship's skin to which an approximation may not be found amongst them; hence I have proposed to call them Neoïds; that is, ship-shape lines.

In the Philosophical Magazine for October 1864, was published a paper which had been read by me to the British Association, containing a summary of the properties of some additional kinds of stream-lines, some in two and others in three dimensions, and of those stream-lines in particular which generate stream-line surfaces of revolution. All these stream-lines also are neoïds, or ship-shape curves.

All the neoïd stream-lines before mentioned are either *unifocal* or *bifocal*; that is to say, they may be conceived to be generated by the combination of a uniform progressive motion with another motion consisting in a divergence of the particles from a certain point or focus, followed by a convergence either towards the same point or towards a second point. Those which are continuous closed curves, when unifocal are circular, and when bifocal are blunt-ended ovals, in which the length may exceed the breadth in any given proportion—for example, the curves marked L B in figs. 2, 3 & 4, Plate XV. To obtain a unifocal or bifocal neoïd resembling a longitudinal line of a ship with sharp ends, such as A, fig. 1, it is necessary to take a part only of a stream-line, and then there is discontinuity of form and of motion at each of the two ends of that line.

The occasion of the investigation described in the present paper was the communication to me by Mr. WILLIAM FROUDE of some results of experiments of his on the resistance of model boats, of lengths ranging from 3 to 12 feet. A summary of those results is published at the end of a Report to the British Association, "On the State of Existing Knowledge of the Qualities of Ships." In each case two models were compared together of equal displacement and equal length; the water-line of one was a wave-line, as at A (Plate XV. fig. 1), with fine sharp ends; that of the other had blunt rounded ends, as at B—suggested, Mr. FROUDE states, by the appearance of water-birds when swimming. At low velocities, the resistance of the sharp-ended boat was the smaller; at a certain velocity, bearing a definite relation to the length of the model, the resistances became equal; and at higher velocities the round-ended model had a rapidly increasing advantage over the sharp-ended model.

Hence it appeared to me to be desirable to investigate the mathematical properties of stream-lines resembling the water-lines of Mr. FROUDE's bird-like models; and I have found that endless varieties of such forms, all closed curves free from discontinuity of form and of motion, may be obtained by using four foci instead of two. They may be called, from this property, *quadrifocal stream-lines*, or, from the idea that suggested

FIG. 2

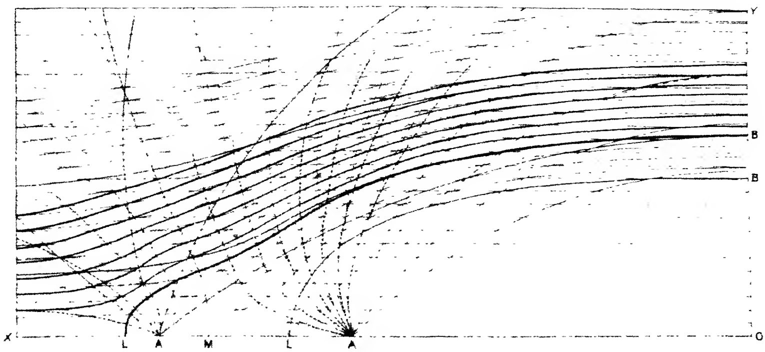


FIG. 3

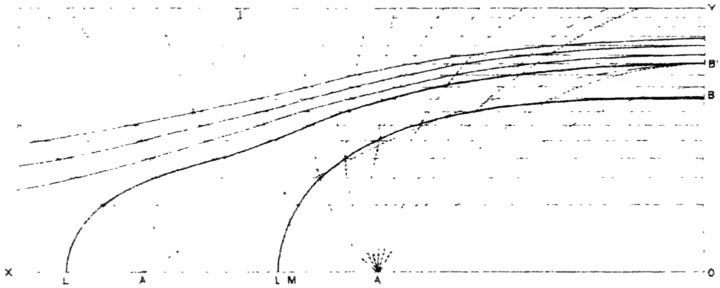


FIG. 5

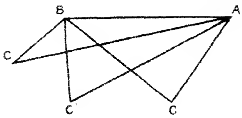
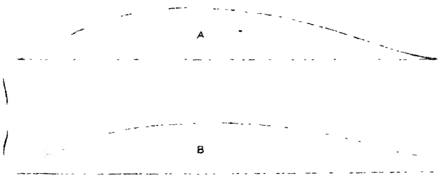


FIG. 1



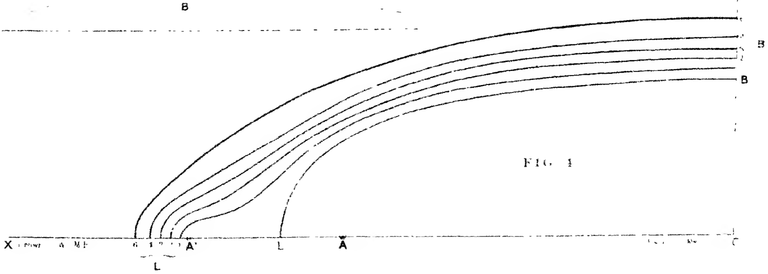


FIG. 4

such shapes to Mr. FROUDE, *Cycnoïds*, or swan-like lines; while the stream-lines in which particles of liquid flow past them may be said to be *Cycnogenous**.

CHAPTER I. *Summary of Cinematical Principles.*

§ 2. *Normal Surfaces to Stream-lines in a Liquid.* (For details on this part of the subject, see STOKES "On the steady Motion of an Incompressible Fluid," Cambridge Transactions, 1842; also RANKINE "On Plane Water-lines in Two Dimensions," Philosophical Transactions, 1863.)—Let a perfectly liquid mass of indefinite extent flow past a solid body in such a manner that, as the distance from the solid body in any direction increases without limit, the motion of the liquid particles approaches indefinitely to uniformity in velocity and direction. Let u, v, and w be the rectangular components of the velocity of any particle; then the condition of constant density requires that the following equation should be fulfilled,

$$\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}=0; \qquad (1)$$

and the condition of perfect fluidity being combined with that of the approximation to uniformity of motion at an indefinite distance requires that the three following equations should be fulfilled:

$$\frac{dv}{dz}-\frac{dw}{dy}=0; \quad \frac{dw}{dx}-\frac{du}{dz}=0; \quad \frac{du}{dy}-\frac{dv}{dx}=0. \qquad (2)$$

These four conditions are fulfilled by making

$$u=\frac{d\phi}{dx}, \quad v=\frac{d\phi}{dy}; \quad w=\frac{d\phi}{dz}, \qquad (3)$$

the *velocity-function*, ϕ, being a function which fulfils the condition

$$\left(\frac{d^2}{dx^2}+\frac{d^2}{dy^2}+\frac{d^2}{dz^2}\right)\phi=0. \qquad (4)$$

The equation

$$\phi=a \text{ (a constant)} \qquad (5)$$

is that of a surface of equal action, which is normal to the direction of motion of every particle that it traverses; in other words, it is normal to all the stream-lines that it cuts. If a series of different values be given to the constant a, the equation (5) represents a series of such normal surfaces; and every stream-line is a normal trajectory to that series of surfaces. In symbols, let ds' denote an elementary arc of a stream-line, and x', y', and z' the coordinates of a fixed point in it, those coordinates being regarded as functions

* Κυκνοειδής, κυκνογενής. It is to be observed that the swan-like curves here described are different from the lines of the vessel which some years ago was built from the designs of Mr. PEACOCK, and described in the Mechanics' Magazine; for the lines of that vessel are oval, and approximate to bifocal neoids, and are wholly without the peculiarly shaped ends that characterize Mr. FROUDE's cycnoid models.

of s'; then we have

$$\left.\frac{\frac{dx'}{ds'}}{\frac{d\phi}{dx}}=\frac{\frac{dy'}{ds'}}{\frac{d\phi}{dy}}=\frac{\frac{dz'}{ds'}}{\frac{d\phi}{dz}}.\right\} \quad . \quad . \quad . \quad . \quad . \quad (6)$$

In short, the stream-lines bear the same relation to the normal surfaces that lines of force bear to equipotential surfaces.

Let the axis of x be taken parallel to the direction of the uniform motion of the particles at an indefinitely great distance from the origin of coordinates, near to which the solid body is supposed to be situated; and let the velocity of that uniform current be taken as the unit of velocity, so that u, v, and w shall represent the ratios of the three components of the velocity of a particle to the velocity at an indefinite distance. Then, when either x, y, or z is indefinitely great, we have

$$u=1;\ v=0;\ w=0;$$

and it is evident that the velocity-function must be of the following form,

$$\phi=x+\phi_1, \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

in which ϕ_1 is a function that vanishes when x, y, or z increases indefinitely. The term x gives, by its differentiations, the expression of a uniform straight current, of the velocity 1. The term ϕ_1 gives, by its differentiations, the three components of the *disturbance* of the velocity from that of the uniform current. Hence, if we suppose the water at an indefinite distance from the disturbing solid to be still, and the solid to move parallel to the axis of x with the velocity -1, the following coefficients,

$$\frac{d\phi_1}{dx},\ \frac{d\phi_1}{dy},\ \frac{d\phi_1}{dz},$$

will represent the components of the velocity of a particle *relatively to still water*.

§ 3. *Stream-line Surfaces in general.*—For some purposes a more convenient way of expressing the properties of stream-lines is, to consider the system of stream-lines in a steadily moving current of liquid as the intersections of two sets of surfaces called *stream-line surfaces*, represented by the two sets of equations

$$\psi=b;\ \chi=c, \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

where b and c are constants, each of which receives a series of different values. Each set of surfaces divides the space in which the current flows into a series of indefinitely thin layers; and the two sets of surfaces divide that space into a series of indefinitely slender *elementary streams**, which are conceived to be of *equal flow*. The uniform current at an indefinite distance from the disturbing solid being, as before, parallel to x, and of the velocity 1, let the transverse area of an elementary stream at an indefinite distance be denoted by σ; the same symbol denotes the volume of the flow in each unit of time along that stream, and therefore along every elementary stream. The areas of

* Note added June 1871.—Called by CLERK MAXWELL "unit-tubes."

the three sections of an elementary stream, made at a given point by three planes parallel to the three coordinate planes respectively, have the following values:

$$\text{parallel to } yz, =\frac{\sigma}{\frac{d\psi}{dy}\cdot\frac{d\chi}{dz}-\frac{d\psi}{dz}\cdot\frac{d\chi}{dy}};$$

and symmetrical expressions for those parallel to zx and to xy respectively.

The three components of the velocity of an elementary stream at a given point are to be found by dividing the volume of flow by the areas of those three sections respectively; hence those components are as follows:—

$$u=\frac{d\psi}{dy}\cdot\frac{d\chi}{dz}-\frac{d\psi}{dz}\cdot\frac{d\chi}{dy}=\frac{d\phi}{dx} \quad \ldots \ldots \ldots \quad (9)$$

(and symmetrical expressions for v and w).

The third member of the equation is introduced in order to show the relations between the stream-line functions ψ and χ, and the velocity-function ϕ.

It is easily ascertained that the preceding values of u, v, and w fulfil the condition of constant density (equation 1); also that the surfaces of equal action ($\phi=a$) cut the stream-line surfaces at right angles, as expressed by the following equations:

$$\left.\begin{aligned}\frac{d\psi}{dx}\cdot\frac{d\phi}{dx}+\frac{d\psi}{dy}\cdot\frac{d\phi}{dy}+\frac{d\psi}{dz}\cdot\frac{d\phi}{dz}=0\,;\\ \frac{d\chi}{dx}\cdot\frac{d\phi}{dx}+\frac{d\chi}{dy}\cdot\frac{d\phi}{dy}+\frac{d\chi}{dz}\cdot\frac{d\phi}{dz}=0.\end{aligned}\right\} \quad \ldots \ldots \ldots \quad (10)$$

The conditions expressed by the three equations (2) take in the present instance the following form:

$$\left.\begin{aligned}0&=\frac{dv}{dz}-\frac{dw}{dy}\\ &=-\frac{d\psi}{dx}\left(\frac{d^2\chi}{dy^2}+\frac{d^2\chi}{dz^2}\right)+\frac{d\psi}{dy}\cdot\frac{d^2\chi}{dxdy}+\frac{d\psi}{dz}\cdot\frac{d^2\chi}{dzdx}\\ &\quad+\frac{d\chi}{dx}\left(\frac{d^2\psi}{dy^2}+\frac{d^2\psi}{dz^2}\right)-\frac{d\chi}{dy}\cdot\frac{d^2\psi}{dxdy}-\frac{d\chi}{dz}\cdot\frac{d^2\psi}{dzdx};\\ 0&=\frac{dw}{dx}-\frac{du}{dz}=(\text{expression formed by symmetry});\\ 0&=\frac{du}{dy}-\frac{dv}{dx}=(\text{expression formed by symmetry}).\end{aligned}\right\} \quad \ldots \quad (11)$$

The preceding set of three equations show the whole conditions which the functions ψ and χ must fulfil, in order that they may represent stream-line surfaces.

In finding the point in a stream-line where a given function F is a maximum, the condition to be fulfilled is

$$\frac{dF}{dt}=\left(u\frac{d}{dx}+v\frac{d}{dy}+w\frac{d}{dz}\right)F=0. \quad \ldots \ldots \quad (11\text{ A})$$

The following formula is an immediate consequence of the equations (2): let dx' denote an elementary line in any direction, and u' the component velocity of a particle along dx'; then

$$\frac{du'}{dt}=\frac{d}{dx'}\left(\frac{u^2+v^2+w^2}{2}\right). \qquad (11\text{ B})$$

In the two previous papers before referred to, a class of stream-lines is described under the name of *Lissoneoïds*, whose characteristic property is that two maxima and one minimum of the velocity coalesce in one point, at the greatest breadth of the figure bounded by the line. The mathematical properties of a lissoneoïd are expressed by the following set of equations:

$$\left.\begin{array}{l} \text{when } x=0;\ \text{let } v=0;\ w=0; \\ \frac{d}{dt}(u^2+v^2+w^2)=0; \\ \frac{d^2}{dt^2}(u^2+v^2+w^2)=0; \end{array}\right\} \qquad (11\text{ C})$$

and it can be shown that for the last two of these equations the following may be substituted in the cases which occur in practice:

$$u\frac{d^2u}{dx^2}+2\frac{du^2}{dy^2}+2\frac{du^2}{dz^2}=0. \qquad (11\text{ D})$$

In order to express the condition that at an indefinitely great distance from the origin v and w shall vanish, and u approximate indefinitely to 1, it is necessary that, when either x, y, or z increases indefinitely, the functions ψ and χ shall approximate indefinitely to two functions of y and z only, which may be denoted by ψ_0 and χ_0, fulfilling the following conditions,

$$\left.\begin{array}{l} \frac{d\psi_0}{dy}\frac{d\chi_0}{dz}-\frac{d\psi_0}{dz}\frac{d\chi_0}{dy}=1; \\ \frac{d\psi_0}{dx}=0;\ \frac{d\chi_0}{dx}=0; \end{array}\right\} \qquad (12)$$

that is to say, first, the surfaces represented by ψ_0 and χ_0 divide the space into elementary streams of equal transverse area; secondly, these surfaces are plane or cylindrical, and parallel to the axis of x; and thirdly, they are asymptotic to the surfaces represented by ψ and χ. Let us now make

$$\psi=\psi_0+\psi_1;\ \chi=\chi_0+\chi_1; \qquad (13)$$

then the equations (9) take the following form:

$$\left.\begin{array}{l} u=1+\frac{d\psi_0}{dy}\cdot\frac{d\chi_1}{dz}+\frac{d\psi_1}{dy}\cdot\frac{d\chi_0}{dz}+\frac{d\psi_1}{dy}\cdot\frac{d\chi_1}{dz}-\frac{d\psi_0}{dz}\cdot\frac{d\chi_1}{dy}-\frac{d\psi_1}{dz}\cdot\frac{d\chi_0}{dy}-\frac{d\psi_1}{dz}\cdot\frac{d\chi_1}{dy} \\ v=\frac{d\psi_0}{dz}\cdot\frac{d\chi_1}{dx}+\frac{d\psi_1}{dz}\cdot\frac{d\chi_1}{dx}-\frac{d\psi_1}{dx}\cdot\frac{d\chi_0}{dz}-\frac{d\psi_1}{dx}\cdot\frac{d\chi_1}{dz}; \\ w=\frac{d\psi_1}{dx}\cdot\frac{d\chi_0}{dy}+\frac{d\psi_1}{dx}\cdot\frac{d\chi_1}{dy}-\frac{d\psi_0}{dy}\cdot\frac{d\chi_1}{dx}-\frac{d\psi_1}{dy}\cdot\frac{d\chi_1}{dx}; \end{array}\right\} \qquad (14)$$

and all the terms in those expressions, except the 1 in the value of u, represent velocities of disturbance produced in a still mass of liquid by the motion of a solid parallel to x with the velocity -1.

The form of the disturbing solid may be represented by an equation of one or other of the following forms:

$$\psi=0; \quad \chi=0; \quad \mathrm{F}(\psi, \chi)=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

In the problems described in the sequel, the first of those expressions is supposed to be used for the figure of the surface of the disturbing solid, viz. $\psi=0$; so that $\psi=b$ with an unlimited series of increasing values of b, expresses the figures of a series of stream-line surfaces lying between successive layers of liquid that enclose the solid within them, like concentric tubes. A series of negative values being given to b, correspond to a set of *internal stream-lines*, which represent currents circulating inside the disturbing solid. In the present investigation, the external stream-lines alone will be considered. The equation $\chi=c$, with a series of values of c, represents a series of stream-line surfaces which meet the surface of the solid ($\psi=0$) edgewise, intersect the surfaces denoted by $\psi=b$, and subdivide the previously mentioned layers of liquid into elementary streams of equal flow.

Two alternative modes of proceeding may be followed in the proposing and solution of problems as to the figures of the stream-line surfaces*. One is as follows: a form is assumed for the function χ, satisfying equations (13) and (12); and thence are deduced, by means of the equations (11), corresponding forms of the function ψ, denoting figures of the disturbing solid and of its enclosing stream-line surfaces; and this is the method which has been followed in previous researches, and which will be followed as regards the quadrifocal stream-lines or cycnogenous neoïds specially treated of in this paper. The other mode of proceeding is to assume for the function ψ a form satisfying equations (13) and (12), and denoting certain figures of the disturbing solid, and of the enclosing stream-line surfaces, and thence to deduce by the aid of the equations (11) the corresponding form and values of the function χ, and the figures of the elementary streams.

From the form of the equations of condition (11) it is easily seen that, if with a given assumed form of either of the functions ψ, χ, there are several forms of the other function which satisfy those equations, then every form obtained by addition or subtraction of those forms will satisfy them also. In symbols, let χ be a given form of one of the functions, and ψ_i any one out of several forms of the other function which, taken along with χ, satisfy the equations; then any function which can be expressed by $\Sigma . \psi_i$ will satisfy them also.

§ 4. *Graphic Construction of Stream-lines.*—Let one side of a piece of paper be taken to represent one of the surfaces whose equation is $\chi=c$. Then the stream-lines which

* Note added in June 1871.—It is to be observed that those methods are tentative only; that is to say, they may fail when tried, and repeated trials may be necessary before a solution is obtained.

are the traces upon that surface of the several surfaces expressed by $\psi=b$* will be represented by lines on that piece of paper; and each of those lines will have an asymptote, being the trace, on the surface $\chi=c$, of a surface whose equation is $\psi_0=b$.

The drawing of such stream-lines is facilitated by the following process invented by Mr. CLERK MAXWELL:—when a function ψ is the sum of two more simple functions, $\psi_0+\psi_1$, draw the series of lines whose equations are $\psi_0=b_0$; then draw the series of lines whose equations are $\psi_1=b_1$; then draw curves diagonally through the angles of the network made by the two former series of curves, in such a manner that at each intersection b_0+b_1 shall be $=b$; the new series of curves will be that represented by the equation $\psi_0+\psi_1=b$. The same process may be extended to curves represented by a function consisting of any number of terms. For example, let the function be one of three terms, $\psi_0+\psi_1+\psi_2$. Draw the two series of lines represented respectively by $\psi_2=b_2$ and $\psi_1=b_1$; through the angles of the network draw the series of lines represented by $\psi_1+\psi_2=b_1+b_2$; then draw a fourth set of lines, being those represented by $\psi_0=b_0$, and through the angles of the network made by the third and fourth series of lines, draw a fifth series of lines, being that represented by

$$\psi_0+\psi_1+\psi_2=b_0+b_1+b_2=b.$$

Figs. 2 and 3 show examples of those processes; and in fig. 4 also the curves have been drawn by means of them, although the network is omitted.

In each case the lines expressed by the function ψ_0 represent a uniform current; and in the figures they are straight and parallel to x. The lines expressed by $\psi-\psi_0$, the sum of the remaining terms of the function, which form a network with the lines of uniform current, may be called *Lines of Disturbance*; for each of them indicates the direction of the motion of disturbance of each particle that it traverses. They are marked with bold dots.

§ 4 A. *Empirical Rule as to the volume enclosed by a Stream-line Surface.*—It has been found by the drawing and measurement of a variety of figures bounded by closed stream-line surfaces, unifocal, bifocal, and quadrifocal, and also by parts of bifocal stream-line surfaces suited for the shapes of vessels, that the following rule gives the volume contained within such a surface to the accuracy of about two per cent.:—multiply the area of midship (or greatest transverse) section by *five sixths* of the longitudinal distance between the pair of transverse sections whose areas are each equal to *one third* of the area of midship section†.

* Note added in June 1871.—The values of b are supposed to be equidifferent.

† This rule was first published as applied to stream-lines in two dimensions, in a treatise entitled 'Shipbuilding, Theoretical and Practical,' by WATTS, RANKINE, NAPIER, and BARNES: Glasgow, 1866, page 107. Its approximate correctness extends to such extreme cases as a sphere on the one hand and a wave-line bow on the other.

CHAPTER II. *Summary of Principal Properties of previously known Special Classes of Stream-lines.*

§ 5. *Stream-lines in two Dimensions, especially those with two Foci.*—Following the first of the two methods mentioned in § 3, let the simplest of all possible forms be assigned to the function χ, viz. $\chi=z$. This form represents the division of the liquid mass into an indefinite number of layers of uniform thickness, by a series of plane stream-line surfaces parallel to x and to y; and it involves the supposition that all the motions of the particles of liquid take place parallel to the plane of x and y.

The equations (9) in this case become the following:

$$u=\frac{d\psi}{dy};\ v=-\frac{d\psi}{dx};\ w=0. \quad \ldots \ldots \quad (16)$$

The equations (11) become the following:

$$\left.\begin{aligned}&\frac{dv}{dz}=-\frac{d^2\psi}{dzdx}=0;\ \frac{du}{dz}=-\frac{d^2\psi}{dydz}=0;\\&\frac{du}{dy}-\frac{dv}{dx}=\frac{d^2\psi}{dx^2}+\frac{d^2\psi}{dy^2}=0.\end{aligned}\right\} \quad \ldots \ldots \quad (17)$$

The equations (12) and (13) are reduced to the following:

$$\left.\begin{aligned}&\frac{d\psi_0}{dy}=1;\\&\text{and therefore } \psi_0=y, \text{ and } \psi=y+\psi_1;\end{aligned}\right\} \quad \ldots \ldots \quad (18)$$

where ψ_1 is a harmonic function in two dimensions; that is, one fulfilling the condition

$$\left(\frac{d^2}{dx^2}+\frac{d^2}{dy^2}\right)\psi_1=0. \quad \ldots \ldots \quad (19)$$

The equations (14) become the following:

$$u=1+\frac{d\psi_1}{dy};\ v=-\frac{d\psi_1}{dx}. \quad \ldots \ldots \quad (20)$$

The preceding equations show that the stream-line surfaces are cylindrical (in the general sense), with generating lines parallel to the axis of z, and that they have asymptotic planes parallel to the plane of zx. The traces of those asymptotic planes on the plane xy are a series of equidistant straight lines parallel to the axis of x, and corresponding to an arithmetical series of values of b in the equation $y=b$, being the stream-lines of a uniform current in a plane layer of uniform thickness.

The simplest case of disturbance of such a current by a solid body is that in which the disturbance may be represented by a radiating current, diverging from an axis in the plane of zx, within the solid body and parallel to z, and converging either towards the same axis, or towards a second axis similarly placed; and this is the mode of production of the bifocal stream-lines in two dimensions, or oögenous neoïds, whose properties are investigated in detail in a paper "On Plane Water-lines," published in the Philosophical

Transactions for 1864, page 369. The traces of the axes of divergence and convergence on the plane of xy are called the *foci*. The construction of such bifocal stream-lines is represented by the finer and fainter network of lines in fig. 2. O X and O Y are the axes of coordinates in the plane of projection, which shows a quadrant of each of the stream-lines, the other three quadrants being symmetrical to that shown. The equidistant straight lines parallel to O X are the asymptotes, corresponding to values of $y=b$. A is one of the foci; and the other is situated at an equal distance from O in the contrary direction. The stream-lines of a current in a plane uniform layer diverging from or converging towards a focus are straight, and make equal angles with each other; and their equation is

$$\psi_1=k\tan^{-1}\frac{x-a}{y}=b; \quad \ldots \ldots \ldots \ldots \quad (21)$$

in which $a=$O A denotes the distance of the focus from the origin, b is a constant having a series of values in arithmetical progression, and k is a constant called the *parameter*; so that $\frac{b}{k}$ is an angle having a series of values in arithmetical progression. This parameter is to be made positive for convergence, and negative for divergence.

If we suppose the diagram extended so as to show both foci, the focus of convergence being in the position $x=+a$, and the focus of divergence in the position $x=-a$, we obtain for the stream-line function representing these motions combined the following expression:

$$\psi_1+\psi_2=k\left(\tan^{-1}\frac{x-a}{y}-\tan^{-1}\frac{x+a}{y}\right)=b. \quad \ldots \ldots \quad (22)$$

The stream-lines or *lines of disturbance* represented by this function are constructed by drawing two similar sets of equiangular radiating straight lines through the two foci, and then drawing curves diagonally through their intersections and through the foci; but as these curves are all circles traversing the foci, it is easier to draw those circles at once, without previously drawing the radiating straight lines; and such is the process described in the paper referred to. The fine arcs which traverse the focus A in fig. 2 are parts of such circular lines of disturbance. Their centres are all in the axis of y; and the radius of any one of them is given by the following formula: let

$$-\frac{b}{k}=\tan^{-1}\frac{x+a}{y}-\tan^{-1}\frac{x-a}{y}=\theta; \text{ then radius of circle}=a\operatorname{cosec}\theta. \ldots \quad (23)$$

The combination of the divergence and convergence with the uniform current gives, for the stream-lines, the comparatively fine curves in fig. 2, which traverse diagonally the network made by the parallel straight lines and the fine circular lines of disturbance that spread from the focus A. The general equation of those stream-lines is

$$\psi=y+k\left(\tan^{-1}\frac{x-a}{y}-\tan^{-1}\frac{x+a}{y}\right)=y-k\theta=b. \quad \ldots \ldots \quad (24)$$

In the particular case $b=0$, this equation has two roots; viz.

$$\left.\begin{array}{l} y=0, \text{ representing the axis O X, and} \\ y=k\theta, \text{ representing the oval of which L B in fig. 2 is a quadrant.} \end{array}\right\} \quad . \quad . \quad (25)$$

That oval is the trace of the cylindrical surface of a solid which will disturb a uniform current in such a way as to produce the whole series of stream-lines; and it is the only one of those lines which is closed and finite, all the others being infinite and having asymptotes. When the two foci coalesce into one, that oval becomes a circle.

The component comparative velocities are as follows:

$$\left.\begin{array}{l} u=\dfrac{d\psi}{dy}=1-\dfrac{k(x-a)}{(x-a)^2+y^2}+\dfrac{k(x+a)}{(x+a)^2+y^2}; \\ v=-\dfrac{d\psi}{dx}=-\dfrac{ky}{(x-a)^2+y^2}+\dfrac{ky}{(x+a)^2+y^2}. \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

In the previous paper already referred to, the parameter here denoted by k is denoted by f; and the comparative velocities here denoted by u and v are denoted by $\frac{u}{c}$ and $\frac{v}{c}$. The origin O is taken midway between the foci for convenience. Should it be placed at unequal distances, let $x=+a'$ for one focus, and $-a''$ for the other; then in the equations, a' is to be put for $-a$, and $+a''$ for $+a$.

Let l denote the half-length O L of the oval stream-line; then by making $u=0$, $y=0$, and $x=l$ in the first of the equations (26), it is found that the following relation exists between the half-length l, the excentricity a, and the parameter k,

$$l^2=a^2-2ka=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26\text{ A})$$

Let y_0 be the greatest half-breadth O B of the oval stream-lines, then we have by equation (24),

$$y_0-2k\tan^{-1}\frac{a}{y_0}=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26\text{ B})$$

§ 6. *Stream-line Surfaces of Revolution.*—To obtain by the first method mentioned in § 3 the equations of stream-line surfaces of revolution, the form of the function χ is to be taken so as to represent a series of longitudinal planes cutting each other at equal angles in the axis of x. Hence we have the following expressions:

$$\left.\begin{array}{l} \chi=\tan^{-1}\dfrac{z}{y};\ \dfrac{d\chi}{dx}=0; \\ \dfrac{d\chi}{dy}=\dfrac{-z}{y^2+z^2};\ \dfrac{d\chi}{dz}=\dfrac{y}{y^2+z^2}; \\ \dfrac{d^2\chi}{dy^2}=\dfrac{d^2\chi}{dz^2}=\dfrac{-2yz}{(y^2+z^2)^2}; \\ \dfrac{d^2\chi}{dydz}=\dfrac{z^2-y^2}{(y^2+z^2)^2}. \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

As it is sufficient to determine the traces of the stream-line surfaces of revolution in any one of those planes, we may take the plane of xy, for which $z=0$; and then we have the following values:

$$\left.\begin{array}{l} \chi=0;\ \frac{d\chi}{dx}=0;\ \frac{d\chi}{dy}=0; \\ \frac{d\chi}{dz}=\frac{1}{y};\ \frac{d^2\chi}{dy^2}=\frac{d^2\chi}{dz^2}=0; \\ \frac{d^2\chi}{dydz}=-\frac{1}{y^2}. \end{array}\right\} \quad \text{(27A)}$$

When the preceding substitutions are made in the equations (9) and (11), they are converted into the following:

The equations (9) become

$$u=\frac{d\psi}{ydy};\ v=-\frac{d\psi}{ydz};\ w=0; \quad (28)$$

and the equations (11) become

$$\left.\begin{array}{l} 0=\frac{dv}{dz}=-\frac{d^2\psi}{ydzdx};\ 0=\frac{du}{dz}=\frac{d^2\psi}{ydydz}\ \text{(and therefore } \frac{d\psi}{dz}=0); \\ 0=\frac{du}{dy}-\frac{dv}{dx}=-\frac{d\psi}{y^2dy}+\frac{1}{y}\left(\frac{d^2\psi}{dx^2}+\frac{d^2\psi}{dy^2}\right). \end{array}\right\} \quad (29)$$

The same substitutions being made in the equations (12) give the following results:

$$\frac{d\psi_0}{ydy}=1;\ \text{and therefore } \psi_0=\frac{y^2}{2}. \quad (30)$$

This last equation shows that the stream-line surfaces which represent a uniform current, and are asymptotes to the actual disturbed stream-line surfaces, are a series of concentric circular cylinders described about the axis of x, the half squares of whose radii are in arithmetical progression. The traces of such a series of cylindrical surfaces are represented in fig. 3 by the straight lines parallel to the axis OX.

The simplest case of the motion of disturbance produced by a solid of revolution whose axis is the axis of x, is represented by a current diverging symmetrically in all directions from a focus in that axis, and afterwards converging towards another such fcous. The stream-line surfaces of revolution about that axis which represent a diverging or converging current alone, as the case may be, are obviously a series of cones with the focus for their common apex, cutting a spherical surface described about that apex into equal zones. The function which represents the traces on the plane of xy of such a series of conical stream-line surfaces is the following:

$$\psi_1=\pm\frac{k^2}{2}\cdot\frac{x-a}{\sqrt{\{(x-a)^2+y^2\}}}=\frac{k^2\cos\theta}{2}; \quad (31)$$

in which a denotes the distance of the focus from the origin of coordinates and $\pm\frac{k^2}{2}$ is a parameter, to be used with the positive sign for convergence and with the negative sign for divergence.

In the second expression for the function, θ denotes the angle made by the trace of the cone with the axis of x.

To draw a set of those traces, describe a circle about the focus; divide the diameter of that circle which lies along the axis of x into a convenient number of equal parts; through the points of division of the diameter draw ordinates perpendicular to it, cutting the circumference; through the points of division of the circumference draw radii; these will be the required traces of the cones.

For the focus of convergence, let $x=+a$, and for the focus of divergence, let $x=-a$; then the following function represents the lines of disturbance, or stream-lines of the combined motions of divergence and convergence,

$$\psi_1+\psi_2=\frac{k^2}{2}\left\{\frac{x-a}{\sqrt{\{(x-a)^2+y^2\}}}-\frac{x+a}{\sqrt{\{(x+a)^2+y^2\}}}\right\}=\frac{k^2}{2}(\cos\theta-\cos\theta'); \quad . \quad . \quad (32)$$

in the last of which expressions θ and θ' denote the angles made with the axis of x by the two lines drawn from the point (x, y) to the foci of convergence and divergence respectively. Those lines of disturbance are constructed graphically by drawing two equal and similar sets of radiating straight lines through the foci, as already described, and then drawing curves through the foci, and diagonally through the angles of the network made by the two sets of radiating straight lines. Those curves are already well-known, being the lines of force of a magnet whose poles are at the foci. The fine curves in fig. 3, which spread from the focus A, are examples of them; they were drawn by the method above described, though the radiating straight lines have been omitted from the Plate to prevent confusion.

The stream-lines which are the traces, on the plane of xy, of the stream-line surfaces of revolution, may be constructed, as before, by drawing them diagonally through the angles of the network made by the parallel straight lines in fig. 3 with the lines of disturbance. Their general equation is as follows:

$$\psi=\frac{y^2}{2}+\frac{k^2}{2}\left\{\frac{x-a}{\sqrt{\{(x-a)^2+y^2\}}}-\frac{x+a}{\sqrt{\{(x+a)^2+y^2\}}}\right\}=b, \quad . \quad . \quad . \quad . \quad . \quad (33)$$

b having a series of values in arithmetical progression. The principal properties of those lines have been stated in the Philosophical Magazine for October 1864; but their detailed investigation has not hitherto been published.

In the particular case $b=0$, equation (33) has two roots, viz.

$$\left.\begin{array}{l} y=0, \text{ representing the axis O X; and} \\ \dfrac{y^2}{k^2}=\cos\theta'-\cos\theta, \text{ representing the oval of which L B in fig. 3 is a quadrant.} \end{array}\right\} \quad . \quad (34)$$

That oval is the trace of the surface of a solid of revolution which will disturb a uniform current in such a way as to produce the whole series of stream-line surfaces whose traces are expressed by equation (33); and that oval surface of revolution is the only surface of the series which is closed and finite—all the others being inde-

finitely long, and having asymptotic cylinders expressed by $\frac{y^2}{2}=b$. To avoid confusion these infinite bifocal stream-line surfaces are not shown in fig. 3. They bear a general likeness to those shown in fig. 2.

When the two foci coalesce into one, the disturbing solid becomes a sphere, whose stream-line surfaces were investigated by Dr. HOPPE (Quarterly Journal of Mathematics, March, 1856).

As to the modification of the formulæ required if the origin is not taken midway between the foci, see the end of § 5.

The component comparative velocities are as follows:

$$\left.\begin{aligned} u&=\frac{d\psi}{ydy}=1+\frac{k^2}{2}\left\{\frac{-x+a}{\{(x-a)^2+y^2\}^{\frac{3}{2}}}+\frac{x+a}{\{(x+a)^2+y^2\}^{\frac{3}{2}}}\right\}, \\ v&=-\frac{d\psi}{ydx}=\frac{k^2y}{2}\left\{-\frac{1}{\{(x-a)^2+y^2\}^{\frac{3}{2}}}+\frac{1}{\{(x+a)^2+y^2\}^{\frac{3}{2}}}\right\}. \end{aligned}\right\} \quad \ldots \ldots \quad (35)$$

Let l denote the half-length O L (fig. 3) of the oval solid. Then by making, in the first of the above equations, $u=0$, $x=l$, and $y=0$, the following relation is found to exist between the half-length, excentricity, and parameter,

$$(l^2-a^2)^2-2k^2la=0. \quad \ldots \ldots \ldots \quad (36)$$

Let y_0 be the extreme half-breadth OB, then by equation (33) we have

$$y_0^6+a^2y_0^4-4k^4a^2=0. \quad \ldots \ldots \ldots \quad (36\text{ A})$$

CHAPTER III. *Special Theory of Quadrifocal Stream-lines, or Cycnogenous Neoïds.*

§ 7. *Quadrifocal Stream-lines in general.*—A quadrifocal stream-line is the trace on a longitudinal diametral plane of a quadrifocal-stream-line surface, belonging either to the cylindrical class or to that of surfaces of revolution. The four foci are situated in an axis parallel to the direction of the uniform current which is disturbed by the solid; and, as in the previous chapters, that axis will be taken for the axis of x, and the transverse axis in the plane of projection for the axis of y.

The general equation of a quadrifocal stream-line may be expressed as follows:

$$\psi=\psi_0+\psi_1+\psi_2+\psi_3+\psi_4=b \quad \ldots \ldots \ldots \quad (37)$$

In that expression ψ_0 is the function representing the uniform current of the velocity 1, which is equal to y or to $\frac{1}{2}y^2$, according as the surfaces are cylindrical or of revolution; ψ_1 expresses the convergence of certain currents towards one of the foci, ψ_2 the divergence of the same currents from a second focus, ψ_3 the convergence of certain currents towards a third focus, ψ_4 the divergence of the same currents from a fourth focus.

The graphic construction of quadrifocal stream-lines is illustrated in figs. 2 and 3. In each of those figures, A is one of the first pair of foci, A′ one of the second pair; the other focus of each pair is supposed to lie at the other side of the origin O, beyond the limits of the drawing.

The lines of disturbance expressed by $\psi_1+\psi_2$, being those due to the first pair of foci,

are the fine curves spreading from A. The lines of disturbance expressed by $\psi_3+\psi_4$, being those due to the second pair of foci, are the fine curves spreading from A′. Both those sets of lines were drawn according to the rules given in sections 5 and 6. The *lines of resultant disturbance*, expressed by the function $\psi-\psi_0=\psi_1+\psi_2+\psi_3+\psi_4$, are drawn diagonally through the angles of the network made by the two former sets of lines. They are marked with strong dots in the figures. They all traverse one or other of the foci, A, A′, with the exception of one line, which meets the axis O X at right angles in the point M*.

The actual stream-lines are drawn diagonally through the network made by the lines of uniform current and the lines of disturbance. They are shown by rather strong lines in figs 2 & 3. In each set of quadrifocal stream-lines there is one only that is finite and closed. It corresponds to the value $\psi=b=0$; and it is the trace of the surface of the solid whose disturbing action produces the whole system of stream-lines. It has rounded ends, cutting the axis of x at right angles. In each of the figures 2 & 3. a quadrant of that curve is shown, marked L′ B′. This is the curve which resembles the water-line of Mr. FROUDE's model B, fig. 1, and is therefore properly a *cycnoïd*, or swan-like curve. The equation $\psi=0$ has another root, viz. $y=0$, representing the axis of x. The other stream-lines of the system, lying outside the curve L′ B′, are infinite, and have for asymptotes the stream-lines of the uniform current. They may be called *cycnogenous* stream-lines, as being produced by the cycnoïd stream-line surface.

In a system of bifocal stream-lines there are two independent constants, on which the dimensions and figures of all the lines of the system depend—the excentricity (being half the distance between the foci) and the parameter (as to which see equation 36). In a system of quadrifocal stream-lines, there are five independent constants, viz.:—the two parameters, for the first and second pair of foci respectively; the eccentricity of the first pair of foci; and the distances of the two foci forming the second pair from a point midway between the first pair. If those distances are equal, the cycnoïd curve and each of the stream-lines produced by it have then two ends symmetrical to each other; if unequal, those ends are unsymmetrical. In all the examples shown in the Plate the ends are symmetrical.

In each of the figures 2 and 3, the bifocal oval stream-line marked B L has been described about the first two foci with the same parameter which is assigned to those foci in describing the quadrifocal closed stream-line B′ L′.

Fig. 4 shows a series of cycnoïds, or quadrifocal closed stream-lines, in two dimensions, described about the same four foci. The parameter for the first pair of foci (one of which is marked A) is constant, and is that of the bifocal oval neoïd B L. The parameter for the second pair of foci (one of which is marked A′) was made successively

* In fig. 2 the quadrifocal stream-lines and their lines of disturbance have been engraved on a plate already covered with bifocal stream-lines and their lines of disturbance; and therefore, in order to avoid confusion, some of the quadrifocal lines of resultant disturbance extending from A towards the axis of Y, in the neighbourhood of the point B′, have been omitted. Enough have been drawn to show the principle of their construction. In fig. 3 the series of quadrifocal lines of disturbance is complete.

equal to $\frac{1}{12}$, $\frac{2}{12}$, $\frac{3}{12}$, $\frac{4}{12}$, and $\frac{6}{12}$ of the first parameter; and thus were drawn the five cycnoïd curves marked respectively 1 1, 2 2, 3 3, 4 4, and 6 6. The lines of uniform current and of disturbance used in drawing these curves are omitted in the engraving.

This last figure illustrates the fact that, with a given set of foci, and a given parameter for the inner pair of foci, the cycnoïd becomes leaner and more hollow at the bow as the parameter for the outer pair of foci diminishes; also that, with large values of the second parameter, that curve is convex throughout, like the line marked 66; and that for some intermediate value the hollowness just vanishes, as is very nearly the case in the line marked 4 4. It is obvious that any degree of fineness may be given to the entrance by increasing the distance of the second foci from the first, and at the same time using a small second parameter.

§ 8. *Cylindric Cycnoïds.—Forms and Velocities of Streams.*—The equation of a system of quadrifocal stream-lines in two dimensions is as follows,

$$\psi=y+k\left(\tan^{-1}\frac{x-a}{y}-\tan^{-1}\frac{x+a}{y}\right)+k'\left(\tan^{-1}\frac{x-a'}{y}-\tan^{-1}\frac{x+a''}{y}\right)=b, \quad . \quad . \quad (38)$$

in which k and k' are the parameters for the inner and outer pairs of foci respectively, a is the excentricity of the inner pair of foci, and a' and a'' are the distances of the outer pair of foci from the origin in opposite directions. The equation of the cycnoïd curve, or trace of the surface of the cylindric solid which generates the series of stream-lines, is $\psi=b=0$. If that solid is symmetrical-ended, we have $a''=a'$. The components of the comparative velocity of a stream at a given point (x, y) are given by the following equations, in which, for brevity's sake, the following notation is used:

$$(x-a)^2+y^2=r_1^2;\ (x+a)^2+y^2=r_2^2:$$
$$(x-a')^2+y^2=r_3^2;\ (x+a'')^2+y^2=r_4^2.$$

$$\left.\begin{aligned} u&=\frac{d\psi}{dy}=1-k\left(\frac{x-a}{r_1^2}-\frac{x+a}{r_2^2}\right)-k'\left(\frac{x-a'}{r_3^2}-\frac{x+a''}{r_4^2}\right);\\ v&=-\frac{d\psi}{dx}=-ky\left(\frac{1}{r_1^2}-\frac{1}{r_2^2}\right)-k'y\left(\frac{1}{r_3^2}-\frac{1}{r_4^2}\right).\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (39)$$

At the extreme breadth of the space bounded by a given stream-line we have $v=0$; and when the cycnoïd is symmetrical-ended, the longitudinal component u at the same point takes the following value, found by making $x=0$,

$$u_0=1+\frac{2ka}{a^2+y_0^2}+\frac{2k'a'}{a'^2+y_0^2}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (39\ \text{A})$$

where y_0 denotes the greatest ordinate or "midship half-breadth" of the stream-line under consideration.

§ 9. *Cylindric Cycnoïds.—Extreme Dimensions.*—The extreme length of a cylindric cycnoïd is made up of the distances of its two rounded ends, where it cuts the axis of x, from the origin of coordinates. Let l be one of those distances; in the expression for

u, equation (39), make $x=l$, $y=0$, $u=0$; then we have the following equation,

$$0=1-k\left(\frac{1}{l-a}-\frac{1}{l+a}\right)-k'\left(\frac{1}{l-a'}-\frac{1}{l+a''}\right),$$

which by ordinary reductions gives the following biquadratic equation:

$$\left.\begin{aligned}0=l^4-l^3(a'-a'')-l^2(a^2+a'a''+2ka+k'(a'+a''))\\+l(a^2-2ka)(a'-a'')+a^2a'a''+2kaa'a''\\+k'a^2(a'+a'').\end{aligned}\right\} \quad \ldots \ldots \quad (40)$$

Of the four roots of this equation, the two greatest, positive and negative respectively, belong to the cylindric cycnoïd; and the sum of their arithmetical values is its length. The two least, positive and negative, belong to an internal stream-line, which is also a closed curve. It passes outside and near to the inner foci, and inside the outer foci, and it is foreign to the purpose of the present investigation.

When the two outer foci are equidistant from the inner foci (that is, when $a''=a'$), equation (40) becomes a quadratic equation in l^2; that is to say, we have

$$0=l^4-l^2(a^2+a'^2+2ka+2k'a')+a^2a'^2+2kaa'^2+2k'a'a^2. \quad \ldots \ldots \quad (40\text{ A})$$

For brevity's sake, let

$$a^2+2ka=\lambda^2, \quad a'^2+2k'a'=\lambda'^2,$$

being in fact, according to equation (26 A), the values of l^2 for two bifocal oval neoïds, with the respective excentricities a and a', and parameters k and k'. Then the solution of equation (40 A) is as follows:

$$l^2=\frac{\lambda^2+\lambda'^2}{2}\pm\sqrt{\left\{\frac{(\lambda^2-\lambda'^2)^2}{4}+4kk'aa'.\right\}} \quad \ldots \ldots \quad (40\text{ B})$$

The greater root is the square of the half-length of the cycnoïd; the lesser root belongs to the internal stream-line already mentioned.

The method of finding the extreme half-breadth in a cycnoïd with unsymmetrical ends, is to make $\psi=0$ in equation (38), and $\frac{v}{y}=0$ in the second equation (39), and, from the pair of equations so obtained, to deduce x and y by elimination. When the ends of the cycnoïd are symmetrical, the extreme half-breadth is midway between the foci; hence, making $x=0$ in equation (38), we have the following transcendental equation,

$$0=y_0-2k\tan^{-1}\frac{a}{y_0}-2k'\tan^{-1}\frac{a'}{y_0}; \quad \ldots \ldots \quad (41)$$

from which y_0 is to be calculated by approximation.

§ 10. *Cycnoïds of Revolution.—Forms and Velocities of Streams.*—The equation of a series of cycnogenous or quadrifocal stream-lines of revolution is as follows:

$$\psi=\frac{y^2}{2}+\frac{k^2}{2}\left(\frac{x-a}{r_1}-\frac{x+a}{r_2}\right)+\frac{k'^2}{2}\left(\frac{x-a'}{r_3}-\frac{x+a''}{r_4}\right)=b; \quad \ldots \ldots \quad (42)$$

in which r_1, r_2, r_3, and r_4 have the same meaning as in equation (38); that is, they are the

distances of the point (x, y) from the four foci respectively. The equation of the cycnoïd of revolution which produces the series of stream-lines is $\psi = b = 0$; and this equation has two roots, viz. $y=0$, denoting the axis of x, and

$$y^2 = k^2(\cos\theta_2 - \cos\theta_1) + k'^2(\cos\theta_4 - \cos\theta_3), \quad \text{(42 A)}$$

in which θ_1, θ_2, θ_3, and θ_4 denote the angles made with the axis of x by lines drawn from the point (x, y) to the four foci.

The component comparative velocities are as follows:

$$\left.\begin{aligned} u &= \frac{d\psi}{ydy} = 1 + \frac{k^2}{2}\left(\frac{-x+a}{r_1^3} + \frac{x+a}{r_2^3}\right) + \frac{k'^2}{2}\left(\frac{-x+a'}{r_3^3} + \frac{x+a''}{r_4^3}\right); \\ v &= -\frac{d\psi}{ydx} = \frac{k^2 y}{2}\left(-\frac{1}{r_1^3} + \frac{1}{r_2^3}\right) + \frac{k'^2 y}{2}\left(-\frac{1}{r_3^3} + \frac{1}{r_4^3}\right). \end{aligned}\right\} \quad (43)$$

When the two ends of the solid are symmetrical, we have $a''=a'$; and the value of u at the midship section, where $v=0$ and $x=0$, is as follows,

$$u_0 = 1 + \frac{k^2 a}{(a^2+y_0)^{\frac{3}{2}}} + \frac{k'^2 a'}{(a'^2+y_0)^{\frac{3}{2}}}; \quad \text{(43 A)}$$

in which y_0 is the midship half-breadth.

§ 11. *Cycnoïds of Revolution.—Extreme Dimensions.*—Let l denote the distance from the origin of one of the points where the cycnoïd surface of revolution cuts the axis of x. Then, in the first of the equations (43), making $u=0$, $y=0$, $x=l$, we obtain the following equation of the eighth order,

$$\left.\begin{aligned} 0 &= (l^2-a^2)^2 \cdot (l-a')^2(l+a'')^2 - 2k^2 la(l-a')^2(l+a'')^2 \\ &\quad - \frac{k'^2}{2}\left\{2l(a''+a') + (a''^2 - a'^2)\right\} \cdot (l^2-a^2)^2. \end{aligned}\right\} \quad (44)$$

The greatest positive and greatest negative real roots of this equation give the ends of the cycnoïd; the other real roots belong to internal stream-lines.

When the ends of the solid are symmetrical, so that $a''=a$, the preceding equation becomes

$$0 = (l^2-a^2)^2(l^2-a'^2)^2 - 2k^2 la(l^2-a'^2)^2 - 2k'^2 la'(l^2-a^2)^2. \quad \text{(44 A)}$$

The greatest half-breadth and its position are to be found in the general case, as before, by deducing values of y and x by elimination from the pair of equations $\psi=0$, $v=0$. When the ends of the solid are symmetrical, the greatest half-breadth is at the origin; hence, making $x=0$, $v=0$, we have the following equation,

$$0 = y_0^2 - \frac{2k^2 a}{\sqrt{(a^2+y_0^2)}} - \frac{2k'^2 a'}{\sqrt{(a'^2+y_0^2)}}; \quad (45)$$

which, when reduced to the form of an algebraic equation with y_0^2 for the unknown quantity, is of the eighth order, as follows:

$$\left.\begin{aligned} 0 &= y_0^8(y_0^2+a^2)^2(y_0^2+a'^2)^2 + 16k^8 a^4(y_0^2+a'^2)^2 \\ &+ 16k'^8 a'^4(y_0^2+a^2)^2 - 8k^4 a^2 y_0^4(y_0^2+a^2)(y_0^2+a'^2)^2 \\ &- 8k'^4 a'^2 y_0^4(y_0^2+a^2)^2(y_0^2+a'^2) \\ &+ 16k^4 k'^4 a^2 a'^2(y_0+a^2)(y_0^2+a'^2). \end{aligned}\right\} \quad \text{(45 A)}$$

Equation (45) may be used to solve the following problem:—Given the midship half-breadth y_0, the excentricities of the two pairs of foci a, a', and the inner parameter k^2; to find the outer parameter k'^2.

Chapter IV. *Dynamical Propositions as to Stream-line Surfaces.*

§ 12. *Resultant Momentum.*—The resultant momentum, parallel to x, of any part of a given elementary stream is equal to that of an undisturbed part of the same stream whose length, projected on the axis of x, is the same. For let σ_0 be the sectional area of an undisturbed part of such a stream and 1 its velocity; then $\sigma_0 dx$ is the momentum of an elementary part of its length.

Let dx also be the projection on the axis of x of an elementary part of the same stream, when disturbed, σ the sectional area of that part on a plane normal to x, and u its component velocity parallel to x; then its component momentum parallel to x is $u\sigma dx$. But $u\sigma$ is the volume of flow along the elementary stream, which is uniform and $=\sigma_0$; therefore

$$u\sigma dx=\sigma_0 dx;$$

so that the component momentum parallel to x of any part of an elementary stream is simply

$$\sigma_0(x_2-x_1);$$

in which x_2 and x_1 are the values of x for its two ends. Consider now an elementary stream of indefinitely great length, so that its two ends lie in one straight line parallel to x, and are at so great a distance from the disturbing solid that its action on the particles at those ends vanishes. The resultant momentum of that stream is the same as if it were undisturbed; and such being the case for every elementary stream, is the case for the whole mass of liquid. This conclusion is expressed by the following equations, in which the integrations extend throughout the whole liquid mass outside the surface of the disturbing solid

$$\left.\begin{array}{l}\iiint(u-1)dx\,dy\,dz=0;\\ \iiint v dx\,dy\,dz=0;\ \iiint w dx\,dy\,dz=0.\end{array}\right\} \quad \ldots\ldots\ldots\ldots \quad (46)$$

The resultant momentum $\iiint u dx\,dy\,dz$ is that of the liquid relatively to the solid, considered as fixed.

If we next consider the centre of mass of the liquid as fixed, the resultant momentum of the liquid becomes

$$\iiint(u-1)dx\,dy\,dz=0;$$

and that of the solid relatively to the liquid, per unit of velocity and density, is represented by $-$D, D denoting the *displacement* of the solid (that is, the volume of liquid which it displaces, and also the mass of the solid supposed equal to that of the displaced liquid).

Thirdly, let the *common centre of mass* of the liquid and solid be taken as a fixed point, and let the momenta of the liquid and solid relatively to that point be taken. Those momenta are equal and opposite—that of the liquid being positive, and that of

the solid negative. The velocity of the centre of mass of the liquid relatively to the solid being still taken as unity, its velocity relatively to the common centre is expressed as follows, L being the total mass of the liquid,

$$\frac{D}{L+D}. \qquad \qquad (46\text{ A})$$

The velocity of the solid relatively to the common centre is

$$\frac{L}{L+D}; \qquad \qquad (46\text{ B})$$

and the respective equal and opposite momenta of the solid and liquid relatively to the same point are expressed by

$$\pm\frac{LD}{L+D}. \qquad \qquad (46\text{ C})$$

When the mass of liquid L becomes indefinitely great, $\frac{D}{L+D}$ becomes indefinitely small, $-\frac{L}{L+D}$ approximates indefinitely to -1, and $\pm\frac{LD}{L+D}$ to $\pm D$; but notwithstanding these indefinitely close approximations, it is necessary to bear in mind that (as is implied in equation 46) the component longitudinal velocity of current u is taken *relatively to the centre of mass of the liquid*, and not relatively to the common centre of mass, the corresponding component relatively to the common centre being

$$u+\frac{D}{L+D}.$$

If the liquid is absolutely free from stiffness and friction, the *resultant pressure* exerted between it and the solid in a horizontal direction is obviously equal to nothing, so long as the velocity is uniform, and only acquires a value in the event of acceleration or retardation; which value is expressed by the rate of change per second in the equal and opposite momenta $\pm\frac{LD}{L+D}$.

To adapt the formulæ of this and the ensuing sections to other velocities and densities than those denoted by unity, let $-V$ be the velocity of the solid, and ϱ the density of the liquid; then quantities denoting velocities are to be multiplied by V, those denoting masses by ϱ, those denoting momentum by $V\varrho$, those denoting heights due to velocities by V^2, those denoting energy, and those denoting intensity of pressure, by $V^2\varrho$.

It is to be observed that, according to the notation of this paper, motion ahead is treated as negative, and motion astern as positive, the latter being the direction of the motion of the liquid relatively to the solid.

§ 13. *Energy of Currents and of Disturbance.*—The energy of the motion of the liquid mass contained within a given space may be taken either relatively to the disturbing solid, considered as fixed, in which case it may be called the *energy of current*, or relatively to the undisturbed liquid, in which case it may be called the *energy of disturbance*. Assuming unity, as before, for the values of the undisturbed velocity and of the density,

it is obvious that the energy of current in an elementary space of the volume $dx\,dy\,dz$ is

$$\tfrac{1}{2}(u^2+v^2+w^2)dx\,dy\,dz, \quad \text{(A)}$$

and that the energy of disturbance is

$$\tfrac{1}{2}(u^2+v^2+w^2-2u+1)\,dx\,dy\,dz. \quad \text{(B)}$$

To find the total energy of current, or of disturbance, as the case may be, in a given finite space, the one or the other of the two preceding expressions is to be integrated throughout that space. In order to solve questions of this kind, recourse must be had to the *velocity-function* (ϕ) well known in hydrodynamics, and already referred to in § 2, equations (1) to (7), and in § 3, equation (9), as representing by its values a series of surfaces which cut all the elementary streams at right angles—and especially to a property of that kind of function which was first demonstrated by GREEN, in his Essay on Potential Functions, and which is expressed as follows:—Let ϕ be a function of x, y and z, which fulfils the condition

$$\frac{d^2\phi}{dx^2}+\frac{d^2\phi}{dy^2}+\frac{d^2\phi}{dz^2}=0\,;$$

let $d\sigma$ be an elementary part of the bounding surface of an enclosed space, and let $\frac{d}{dn}$ denote differentiation relatively to the normal to that elementary part, dn being positive outwards; then (under certain limitations which do not affect the subject of the present paper)* we have

$$\iiint\left(\frac{d\phi^2}{dx^2}+\frac{d\phi^2}{dy^2}+\frac{d\phi^2}{dz^2}\right)dx\,dy\,dz=\iint\phi\frac{d\phi}{dn}\,d\sigma, \quad \text{(C)}$$

the double integral extending to all parts of the bounding surface. Observing now that

$$u=\frac{d\phi}{dx},\ v=\frac{d\phi}{dy},\ w=\frac{d\phi}{dz},$$

let E_C denote the energy of current, and E_D the energy of disturbance, within a given space, corresponding to the undisturbed velocity 1 and density 1; then we have

$$E_C=\tfrac{1}{2}\iint\phi\frac{d\phi}{dn}\,d\sigma\,; \quad (47)$$

$$E_D=\tfrac{1}{2}\iint\phi\frac{d\phi}{dn}\,d\sigma-\iiint\left(\frac{d\phi}{dx}-\tfrac{1}{2}\right)dx\,dy\,dz. \quad (47\text{A})$$

It is next to be observed that, because the velocity-function ϕ expresses a series of surfaces cutting all the stream-lines at right angles, the coefficient $\frac{d\phi}{dn}$ (denoting the component velocity normal to the elementary surface $d\sigma$) is *nothing* for all bounding surfaces and parts of bounding surfaces that coincide with stream-line surfaces,—and therefore that, in finding the integral E_C which expresses the energy of current within a given

* As to the limitations to which this proposition is subject, see a paper by HELMHOLTZ, in CRELLE's Journal for 1858, "Ueber Integrale der hydrodynamischen Gleichungen, welche den Wirbelbewegungen entsprechen;" also THOMSON on Vortex-Motion, Trans. Roy. Soc. Edin. 1867–68, pp. 239 *et seqq.*

space, it is necessary to take into account *those boundaries only of that space which intersect the stream-lines.*

The values of the velocity-function ϕ for quadrifocal stream-line surfaces are obviously the following:—For cylindrical stream-line surfaces,

$$\phi = x + k \text{ hyp log} \frac{r_2}{r_1} + k' \text{ hyp log} \frac{r_4}{r_3}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (48)$$

for stream-line surfaces of revolution,

$$\phi = x + \frac{k^2}{2}\left(\frac{1}{r_1} - \frac{1}{r_2}\right) + \frac{k'^2}{2}\left(\frac{1}{r_3} - \frac{1}{r_4}\right); \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (49)$$

in which r_1, r_2, r_3, and r_4 denote, as before, the distances of a point from the four foci. These expressions may be made applicable to bifocal surfaces by making $k'=0$, and might be extended to surfaces with any number of pairs of foci by increasing the number of terms and parameters.

When a pair of foci coalesce, the function of r belonging to those foci is to undergo the operation $-\mathrm{A}\,\frac{d}{dx}$, in which A is an arbitrary constant of one dimension—thus giving, for cylindrical surfaces, a term of the form $-\frac{k\mathrm{A}x}{r^2}$, and for surfaces of revolution a term of the form $-\frac{k^2\mathrm{A}x}{2r^3}$.

In the foregoing investigations, and in their appplications which are to follow, the energy of disturbance is taken relatively to the centre of mass of the liquid. If taken relatively to the common centre of mass of the liquid and solid, it would be increased by a quantity whose value for the whole mass of liquid, per unit of undisturbed velocity and of density, is

$$\frac{\mathrm{D}^2\mathrm{L}}{2(\mathrm{L}+\mathrm{D})^2} = \frac{\mathrm{D}^2}{2(\mathrm{L}+\mathrm{D})^2}\iiint dx\,dy\,dz; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (49\ \mathrm{B})$$

but when the extent of the liquid is unlimited, that quantity vanishes as compared with the quantity given by equation (47 A).

§ 14. *Energy in an Elementary Stream.*—In order to apply the principles of the preceding article to the whole or to a given part of an elementary stream, let σ_0 be the transverse sectional area of that stream when undisturbed, measured on a plane normal to x, σ the sectional area on such a plane at a given point, x_1 and x_2 the values of x, σ_1 and σ_2 the values of σ, and ϕ_1 and ϕ_2 the values of ϕ, for the two ends of the part of the stream under consideration; and let x_2 be greater than x_1. Then the energy of current, per unit of undisturbed velocity and of density, is found by taking the integral in equation (47) for those two ends only; that is to say,

$$\mathrm{E_C} = \tfrac{1}{2}\phi_2 \frac{d\phi_2}{dx_2}\sigma_2 - \tfrac{1}{2}\phi_1 \frac{d\phi_1}{dx_1}\sigma_1;$$

but $\frac{d\phi}{dx}\sigma = u\sigma = \sigma_0$; and therefore we have simply, for the energy of current,

$$\mathrm{E_C} = \frac{\sigma_0}{2}(\phi_2 - \phi_1). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (50)$$

The energy of disturbance per unit of velocity and density is expressed by

$$E_D = E_C - \int_{x_1}^{x_2} (u - \tfrac{1}{2})\sigma dx) = \frac{\sigma_0}{2}\left\{\phi_2 - \phi_1 - 2x_2 + 2x_1 + \int_{x_1}^{x_2} \frac{dx}{u}\right\}. \quad . \quad . \quad . \quad . \quad . \quad (51)$$

§ 15. *Total Energy of Disturbance.*—In order to find the total energy of disturbance throughout the indefinitely extended mass of liquid, the most convenient method is to find the limit to which E_D in equation (47 A) approximates when the integrals are taken throughout a circular cylinder for cylindric cycnoïds, or a sphere for cycnoïds of revolution, and the radius of the cylinder or of the sphere is indefinitely increased. The coefficient $\frac{d\phi}{dn}$ is $=0$ at every point of the surface of the disturbing solid; therefore no integration has to be performed over that surface. The triple integral in the second term of the equation, viz.

$$-\iiint (u - \tfrac{1}{2}) dx\, dy\, dz,$$

may be simplified by considering that, because the integration extends throughout the unlimited mass of the liquid, we have, by equation (46),

$$\iiint (u - 1) dx\, dy\, dz = 0,$$

and consequently

$$-\iiint (u - \tfrac{1}{2}) dx\, dy\, dz = -\tfrac{1}{2}\iiint dx\, dy\, dz.$$

Now this is obviously the half difference, with the sign reversed, between the volume of the indefinitely large cylinder or sphere, as the case may be, and the displacement or volume of the disturbing solid, denoted by D. Moreover, in the first term of the equation, we have $dn = dr$; $\int d\sigma = r d\theta$ for a cylinder, or $2\pi r^2 \sin\theta d\theta$ for a sphere, θ being the angle which r makes with the axis of x; and the limits of integration are from $\theta = 0$ to $\theta = 2\pi$ for a cylinder, and from $\theta = 0$ to $\theta = \pi$ for a sphere. Hence we have the following expressions:—For indefinitely deep cylindrical solids,

$$E_D = \tfrac{1}{2}\int_0^{2\pi} \left(\phi \frac{d\phi}{dr} - \frac{r}{2}\right) r d\theta + \frac{D}{2}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (52)$$

for solids of revolution,

$$E_D = \tfrac{1}{2}\int_0^{\pi} \left(\phi \frac{d\phi}{dr} - \frac{r}{3}\right) 2\pi r^2 \sin\theta d\theta + \frac{D}{2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (53)$$

In taking the values of ϕ and $\frac{d\phi}{dr}$ corresponding to an indefinitely great value of r, it is to be observed that the distance $2a$, or $a' + a''$, between a given pair of foci, becomes indefinitely small compared with r, and that consequently, if F be a function of the distance from a focus, and ΔF the difference of its values for a pair of foci whose distance apart is $2a$, we are to make

$$\Delta F \text{ sensibly} = -2a \frac{dF}{dx}.$$

Hence $\left(\text{observing that } \frac{dx}{dr}=\frac{x}{r}\right)$ we have for an indefinitely large cylinder,

$$\left.\begin{aligned}\phi&=x-\Sigma.\frac{2kax}{r^2},\\ \frac{d\phi}{dr}&=\frac{x}{r}+\Sigma.\frac{2kax}{r^3};\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (54)$$

and for an indefinitely large sphere,

$$\left.\begin{aligned}\phi&=x-\Sigma.\frac{k^2ax}{r^3},\\ \frac{d\phi}{dr}&=\frac{x}{r}+\Sigma.\frac{2k^2ax}{r^4};\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (55)$$

in each of which expressions Σ denotes the summation of terms belonging to the several pairs of foci, if there are more than one pair—each term containing its proper parameter, k or k^2, and its proper double excentricity, $2a(=a'+a''$ when those two distances are unequal).

Substituting $\cos\theta$ for $\frac{x}{r}$, the functions within brackets in the integrals of equations (52) and (53) are found to have the following values:—

Cylinder:

$$\phi\frac{d\phi}{dr}-\frac{r}{2}=r(\cos^2\theta-\tfrac{1}{2}-\text{terms in } \tfrac{1}{r^4}\text{ \&c.}). \quad . \quad . \quad . \quad (56)$$

Sphere:

$$\phi\frac{d\phi}{dr}-\frac{r}{3}=r(\cos^2\theta-\tfrac{1}{3}+\Sigma.\frac{k^2a\cos^2\theta}{r^2}-\text{terms in } \tfrac{1}{r^6}\text{ \&c.}). \quad . \quad . \quad (57)$$

The terms in $\frac{1}{r^4}$ and higher powers of $\frac{1}{r}$ vanish, because of the indefinite increase of r. The terms in $\cos^2\theta-\frac{1}{2}$ and $\cos^2\theta-\frac{1}{3}$ disappear from the integration. Hence the integral in equation (52) vanishes altogether; and that in equation (53) has for its value

$$\frac{1}{2}\int_0^\pi\Sigma\left(\frac{k^2a\cos^2\theta}{r^2}\right).2\pi r^2\sin\theta d\theta=-\frac{2\pi}{3}\Sigma k^2a; \quad . \quad . \quad . \quad (58)$$

so that we obtain finally, for *the total energy of disturbance per unit of velocity and of density*, if the disturbing solid is an indefinitely deep cylinder,

$$E_D=\tfrac{1}{2}D; \quad . \quad . \quad . \quad . \quad (59)$$

and if it is a solid of revolution,

$$E_D=\tfrac{1}{2}\left(D-\frac{4\pi}{3}\Sigma k^2a\right). \quad . \quad . \quad . \quad . \quad (60)$$

The *ratio borne by the total energy of disturbance to the energy of the disturbing solid* is:—

for indefinitely deep cylinders,

$$\frac{2E_D}{D}=1; \quad . \quad . \quad . \quad . \quad (59\text{ A})$$

for solids of revolution,

$$\frac{2E_D}{D}=1-\frac{4\pi}{3D}\cdot\Sigma k^2 a; \quad \text{(60 A)}$$

observing, in the last expression, that for any pair of foci whose distances from the origin a' and a'' are unequal, the mean of those distances, $\frac{a'+a''}{2}$, is to be taken as the value of a.

When the disturbing solid is a sphere of the radius l, its displacement is $D=\frac{4\pi l^3}{3}$. It has one focus at its centre, produced by the coalescence of a pair of foci; k^2 becomes indefinitely great, and a indefinitely small; but their product has a finite value, $k^2 a=\frac{l^3}{2}$. Hence in this case we have

$$\frac{2E_D}{D}=1-\tfrac{1}{2}=\tfrac{1}{2}; \quad \text{(60 B)}$$

that is to say, *the total energy of the disturbance produced by a sphere is equal to half the energy of the sphere.*

When the solid is an oval or bifocal neoïd of revolution, and the excentricity a increases indefinitely as compared with the parameter k^2, the displacement approximates upwards towards that of a cylinder of revolution of the length $2a$ and transverse section $2\pi k^2$ (that is, towards $4\pi k^2 a$); so that in this case we have for the upper limit of the ratio of the total energy of disturbance to the energy of the solid, the following value:—

$$\frac{2E_D}{D}=1-\tfrac{1}{3}=\tfrac{2}{3}. \quad \text{(60 C)}$$

For all neoïds of revolution, oval and cycnoïd, the ratio in question lies between the limits $\frac{1}{2}$ and $\frac{2}{3}$. Its value in any particular case may always be determined to any required degree of approximation by constructing the figure of the disturbing solid and measuring its displacement. For example, in fig. 3 it is found to be, for the oval neoïd of revolution L B, 0·56; and for the cycnoïd of revolution L′ B′, 0·6 nearly.

The principles of this and the three preceding sections (§ 12, 13, and 14) are applicable not only to bifocal, quadrifocal, and other stream-line surfaces having foci situated in one axis, but to all stream-line surfaces which can be generated by combining a uniform current with disturbances generated by pairs of foci arranged in any manner whatsoever, or having, instead of detached focal points, *focal spaces*; the disturbance-functions belonging to which are to be found by integrating the corresponding functions belonging to the points contained in those spaces, a process similar to that of finding the potential of a solid*.

§ 16. *Disturbance of Pressure and Level.*—It is well known that in all cases of the steady flow of a liquid, the sum of the height due to velocity, and the height due to elevation and pressure combined, is constant in a given elementary stream; that is to

* Note by the Reporter.—See paper, Professor C. NEUMANN, in CRELLE's Journal for 1861, on the equation $\frac{d^2u}{dx^2}+\frac{d^2u}{dy^2}=0$.

say, let h_0 be the *head*, or height due to elevation and pressure, in a given elementary stream, at a point where the velocity is that of the undisturbed uniform current; let V, as before, denote that velocity, so that uV, vV, and wV are the components of the velocity at any other point; then, at that other point, the head is given by the following equation,

$$h=h_0+\frac{V^2}{2g}(1-u^2-v^2-w^2); \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (61)$$

and the following difference may be called the *disturbance of head*,

$$h-h_0=\frac{V^2}{2g}(1-u^2-v^2-w^2).$$

Some of the general consequences of this principle have been pointed out in the paper "On Plane Water-lines" already referred to; and its bearing on the laws of the resistance of ships has been shown in a paper "On the Computation of the probable Engine-power and Speed of proposed Ships," published in the Transactions of the Institution of Naval Architects for 1864.

In connexion with the subject of the present paper, it is sufficient to state that, when a current of a perfect liquid of unlimited extent in all directions flows past a solid, the disturbance of head takes the form of variation of pressure only, the energy of a given particle of an elementary stream changing its form between energy of motion and energy of pressure as the velocity varies—so that points of minimum velocity of current are points of maximum pressure, and points of maximum velocity of current are points of minimum pressure,—but that where the current is bounded above by a free upper surface, exposed to the air, that surface continues to be everywhere a surface of uniform pressure, and the disturbances of head take the form of disturbances of level, places of minimum velocity being marked by a swell, and those of maximum velocity by a hollow. For example, when a floating solid body, as a ship, moves through still water, the surface of the water is raised at those points where the particles of water are pushed or drawn ahead by the ship, and depressed at those points where they run astern past her sides in order to fill up the space in her wake.

The aggregate disturbance of head throughout the whole liquid mass is expressed as follows,

$$\iiint(h-h_0)dx\,dy\,dz=-\frac{V^2}{2g}\cdot E_D, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (62)$$

being obviously equal, but of contrary sign, to the total energy of disturbance per unit of density (see equation 47 A).

Let the whole volume of the liquid mass be denoted by $L=\iiint dx\,dy\,dz$; then

$$\frac{\iiint(h-h_0)dx\,dy\,dz}{\iiint dx\,dy\,dz}=-\frac{V^2E_D}{2gL} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (63)$$

expresses a depression of the centre of gravity of that mass relatively to the surface of the liquid at an indefinite distance from the disturbing solid—in other words, *an eleva-*

tion of the surface of the liquid at an indefinite distance from the solid above the centre of mass of the liquid; so that the disturbance of head at any point *relatively to that centre of mass* is expressed as follows:—

$$h-h_0+\frac{V^2E_D}{2gL}. \qquad (64)$$

The last term vanishes when the volume of liquid L increases indefinitely.

When the trace of a disturbing solid, together with its external stream-lines and lines of disturbance, has been drawn, as in figs. 2 and 3, the manner in which the disturbances of motion and of head vary at different points may be represented to the eye by means of a diagram like fig. 5, constructed as follows. Draw a straight line A B to represent the velocity of the undisturbed current (equal and opposite to the velocity of the ship). From A draw a series of straight lines, such as A C, A C′, A C″, parallel to a series of tangents at a series of points in the trace of the solid. From B draw a series of straight lines, such as B C, B C′, B C″, parallel to the tangents of the lines of disturbance at the same series of points, cutting the first-mentioned series of lines in C, C′, C″, Then in each of the triangles in the diagram, such as A B C, corresponding to a given point in the trace of the solid, B C will represent the direction and velocity of the disturbance, A C the direction and velocity of the elementary stream of liquid relatively to the solid; and the disturbance of head, positive upwards, will be expressed by $\frac{AB^2-AC^2}{2g}$.

At the points marked L and L′ in figs. 2, 3, and 4, the disturbance of head is simply the height due to the velocity of the disturbing solid.

When the disturbances of head, as in a liquid with a free upper surface, take the form of disturbances of level, they produce two effects—alteration of the forms and motion of the elementary streams, and the formation of waves; which waves may give rise to a particular kind of resistance. In the present paper it is assumed that the dimensions of the disturbing solid are so large, or its motion so slow, that the effects of the disturbances of level on the forms and motions of the elementary streams may be neglected; and the investigation in the ensuing sections is confined to the action of those disturbances in producing waves and wave-resistance.

§ 17. *Virtual Depth and Speed of Waves.*—The term *virtual depth of longitudinal disturbance*, or, more briefly, *virtual depth*, is used to denote the depth found by integrating the velocity of longitudinal disturbance throughout a vertical column of a liquid mass, and dividing the integral by the value of that velocity at the free upper surface of the mass. For example, let $u-1$ be the velocity of longitudinal disturbance in a given indefinitely slender vertical column at the depth z, and u_1-1 its value at the surface; and let Z be the virtual depth; then

$$Z=\frac{\int(u-1)dz}{u_1-1}; \qquad (65)$$

when the column is a prism of finite dimensions, the *mean virtual depth* is as follows:—

$$Z_m = \frac{\iiint (u-1)dx\,dy\,dz}{\iint (u_1-1)dx\,dy}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (66)$$

When the disturbance is that produced by the longitudinal advance of a solid whose figure is a stream-line surface of revolution, with any number of pairs of foci, floating immersed to the axis in a liquid of indefinite depth, the integrations indicated in the preceding equations give the following results:—

Virtual depth at a given point,

$$Z = \frac{\Sigma k^2\left(\frac{a-x}{r_1^2}+\frac{a+x}{r_2^2}\right)}{\Sigma k^2\left(\frac{a-x}{r_1^3}+\frac{a+x}{r_2^3}\right)}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (67)$$

the notation being the same as in equations (48) and (49).

Mean virtual depth throughout the whole mass,

$$Z_m = \frac{D}{S}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (68)$$

in which D is the displacement of the floating solid, and S the area of its *water-section* (that is, of its horizontal section in the plane of the surface of the water); so that the mean virtual depth is equal simply to the mean depth of immersion of the solid. Here it must be explained that, when the disturbances relatively to the centre of mass of the liquid are integrated, equation (68) takes the form $Z_m = \frac{0}{0}$, and that the value $\frac{D}{S}$ is obtained by taking the disturbances relatively to the common centre of mass of the liquid and solid.

At the two ends of the floating solid, where $x=l$ and $y=0$, the virtual depth takes the following value.

$$Z_l = \frac{\Sigma \cdot \frac{k^2 a}{l^2-a^2}}{2l\Sigma \cdot \frac{k^2 a}{(l^2-a^2)^2}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (69)$$

When there is but one pair of foci, this is reduced to $\frac{l^2-a^2}{2l}$.

At the midship section (B and B′ in the figures) the virtual depth is

$$Z_{y_0} = \frac{\Sigma \cdot \frac{k^2 a}{a^2+y_0^2}}{\Sigma \cdot \frac{k^2 a}{(a^2+y_0^2)^{\frac{3}{2}}}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (70)$$

y_0 being the extreme half-breadth. When there is but one pair of foci, this becomes simply $\sqrt{a^2+y_0^2}$.

For the disturbance caused by a sphere, half immersed, equation (67) takes the following form,

$$Z=r\cdot\frac{2\cos^2\theta-1}{3\cos^2\theta-1}; \qquad (71)$$

in which r is the horizontal distance from the centre of the sphere, and θ the angle that r makes with the axis of x; and the same value of Z is approximated to at distances from a disturbing solid of any figure which are very great compared with the dimensions of the solid.

The following examples are calculated for the oval neoïd of revolution LB, and for the cycnoïd of revolution L'B', shown in fig. 3, the unit of measure being one tenth part of the distance from the axis OX to the nearest of the straight lines that are parallel to it,—also for a sphere of the radius 1.

	Oval.	Cycnoïd.	Sphere.
Half-length l	64	95	1
Extreme half-breadth y_0=greatest depth of immersion	26	31·6	1
Mean virtual depth Z_m	19·1	20·5	$\frac{2}{3}$
Virtual depth at ends Z_l	13·2	13·5	$\frac{1}{2}$
Virtual depth amidships Z_{y_0}	55·5	62·4	1

When *a wave of a given length* travels in water of unlimited depth, the virtual depth of disturbance is equal to the radius of a circle whose circumference is equal to the length of the wave. For *a wave of a given periodic time*, in water of unlimited depth, the virtual depth is equal to the height of a revolving pendulum which makes one revolution in the period of a wave. For *a wave travelling at a given speed*, under all circumstances whatsoever, the virtual depth is twice the height due to the speed; and conversely, *for a given virtual depth*, under all circumstances, the speed is that acquired during a fall through half the depth. (See Proceedings of the Royal Society, 16th April, 1868, page 345.) These laws are expressed as follows. Let W be the speed of advance of a wave in a horizontal direction perpendicular to the line of its crest, λ its length, T its period; then we have

in water of any depth, limited or unlimited,

$$W^2=gZ; \qquad (72)$$

and in water of unlimited depth.

$$T=2\pi\sqrt{\frac{Z}{g}}; \qquad (72\text{ A})$$

$$\lambda=WT=2\pi Z; \qquad (72\text{ B})$$

and therefore

$$W^2=\frac{g\lambda}{2\pi}; \qquad (72\text{ C})$$

$$W=\frac{gT}{2\pi}. \qquad (72\text{ D})$$

§ 18. *Probable Laws of Wave-resistance.*—It has been proved by observation that a floating solid, such as a ship, is accompanied by waves, originating in the disturbances which it produces in the level of the water. None of those waves (at all events none whose energy is appreciable) travel faster than the floating solid. Some travel at the same speed, and some slower—each wave having its velocity in a direction normal to its crest regulated by its virtual depth, according to equation (72).

Those waves may be divided into three classes. The *first class*, whose properties were pointed out by Mr. SCOTT RUSSELL about twenty-five years ago, have a speed, and therefore a virtual depth, depending on the periodic time which elapses between the raising of a swell by the fore body and after body of the vessel respectively.

In the *second class* the virtual depth is regulated by the mean virtual depth of the whole longitudinal disturbance (68)—that is, by the mean depth of immersion of the vessel; the existence of these waves has been proved by observations of several actual vessels, some of which are described in a paper read to the British Association in 1868 (see the Reports for that year, p. 194; also the Transactions of the Institution of Naval Architects, 1868, p. 275; and the 'Engineer' for the 28th August and 30th October, 1868).

The waves which have been found by observation most distinctly to follow this law. are a pair of diverging waves which closely follow the stern of the vessel.

The *third class* of waves appear to depend on the several virtual depths of disturbance at various points in the neighbourhood of the vessel, especially at and near the bow. They diverge at various angles; and travelling into water in which the virtual depth increases, they become accelerated, so that their ridges are gradually curved forward. The general theory of this class of waves has been stated in the papers already referred to in connexion with the second class; but, so far as I know, they have not yet been subjected to exact observation, for which perfectly smooth water is necessary.

When a wave accompanies a disturbing body whose speed is greater than that of the wave, the direction of advance of the wave, which is perpendicular to its ridge-line, adjusts itself so as to make with the direction of advance of the vessel an angle whose cosine is the ratio borne by the speed of the wave to the speed of the ship; that is to say, let W be the speed of the wave, V that of the ship, α the angle of obliquity of the advance of the wave, then

$$\cos\alpha=\frac{W}{V} \qquad \ldots\ldots\ldots\ldots \qquad (73)$$

(see Transactions of the Institution of Naval Architects for 1864, vol. v. p. 321; also WATTS, RANKINE, NAPIER, and BARNES, on 'Shipbuilding,' p. 79). The effect of the divergence of a wave is to disperse, to distant parts of the water, a certain quantity of energy which is never restored to the vessel, and thus to cause a kind of resistance which may be called *wave-resistance.* It has been suggested, as a probable law of the rate at which a diverging wave disperses energy, that this rate is proportional to the *breadth of new wave* raised in a second; which breadth is equal to the speed of the vessel multiplied

by the sine of the angle of obliquity of the wave; that is,

$$V \sin \alpha = \sqrt{V^2 - W^2}; \quad \ldots \ldots \ldots \ldots \quad (74)$$

and if this be correct, the resistance arising from the dispersion of energy by a set of waves of a given speed may be expressed as follows:—Let R′ be the propelling force which would be required in order to produce the disturbance constituting the wave-motion, if the whole of the energy of that motion were dispersed; then the actual propelling force required in order to restore the energy dispersed by those waves will be

$$R' \sin \alpha = R' . \sqrt{\left(1 - \frac{W^2}{V^2}\right)}. \quad \ldots \ldots \ldots \ldots \quad (75)$$

The total wave-resistance of a ship, according to this hypothesis, is the sum of a set of terms similar to the above expression, each term belonging to a different set of waves and containing its proper values of R′ and of W′.

Each value of R′ is probably proportional to the square of the speed of the ship, and to some function of her dimensions and of the position of that part of her where the set of waves in question originates, and may therefore be expressed in units of weight by $\frac{\omega V^2 \varrho}{2g}$, where ω is such a function, and ϱ the density of the water. Hence the total wave-resistance may be expressed as follows:

$$\Sigma . R' \sin \alpha = \frac{V^2 \varrho}{2g} . \Sigma \left\{ \omega \sqrt{\left(1 - \frac{W^2}{V^2}\right)} \right\}. \quad \ldots \ldots \ldots \quad (76)$$

For waves of the first class the value of W is that given by equation (72 D), the period T_1 being expressed as follows,

$$T_1 = \frac{f_1 l_1 + f_2 l_2}{V}; \quad \ldots \ldots \ldots \ldots \quad (77)$$

where l_1 and l_2 are the lengths of the fore body and after body respectively, and f_1 and f_2 two coefficients, depending on the forms of those bodies. From the practical results of the rules given by Mr. Scott Russell, there seems to be reason to believe that those coefficients are sensibly equal to, or not very different from, the *coefficients of fineness*, found by dividing the displacement of the fore body and after body respectively by the area of midship section. The speed of waves of the first class is thus given by the following formula,

$$W_1 = \frac{g(f_1 l_1 + f_2 l_2)}{2\pi V}; \quad \ldots \ldots \ldots \ldots \quad (78)$$

and in order that such waves may not disperse energy by their divergence, it is necessary that W_1 should be equal to or greater than V; that is to say, that

$$f_1 l_1 + f_2 l_2 = \text{or} > \frac{2\pi V^2}{g}. \quad \ldots \ldots \ldots \ldots \quad (79)$$

It appears further, from results of practice, that it is advisable that the two terms of the left-hand member of this equation should be equal to each other; that is to say,

$$f_1 l_1 = f_2 l_2 = \text{or} > \frac{\pi V^2}{g}; \quad \ldots \ldots \ldots \ldots \quad (80)$$

and if we make $f_1=\frac{1}{2}$ and $f_2=\frac{3}{4}$, this becomes Mr. SCOTT RUSSELL'S rule for the least lengths of fore and after body suited to enable a ship to be driven economically at a given speed.

It is well known that in water that is shallow, compared with the length of a wave, waves of a given period are retarded according to a certain law (see AIRY on Tides and Waves). Hence the fact, which has often been observed, that a length which is sufficient for a given speed in deep water, becomes insufficient in shallow water—the waves of the first class becoming divergent, and the swell under the after body lagging behind, so as to make the stern of the vessel "squat," as it is called.

For waves of the second class, the value of W is given by equation (72), putting for Z the value given by equation (68)—that is, the mean depth of immersion $D \div S$. Hence we have

$$W^2=\sqrt{\frac{gD}{S}}; \qquad (81)$$

and this is probably unaltered in shallow water. The period of these waves is the same with that of the dipping, or vertical oscillation of the ship, whose value in deep water is

$$T_2=2\pi\sqrt{\frac{D}{gS}}. \qquad (82)$$

Waves of the third class are observed to have, as theory indicates, a great angle of obliquity at and near the bow of the vessel, gradually diminishing as they travel to more distant masses of water where the virtual depth is greater. Beyond this general agreement, their precise laws are not yet known, for want of a sufficient number of precise observations.

The general nature of the phenomena of wave-resistance, as indicated both by theory and by observation, are as follows. When either the speed of the vessel is so small, or her dimensions so great, as to make the ratio $\frac{W}{V}$ of the speed of each set of waves to that of the vessel greater than or equal to unity, in other words, to make the ratio $\sqrt{\left(1-\frac{W^2}{V^2}\right)}$ of the breadth of new wave raised per second to the speed of the ship nothing or imaginary, there is no wave-resistance, and the only resistances to be overcome in driving the ship at a uniform speed are that due to stiffness or viscosity, and that due to friction or "skin-resistance." The first of these increases simply as the speed; and at the velocities usual in navigation, it becomes almost inappreciable when compared with the resistance due to friction. At very low speeds it is the principal resistance. Its laws have been fully investigated by Mr. STOKES.

The resistance due to friction increases sensibly as the square of the speed. Some remarks on this kind of resistance will be added in the next section.

So soon as the ratio $\frac{W}{V}$ becomes less than unity for any set of waves, wave-resistance

begins to be felt, and shows its nature by increasing more rapidly than the square of the speed; and its effects become more and more conspicuous as additional sets of waves come successively into operation as means of dispersing energy.

When the speed of the disturbing body becomes so great that, for all or for most of the sets of waves, the ratio $\frac{W}{V}$ becomes a very small fraction, the whole, or nearly the whole of the energy of disturbance is dispersed and wasted, and wave-resistance becomes the principal, or it may be the only appreciable resistance. In this extreme case it is possible to make a theoretical estimate of the amount of that resistance, as follows. The whole energy of disturbance is expressed, in absolute units, by

$$V^2 \varrho E_D,$$

a function of which values have been given in equations (47 A), (53), (60), &c.

The total dispersion of that quantity of energy, and its reproduction by the disturbing action of the solid, may be considered as taking place while the midship section M sweeps through a space equal to the displacement D of the solid—that is, while the solid advances through the distance $\frac{D}{M}$; and hence the propelling force required to overcome wave-resistance will probably have the following value, in *units of weight*.

$$\Sigma \, . \, R' = \frac{V^2 \varrho E_D M}{g D}; \quad \dots \dots \dots \dots \quad (83)$$

and the resistance will again increase as the square of the velocity.

The only solid *of continuous figure* on which experiments have been made suitable for comparison with this formula is the sphere. For that body, equation (60 B) informs us, we have $\frac{2E_D}{D} = \frac{1}{2}$, and $\frac{E_D}{D} = \frac{1}{4}$; therefore the extreme wave-resistance is

$$\Sigma \, . \, R' = \frac{V^2 \varrho M}{4g}; \quad \dots \dots \dots \dots \quad (83\text{A})$$

that is to say, it is equal to *the weight of a column of liquid of half the height due to the speed, on a base equal to the midship section*,—a result which agrees very closely with experiment.

Since a propelling instrument which acts by the reaction of the water, as a paddle, a screw, an oar, or a jet, drives the particles of water astern, it tends to diminish the height of the crest of a wave, and to increase the depth of a trough or hollow;—in the former case diminishing, and in the latter increasing the energy of the wave, which partly goes to waste in the case of divergence; and hence it follows that it is favourable to economy of power that such a propelling instrument should act on the crest, rather than on the hollow of a wave. This fact is well known in practice.

The production of diverging waves is not prevented by totally submerging the disturbing body; but those waves are of less height at the surface of the water, the more deeply the body is covered. The virtual depth, and consequently the speed, of the waves of the second and third classes increases, and their angle of divergence diminishes, with

increased submergence of the body; but the speed, and consequently the angle of divergence, of the waves of the first class is unaltered, because they depend on the time occupied by the solid in moving through a certain portion of its length.

§ 19. *Remarks on the Skin-resistance.*—It is well known through observation:—that the friction between a ship and the water acts by producing a great number of very small eddies in a thin layer of water close to the skin of the vessel, and also an advancing motion in that layer of water; that this *frictional layer* (as it may be called) is of insensible thickness at the cutwater, and gradually increases in thickness towards the stern, by the communication of the combined whirling and progressive motion to successive streams of particles; and that, finally, the various elementary streams of which the frictional layer is composed, uniting at the stern of the ship, form her *wake*—that is, a steady or nearly steady current, full of small eddies, which follows the ship, but at a speed relatively to still water which is less than the speed of the ship.

The central stream of the wake has the greatest velocity ahead; and other parts of it have velocities diminishing from the centre towards the circumference. If the friction between the water and a given area of the skin of the ship is equal to that of an equal area of one layer of water upon another at a given velocity, the *mean* forward velocity of the whole wake relatively to still water, and its mean backward velocity relatively to the ship, are each of them equal to one half of her speed.

The effect of discontinuity of form, as when the figure of the vessel presents angles to the water, is to produce eddies which are dragged along with the ship, and thus to add to the wake; and hence the resistance arising from discontinuity of form is analogous in its laws to that arising from friction; and both those forces are comprehended under the name of *eddy-resistance.* Bodies of discontinuous forms, however, are foreign to the subject of this paper.

Let V, as before, be the velocity of the ship; let W′ denote the mean velocity of the wake, and C its area of cross section, both taken at a distance astern of the vessel sufficient for the wake to have become a steady forward current. Let R be the amount of the skin-resistance in units of weight, and ϱ the density of water. Then the mass of water added to the wake in each second is $\varrho C(V-W')$; and the velocity impressed on that mass by the force R is W′; whence we have the following equation,

$$R=\frac{1}{g}\cdot\varrho C(VW'-W'^2); \quad \dots\dots\dots \quad (84)$$

and if the mean velocity of the wake is half the velocity of the ship, that equation becomes

$$R=\frac{\varrho CV^2}{4g}. \quad \dots\dots\dots \quad (84\text{ A})$$

It is obvious from equation (84) that, for a given amount of skin-resistance, the wake has the *least possible sectional area* when its mean speed is half that of the ship.

The work done by the ship on the water per second in producing the wake is RV; the actual energy of the current of the wake is increased in each second by the amount

$$\frac{\varrho C}{g}\cdot(V-W')\frac{W'^2}{2};$$

and the difference between those quantities—that is,

$$RV-\frac{\varrho C}{g}(V-W')\frac{W'^2}{2}=\frac{\varrho C}{g}(V-W')\left(VW'-\frac{W'^2}{2}\right), \quad . \quad . \quad . \quad . \quad (84\text{ B})$$

is the energy added to that of the eddies in each second. If, as before, we have $W'=\frac{1}{2}V$, the preceding equation takes the following value,

$$RV-\frac{\varrho CV^3}{16g}=\frac{3\varrho CV^3}{16g}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (84\text{ C})$$

so that one fourth of the work of friction is expended in producing the current in the wake, and the other three fourths in producing eddies.

If the velocities V and W′ of the ship and her wake, and the amount of eddy-resistance R, are given, the sectional area C of the wake may be calculated from equation (76).

The elementary streams of which the wake is composed move astern relatively to the ship with a velocity less than that of an undisturbed current in the ratio expressed by $\frac{V-W'}{V}$; and hence they occupy a transverse area greater than they would do in the undisturbed state in the ratio expressed by

$$\frac{V}{V-W'}=1+\frac{W'}{V-W'}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (85)$$

which, when $W'=\frac{1}{2}V$, becomes $=2$. This causes a certain modification in the forms of the stream-lines outside the wake, which might be represented by taking for the surface of an imaginary disturbing solid a surface midway between the skin of the vessel and the outer surface of the frictional layer, followed by an indefinitely long cylindrical tail of one half of the sectional area of the wake; but the detailed investigation of this will not now be entered on.

Mr. Froude a few years ago pointed out that the most perfect propeller for driving a ship against skin-resistance, would be one which should act solely on the particles of the wake, driving them astern so as just to take away their forward velocity and no more. The velocity of such a propeller relatively to the ship would be equal and opposite to her speed V; and the energy expended in working it would be simply RV, equal to the work done by the ship, through friction, on the water. It would thus be a propeller free from "slip" and free from waste of power. It would stop the following current in the wake, and would at the same time impress on the water an additional quantity of energy in the form of eddy-motion, equal to the energy taken away in stopping the current; so that the total energy impressed on the water in each second would be the same as before.

It would preserve to the stream-lines the shape which they would have in the absence of friction.

A propeller of the most efficient kind possible, producing the same forward thrust R, by acting on previously undisturbed water so as to impress a backward velocity W″ on a current of the sectional area B, would move it astern relatively to the ship with the

velocity $V+W''$; and besides expending in each second the quantity of energy RV in driving the ship ahead, it would expend the additional quantity RW'' in driving the water astern. The relation between the sectional area and the velocity of the current produced by such a propeller is given by the following equation,

$$R=\frac{\varrho B}{g}(V+W'')W'', \quad \ldots \ldots \quad (86)$$

because $\varrho B(V+W'')$ is the mass of water acted on in each second, and W'' the velocity impressed on it. The *counter-efficiency*, being the ratio in which the total work done exceeds the useful work, is

$$1+\frac{W''}{V}. \quad \ldots \ldots \quad (86\text{ A})$$

In previous writings* it has been shown that the amount of skin-resistance is probably expressed by a formula of the following kind,

$$R=\frac{\varkappa\varrho V^2}{2g}\cdot\iint q^3 d\omega; \quad \ldots \ldots \quad (87)$$

in which $d\omega$ is the area of an elementary portion of the skin of the ship, $q=\sqrt{(u^2+v^2+w^2)}$ the ratio borne by the velocity with which the particles of water glide over that elementary area, to the velocity of the ship (V), ϱ the density of water, and $\varkappa$ a coefficient of friction, whose value, as deduced from the performance of actual ships, is about ·0036 or ·004 for a clean surface of painted iron.

The integral $\iint q^3 d\omega$ is called the *augmented surface*; and the ratio

$$\frac{\iint q^3 d\omega}{\int G dx} \quad \ldots \ldots \quad (87\text{ A})$$

is called the *coefficient of augmentation*. The denominator, $\int G dx$, is what may be called the girth-integral, G denoting the immersed girth of a given cross section of the vessel. The augmented surface and coefficient of augmentation can be calculated for any particular stream-line surface by drawing it, constructing such a diagram as that shown in fig. 5, and finding approximate values of the definite integrals by SIMPSON'S Rules; but to give exact general symbolic expressions for them involves difficulties which have not yet been overcome.

The following are particular cases in which exact expressions have been found:—

Indefinitely deep circular cylinder of radius l, $q=2\sin\theta$;
Augmented surface per unit of depth, $21\frac{1}{3}l$;
Coefficient of augmentation, $\frac{16}{3}=5{\cdot}3$ nearly.
Sphere of radius l, $q=\frac{3}{2}\sin\theta$;

* Philosophical Transactions 1863, p. 134; 1864, p. 384; Civil Engineer and Architect's Journal, October 1861; Transactions of the Institution of Naval Architects for 1864, vol. v. p. 322; Shipbuilding, Theoretical and Practical.

Augmented surface, $\frac{8}{3}\frac{1}{2}\pi^2 l^2 = 25 l^2$ nearly;
Girth-integral, $\int G dx = \pi^2 l^2 = 9{\cdot}87 l^2$ nearly;
Coefficient of augmentation, $\frac{8}{3}\frac{1}{2} = 2{\cdot}531$ nearly.

In each case θ denotes the angle made by a given radius with the direction of motion.

For a sphere half-immersed the augmented surface and girth-integral have respectively half the values given above.

For an approximately trochoidal riband of uniform breadth, it has been elsewhere shown (Philosophical Transactions, 1863, p. 134) that the coefficient of augmentation is very nearly $1+4\sin^2\beta+\sin^4\beta$, β being the angle of greatest obliquity of the riband to the direction of motion.

With a view to the calculation of the augmented surface by numerical definite integration in particular cases, the following values of the elementary surface $d\omega$ and of its first integral are given. As to the function χ, see § 3.

General case:

$$d\omega = \sqrt{\{dy^2dz^2+dz^2dx^2+dx^2dy^2\}}. \quad (88)$$

Cylindrical surface of indefinite depth; $d\omega$ per unit of depth $=\frac{q dx}{u}$. . . . . (88 A)

Surface of revolution, *half-immersed*; $\int d\omega$ for a zone or belt measuring dx lengthwise $=\frac{\pi y q dx}{u}$. . . . . (88 B)

§ 20. *General Remarks.*—The dynamical investigations contained in this chapter are partly certain and exact, partly approximate, and partly conjectural. The results arrived at in §§ 12 to 15 as to momentum and energy of current and of disturbance, are all certain and exact when applied to the case of a solid body of any figure past which a fluid can glide continuously, immersed in an unlimited mass of liquid, and approximate when applied to cases such as those described in § 16, in which these conditions are approximately fulfilled. The results as to virtual depth of disturbance, and as to speed of waves, in § 17, are partly exact, and partly approximate. The probable laws of wave-resistance and of skin-resistance, in §§ 18 and 19, are partly conjectural, and require the aid of much additional experimental research to test and verify them, and to make them definite; but still they have already to a certain extent been verified by observations of the performance of ships. The whole body of results, whether certain or conjectural, are set forth in the hope that they may prove useful in deducing general principles from the data of experiment and observation, and in suggesting plans for further research.

SUMMARY OF THE CONTENTS.

CONTENTS OF SUPPLEMENT.

Supplement to a paper on the Mathematical Theory of Stream-lines. By WILLIAM JOHN MACQUORN RANKINE, *C.E., LL.D., F.R.SS. Lond. & Edin.*

Received January 8,—Read February 10, 1870.

I. *Addendum to* § 16.—*Points of no Disturbance of Pressure.* Mr. BERTHON'S *Log.*—The points in the surface of the disturbing solid, and elsewhere, at which there is no disturbance of pressure, are given by the equation

$$q^2=u^2+v^2+w^2=1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (a)$$

Such points can be found graphically for a given stream-line surface, by constructing a diagram such as fig. 5, and finding by trial the points for which AC=AB.

At the surface of a sphere it is easily shown that we have

$$q=\tfrac{3}{2}\sin\theta; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (b)$$

in which θ is the angle made by a radius of the sphere with the direction of motion. Hence, on the surface of a sphere, the points of no disturbance of pressure are contained in the circle given by the equation

$$\theta=\sin^{-1}\tfrac{2}{3}=41° \; 59' \text{ nearly.}$$

In February 1850 there was communicated to the Royal Society a paper by the Reverend E. L. BERTHON, describing an instrument invented by him, called a "hydrostatic log;" and a more detailed account of that invention was read by Mr. VAUGHAN PENDRED to the Society of Engineers on the 6th of December 1869. One part of that instrument consists of a vertical cylindrical tube, with a closed flat bottom, and having, in the front part of the cylindrical surface, near the bottom, a small hole, whose angular position, relatively to the direction in which the tube is moved through the water, is so adjusted that the pressure of the water outside produces no disturbance of the level of the column inside the tube. Mr. BERTHON ascertained solely by experiment the "zero-angle" or "neutral angle," as it has been called, and found it to be 41° 30′—a result with which the theoretical value for a sphere agrees almost exactly. That agreement shows that the disturbance in the water caused by the short vertical flat-bottomed cylinder employed by Mr. BERTHON was sensibly identical with that produced by a sphere, and also that, from the foremost points of the tube, as far round each way as the zero-angle, the disturbance of pressure was not sensibly affected by wave-motion, viscosity, or friction.

II. *Addendum to* § 17.—*Interference of Waves.* It was suggested to me by Mr. WILLIAM FROUDE, in a letter dated the 11th November, 1869, that one of the circumstances in the figure of a vessel on which the smallness of wave-resistance depends, is the interference of waves originating at different parts of the vessel's surface, so as wholly or

partially to neutralize each other. Suppose, for example, that at a certain point (which may be denoted by A), at or near the bow of the ship, there is a disturbance producing a wave-ridge. This first ridge is followed by a series of other wave-ridges, at distances apart depending on the period and virtual depth of the original disturbance, and diverging at an angle depending on the ratio borne by the speed of the waves to the speed of the ship.

Mr. FROUDE remarks, as an observed fact, that the *second* wave-ridge of the series is that which appears to carry away the most energy. This wave-ridge is in contact with the side of the vessel at a point which we may call B, at a distance astern of A depending on the dimensions and position of the waves. Suppose, now, that the surface of the vessel is so shaped that the disturbance impressed by it on the water has a tendency to produce a *wave-trough* at B; this disturbance will, to a certain extent, neutralize, by interference, the disturbance originating at A; that is, to use Mr. FROUDE'S words, the wave-troughs originating at B will "swallow" the second and following ridges of the series of waves originating at A, leaving unaltered the first wave-ridge only of that series —thus diminishing the quantity of energy which is carried away by diverging waves. It is obvious that the greatest effect of the interference of two given series of waves can be realized at one particular speed of the ship only, because of the influence of the speed of the ship on the positions of the waves; and this may account for the diminutions of the rate of increase of resistance with speed which occur at certain particular velocities of a given vessel. As regards this and many other questions of the resistance of vessels and of the motions which they impress on the water, a great advancement of knowledge is to be expected from the publication in detail of the results of the experiments on which Mr. FROUDE has long been engaged.

XI. *Records of the Magnetic Phenomena at the Kew Observatory.*—No. IV. *Analysis of the principal Disturbances shown by the Horizontal and Vertical Force Magnetometers of the Kew Observatory, from* 1859 *to* 1864. *By General Sir* Edward Sabine, *K.C.B., President.*

Received June 15,—Read June 15, 1871.

This paper contains an analysis of the first portion of the Automatic Records obtained at the Kew Observatory by means of the self-recording Horizontal and Vertical Force Magnetometers devised by the late Superintendent of that Observatory, Mr. John Welsh, aided by Mr. Beckley, Engineer in that establishment.

The Record of the Horizontal Force commenced on the 1st of January, 1858, and that of the Vertical Force a year later, viz. on the 1st of January, 1859. The present notice includes the records of both to the 31st of December, 1864,—making seven complete years of the Horizontal Force, and six complete years of the Vertical Force. The daily Photograms, from which the results have been derived, are carefully preserved at the Observatory, and hitherto, at least, appear to have suffered little or no deterioration. The apparatus and methods by which the photograms are obtained have been already fully described. The Observatory had the great misfortune of losing the Superintendence of Mr. Welsh by his decease in 1859; but the instruments and methods which he had so ably devised, have continued and still continue in use, unchanged. It was originally his intention to discuss the results obtained with the self-recording magnetic instruments in successive periods, each of six years, beginning with 1858; but in consequence of the illness which preceded his death, no preparations had been made for the commencement of this work. The photograms were indeed preserved with all suitable care, but were liable, as must necessarily be the case even under the most favourable conditions, to deterioration. Under these circumstances I ventured in 1862 to propose to Mr. Gassiot, Chairman of the Kew Committee (of which I was myself a member), that the photograms of the First of the Periods contemplated by Mr. Welsh, viz. from January 1, 1858, to December 31, 1864, should be entrusted to me, to be tabulated by the non-commissioned Officers at the Woolwich Magnetic Office, and subsequently discussed by myself on the same plan as the Eye-observations at the Colonial Magnetic Observatories had been discussed. This arrangement, so far as the tabulation was concerned, was speedily effected, and the Photograms were forthwith returned* for safe custody to Kew, where they remain and are in good preservation. My own time, and that of the Office, having been much occupied latterly in preparing the Maps of the Magnetic Elements corresponding to the Epoch of 1842.5, the deductions from the Photograms, from January 1858 to December 1864, have not been completed at so early a

date as I could have wished, and should otherwise have accomplished ; but I now present them as the first instalment of a research which I hope we may regard as permanently established and provided for by Mr. GASSIOT's munificent donation.

The method by which the tabulated hourly records of the Horizontal and Vertical Forces have been separated into their two categories of disturbed and (comparatively) undisturbed values, was described originally in the 3rd Volume of the Magnetical Observations at the Toronto Observatory, published in 1847, and has since been frequently employed and discussed in papers printed in the Philosophical Transactions and elsewhere. It is the same which was subsequently adopted by Dr. ALEXANDER DALLAS BACHE, For. Member of the Royal Society, in his admirable discussion of the Results of the Magnetical Observations made at the Girard College Observatory in Philadelphia, in the years 1840 to 1847 inclusive, published in the Smithsonian Contributions to Knowledge, vols. xii. and xiii. In the case of the hourly records of the horizontal Force at Kew between January 1, 1858, and December 31, 1864, the number of instances in which the records indicated an amount of disturbance equalling or exceeding ·150 of an inch, taken as the separating value, was 5932, being about 1 in 10; and in the case of the records of the Vertical Force between the 1st of January 1859 and the 31st of December 1864, 6957, being about 1 in 7 of the whole body. The value of ·150 of an inch in parts of the respective forces in the different years was as follows:—

	Horizontal Force.	Vertical Force.
1858	0·00124	———
1859	0·00141	0·000357
1860	0·00152	0·000377
1861	0·00157	0·000389
1862	0·00162	0·000381
1863	0·00168	0·000377
1864	0·00174	0·000399
Means . .	0·00154	0·000380

Distributed into the several years of their occurrence, these showed the proportions in each year to be as follows:

In the Horizontal Force.

Year ending December 31, 1858	267·893	inches.
,, ,, 1859	369·286	,,
,, ,, 1860	270·349	,,
,, ,, 1861	206·748	,,
,, ,, 1862	183·645	,,
,, ,, 1863	114·642	,,
,, ,, 1864	114·725	,,
Total in the 7 years . . .	1527·288	,,
Mean Annual Value $\frac{1527{\cdot}288}{7}$	=218·184	,,

In the Vertical Force.

Year ending December 31, 1859	. . . .	540·235 inches.
„ „ 1860	. . . .	364·208 „
„ „ 1861	. . . .	226·625 „
„ „ 1862	. . . .	358·756 „
„ „ 1863	. . . .	174·712 „
„ „ 1864	. . . .	238·597 „
Total in the 6 years	. . .	1903·133 „
Mean Annual Value $\frac{1903 \cdot 140}{6}$		=317·189 „

Hence it appears that of the years comprised in the Record, 1859 was manifestly the year of greatest disturbance both in the Horizontal and Vertical Forces; and 1863, or we might say 1863 and 1864, the year or years of least disturbance*.

I proceed to the details in the different years:—

Horizontal Force Disturbances, from 1st January 1858 to 31st December 1864.

The number of the Bifilar observations in which the amount of disturbance equalled or exceeded ·150 inch in the 7 years was 5932, being about 1 in 10 of the whole body of the observations taken from the traces (60491).

* The earliest indication of what is now so generally recognized as "the Disturbance Period of the Magnetic Variations" is derivable from Arago's Observations of the Declination in Paris, commenced in 1820 and terminated in 1830. [Editor's Note, pages 355–357, in the English translation of Metereological Essays by François Arago (Longman), 1855.] The Epoch of Minimum shown by those observations occurred in 1823–24, and the Epoch of Maximum in 1829. The annual increase was progressive and continuous from the minimum to the maximum; and the years preceding 1823 and following 1829 showed portions of a corresponding variation.

From the period of the establishment of the British Colonial Observatories in 1841, however, we may date the existence of a more full and systematic investigation of the phenomena of the Decennial Magnetic Variation, not limited to a single element, viz. the Declination, as in the case of M. Arago's observations, but including also the phenomena of the Horizontal and Vertical Forces, observed or recorded by suitable methods. From the concurrent testimony obtained from localities widely separated from each other, such as Toronto in Canada, Hobarton in Tasmania, Munich in Germany, and Girard College in the United States of America, we have learnt that 1843–44 was an Epoch of Minimum, and 1848–49 an Epoch of Maximum Disturbance in each of the three Magnetic Elements. The further evidence contained in this paper, derived from the Automatic Records of the Horizontal and Vertical Forces at the Kew Observatory, shows that 1858–59 was also an Epoch of Maximum, and 1863–64 an Epoch of Minimum.

The facts thus brought together exhibit an accordance (too close to be regarded as accidental) with the Decennial Variation in the phenomena of the Solar Spots observed by Hofrath Schwabe, Hon. F.R.S., as announced by himself in the following words:—"The numbers in the accompanying Table leave no doubt that from 1826 to 1850 the occurrence of spots has been so far characterized by periods of ten years, that its maxima have fallen in 1828, 1837, 1848, and its minima in 1833 and 1843." [Humboldt's Cosmos (Longman), vol. iii. pages 291 and 292.]

Table I.—Aggregate Values of the disturbed observations.

Year ending December 31st, 1858	. . . .	267·893 inches.
" " 1859	. . . .	369·286 "
" " 1860	. . . .	270·349 "
" " 1861	. . . .	206·748 "
" " 1862	. . . .	183·645 "
" " 1863	. . . .	114·642 "
" " 1864	. . . .	114·725 "
Total in the 7 years	. . .	1527·288 "
Mean Annual Value	$\frac{1527\cdot288}{7}$	=218·184 "

Table II.—Ratios in each year to the Mean Annual Value.

Year ending December 31st, 1858	. . . .	1·23
" " 1859	. . . .	1·69
" " 1860	. . . .	1·24
" " 1861	. . . .	0·95
" " 1862	. . . .	0·84
" " 1863	. . . .	0·53
" " 1864	. . . .	0·53

Table III.—Aggregate Values in the different years divided into disturbances increasing the Force, and disturbances decreasing the Force.

	Increasing.	Decreasing.
Year ending December 31st, 1858	74·949	192·944 inches.
" " 1859	85·256	284·030 "
" " 1860	86·114	184·235 "
" " 1861	27·221	179·527 "
" " 1862	46·342	137·303 "
" " 1863	21·192	93·450 "
" " 1864	20·432	94·293 "
Total in the 7 years	361·506	1165·782 "

The ratio of the value of the disturbances decreasing the Force to those which increased it was, on the average of the 7 years, nearly as 3·23 to 1.

TABLE IV.—Aggregate Values of the disturbed observations distributed into the several *months* of their occurrence, with the ratios which the values in the preceding column (or the sums in the 7 years) bear to the mean monthly value, or average of all the months.

Months.	Years ending December 31.							Sums in the 7 years.	Ratios.	Months.
	1858.	1859.	1860.	1861.	1862.	1863.	1864.			
	in.	in.	in.	in.	in.	in.	in.	in.		
Jan.	3·557	15·969	8·204	42·588	12·846	17·974	1·071	102·209	0·80	Jan.
Feb.	24·035	25·711	16·041	28·481	8·217	11·089	5·256	118·830	0·93	Feb.
Mar.	44·388	10·497	38·120	18·924	7·195	9·491	8·643	137·258	1·08	Mar.
April	43·457	39·991	26·714	11·169	5·330	10·599	7·420	144·680	1·14	April
May	32·268	14·661	16·990	7·610	6·146	5·511	10·646	93·832	0·74	May
June	18·750	17·897	12·773	7·080	5·458	5·473	21·756	89·187	0·70	June
July	17·574	29·487	39·631	17·260	14·758	11·538	11·901	142·149	1·12	July
Aug.	7·569	45·249	62·801	12·501	37·801	12·265	15·440	193·626	1·52	Aug.
Sept.	12·472	61·234	16·212	8·772	13·781	10·823	9·438	132·732	1·04	Sept.
Oct.	26·163	40·283	13·188	20·996	35·139	8·395	10·029	154·193	1·21	Oct.
Nov.	9·050	31·288	5·223	13·239	8·395	9·364	7·949	84·508	0·66	Nov.
Dec.	28·610	37·019	14·452	18·128	28·579	2·120	5·176	134·084	1·05	Dec.
Total in the 7 years								1527·288		
Mean monthly value $\frac{1527·288}{12}=$								127·274=1·00		

Tables V. and VI. exhibit the aggregate monthly values in the different years separated into disturbances increasing the Force and disturbances decreasing the Force.

TABLE V.—Disturbances increasing the Force.

Months.	Years ending December 31.							Sums in the 7 years.	Ratios.	Months.
	1858.	1859.	1860.	1861.	1862.	1863.	1864.			
	in.	in.	in.	in.	in.	in.	in.	in.		
Jan.	1·345	1·675	1·655	1·566	0·818	1·115	0·158	8·332	0·28	Jan.
Feb.	5·731	1·594	0·625	2·222	0·287	0·942	0·355	11·756	0·39	Feb.
Mar.	10·289	1·916	4·637	2·690	0·340	0·743	1·205	21·820	0·72	Mar.
April	12·252	5·593	7·812	2·625	0·824	4·148	1·691	34·945	1·16	April
May	13·791	7·276	11·926	4·711	3·229	2·441	3·437	46·811	1·55	May
June	10·861	6·379	4·447	3·158	4·563	2·575	3·596	35·579	1·18	June
July	9·368	6·879	14·452	1·029	4·111	2·578	3·584	42·001	1·39	July
Aug.	1·929	4·862	33·578	3·809	14·774	2·667	1·947	63·566	2·11	Aug.
Sept.	3·175	13·568	4·281	2·035	8·273	2·317	1·106	34·755	1·15	Sept.
Oct.	2·793	7·885	1·253	1·561	4·321	0·493	1·252	19·558	0·65	Oct.
Nov.	1·792	10·238	0·301	0·300	1·816	0·654	1·162	16·263	0·54	Nov.
Dec.	1·623	17·391	1·147	1·515	2·986	0·519	0·939	26·120	0·87	Dec.
Total in the 7 years								361·506		
Mean monthly value $\frac{361·506}{12}=$								30·125=1·00		

TABLE VI.—Disturbances decreasing the Force.

Months.	Years ending December 31.							Sums in the 7 years.	Ratios.	Months.
	1858.	1859.	1860.	1861.	1862.	1863.	1864.			
	in.	in.	in.	in.	in.	in.	in.	in.		
Jan.	2·212	14·294	6·549	41·022	12·028	16·859	0·913	93·877	0·97	Jan.
Feb.	18·304	24·117	15·416	26·259	7·930	10·147	4·901	107·074	1·10	Feb.
Mar.	34·099	8·581	33·483	16·234	6·855	8·748	7·438	115·438	1·19	Mar.
April	31·205	34·398	18·902	8·544	4·506	6·451	5·729	109·735	1·13	April
May	18·477	7·385	5·064	2·899	2·917	3·070	7·209	47·021	0·48	May
June	7·889	11·518	8·326	3·922	0·895	2·898	18·160	53·608	0·55	June
July	8·206	22·608	25·179	16·231	10·647	8·960	8·317	100·148	1·03	July
Aug.	5·640	40·387	29·223	8·692	23·027	9·598	13·493	130·060	1·34	Aug.
Sept.	9·297	47·666	11·931	6·737	5·508	8·506	8·332	97·977	1·01	Sept.
Oct.	23·370	32·398	11·935	19·435	30·818	7·902	8·777	134·635	1·39	Oct.
Nov.	7·258	21·050	4·922	12·939	6·579	8·710	6·787	68·245	0·70	Nov.
Dec.	26·987	19·628	13·305	16·613	25·593	1·601	4·237	107·964	1·11	Dec.
Total in the 7 years								1165·782		
Mean monthly value $\frac{1165·782}{12}=$								97·149=1·00		

TABLE VII.—Aggregate Values of the disturbed observations distributed into the several *hours* of their occurrence, together with the ratios of the values at the different hours to the mean hourly value or average of all the hours.

Kew Astronomical hours.	Years ending December 31.							Sums in the 7 years.	Ratios.	Kew Civil Time.
	1858.	1859.	1860.	1861.	1862.	1863.	1864.			
	in.	in.	in.	in.	in.	in.	in.	in.		
18	9·766	14·521	9·441	6·209	5·754	3·618	3·800	53·109	0·83	6 A.M.
19	10·897	15·639	12·484	6·257	7·291	4·126	5·933	62·627	0·98	7 „
20	11·556	14·883	12·476	7·499	8·536	5·144	6·566	66·660	1·05	8 „
21	12·396	14·950	14·853	8·239	8·078	6·502	8·494	73·512	1·16	9 „
22	14·117	16·340	14·447	9·459	8·377	7·182	6·781	76·703	1·21	10 „
23	12·962	16·527	14·802	8·632	7·691	5·548	5·824	71·986	1·13	11 „
0	11·512	18·032	12·799	7·004	6·718	4·464	3·923	64·452	1·01	Noon.
1	11·207	11·284	8·370	7·099	5·418	5·296	4·727	53·401	0·84	1 P.M.
2	10·679	12·367	9·701	7·687	6·188	5·031	4·382	56·035	0·88	2 „
3	10·533	16·711	15·310	8·944	5·702	3·708	3·530	64·438	1·01	3 „
4	10·951	15·307	13·413	7·954	6·611	4·546	4·924	63·706	1·00	4 „
5	10·630	18·109	15·480	7·409	7·210	4·808	4·190	67·836	1·07	5 „
6	12·219	13·161	11·239	9·311	8·644	5·131	4·540	64·245	1·01	6 „
7	12·141	13·080	8·746	8·266	9·501	3·074	4·405	59·213	0·93	7 „
8	10·504	15·408	8·527	11·210	10·296	4·596	3·759	64·300	1·01	8 „
9	10·700	13·984	9·911	12·340	9·277	5·228	5·255	66·695	1·05	9 „
10	12·210	15·411	10·805	11·059	10·088	7·063	4·823	71·459	1·12	10 „
11	14·210	17·396	10·082	13·423	9·024	5·818	6·594	76·547	1·20	11 „
12	12·960	17·552	11·286	10·882	9·471	5·073	6·683	73·907	1·16	Midnight.
13	9·551	18·324	10·158	10·536	8·151	4·870	5·008	66·598	1·05	1 A.M.
14	9·312	16·828	9·211	7·942	7·208	3·494	3·022	57·017	0·90	2 „
15	10·663	13·844	9·392	6·794	6·479	2·870	2·263	52·305	0·82	3 „
16	9·129	14·315	9·961	5·877	5·027	3·279	2·732	50·320	0·79	4 „
17	7·088	15·313	7·455	6·716	6·905	4·173	2·567	50·217	0·79	5 „
Total in the 7 years								1527·288		
Mean hourly value $\frac{1527·288}{24}=$								63·637=1·00		

Tables VIII. and IX. exhibit the aggregate values at the different hours separated into disturbances increasing the Force and disturbances decreasing the Force, and also the ratios of the values at each hour of both kinds of disturbance to their respective mean hourly values.

TABLE VIII.—Disturbances increasing the Force.

Kew Astronomical hours.	Years ending December 31.							Sums in the 7 years.	Ratios.	Kew Civil Time.
	1858.	1859.	1860.	1861.	1862.	1863.	1864.			
	in.	in.	in.	in.	in.	in.	in.	in.		
18	1·654	1·537	2·476	0·691	1·262	0·645	0·392	8·657	0·57	6 A.M.
19	1·452	1·644	3·358	0·540	1·059	0·562	0·369	8·984	0·60	7 „
20	1·660	2·091	2·663	0·538	1·216	0·519	0·473	9·160	0·61	8 „
21	1·559	2·550	2·627	1·303	1·955	0·154	0·163	10·311	0·68	9 „
22	2·660	2·542	2·912	1·198	1·491	0·821	0·506	12·130	0·81	10 „
23	3·634	3·140	2·971	1·614	1·002	0·690	0·680	13·731	0·91	11 „
0	3·127	3·111	2·973	1·967	1·497	0·156	0·362	13·193	0·88	Noon.
1	3·920	4·631	2·781	1·794	1·961	0·667	0·513	16·267	1·08	1 P.M.
2	3·020	6·038	5·121	0·877	2·425	0·869	0·853	19·203	1·27	2 „
3	4·488	9·391	10·419	1·719	2·576	1·052	0·748	30·393	2·02	3 „
4	5·383	8·398	8·438	2·792	2·180	1·829	1·540	30·560	2·03	4 „
5	7·524	11·107	10·866	1·940	3·060	1·520	1·461	37·478	2·49	5 „
6	6·307	4·764	6·232	2·903	2·518	1·895	1·243	25·862	1·72	6 „
7	4·433	4·139	3·183	0·951	1·371	0·709	1·815	16·601	1·10	7 „
8	1·945	2·538	2·197	0·742	3·318	1·032	1·380	13·152	0·87	8 „
9	2·044	1·787	1·217	1·333	3·004	1·537	1·405	12·327	0·82	9 „
10	1·633	1·771	1·305	0·565	2·008	1·560	1·397	10·239	0·68	10 „
11	3·818	2·571	2·335	0·929	1·611	0·747	1·480	13·491	0·90	11 „
12	2·366	2·088	1·885	0·514	2·370	0·596	1·229	11·048	0·73	Midnight.
13	1·981	1·535	2·645	0·320	1·703	0·335	0·836	9·355	0·62	1 A.M.
14	2·991	2·224	2·296	0·000	1·357	0·712	0·458	10·038	0·67	2 „
15	3·070	1·111	2·006	0·629	1·879	1·132	0·599	10·426	0·69	3 „
16	2·221	1·862	1·566	0·746	1·754	0·594	0·380	9·123	0·61	4 „
17	2·059	2·686	1·642	0·616	1·765	0·859	0·150	9·777	0·65	5 „
Total in the 7 years								361·506		
Mean hourly value $\frac{361·506}{24}=$								15·063=1·00		

Table IX.—Disturbances decreasing the Force.

Kew Astronomical hours.	Years ending December 31. 1858.	1859.	1860.	1861.	1862.	1863.	1864.	Sums in the 7 years.	Ratios.	Kew Civil Time.
	in.	in.	in.	in.	in.	in.	in.	in.		
18	8·112	12·984	6·965	5·518	4·492	2·973	3·408	44·452	0·92	6 A.M.
19	9·445	13·995	9·126	5·717	6·232	3·564	5·564	53·643	1·10	7 ,,
20	9·896	12·792	9·813	6·961	7·320	4·625	6·093	57·500	1·18	8 ,,
21	10·837	12·400	12·226	6·936	6·123	6·348	8·331	63·201	1·30	9 ,,
22	11·457	13·798	11·535	8·261	6·886	6·361	6·275	64·573	1·33	10 ,,
23	9·328	13·387	11·831	7·018	6·689	4·858	5·144	58·255	1·20	11 ,,
0	8·385	14·921	9·826	5·037	5·221	4·308	3·561	51·259	1·06	Noon.
1	7·287	6·653	5·589	5·305	3·457	4·629	4·214	37·134	0·76	1 P.M.
2	7·659	6·329	4·580	6·810	3·763	4·162	3·529	36·832	0·76	2 ,,
3	6·045	7·320	4·891	7·225	3·126	2·656	2·782	34·045	0·70	3 ,,
4	5·568	6·909	4·975	5·162	4·431	2·717	3·384	33·146	0·68	4 ,,
5	3·106	7·002	4·614	5·469	4·150	3·288	2·729	30·358	0·62	5 ,,
6	5·912	8·397	5·007	6·408	6·126	3·236	3·297	38·383	0·79	6 ,,
7	7·708	8·941	5·563	7·315	8·130	2·365	2·590	42·612	0·88	7 ,,
8	8·559	12·870	6·330	10·468	6·978	3·564	2·379	51·148	1·05	8 ,,
9	8·656	12·197	8·694	11·007	6·273	3·691	3·850	54·368	1·12	9 ,,
10	10·577	13·640	9·500	10·494	8·080	5·503	3·426	61·220	1·26	10 ,,
11	10·392	14·825	7·747	12·494	7·413	5·071	5·114	63·056	1·30	11 ,,
12	10·594	15·464	9·401	10·368	7·101	4·477	5·454	62·859	1·29	Midnight.
13	7·570	16·789	7·513	10·216	6·448	4·535	4·172	57·243	1·18	1 A.M.
14	6·321	14·604	6·915	7·942	5·851	2·782	2·564	46·979	0·97	2 ,,
15	7·593	12·733	7·386	6·165	4·600	1·738	1·664	41·879	0·86	3 ,,
16	6·908	12·453	8·395	5·131	3·273	2·685	2·352	41·197	0·85	4 ,,
17	5·029	12·627	5·813	6·100	5·140	3·314	2·417	40·440	0·83	5 ,,
Total in the 7 years								1165·782		
Mean hourly value $\frac{1165·782}{24}=$								48·574=1·00		

Vertical Force Disturbances, from 1st January 1859 to 31st December 1864.

The number of the Vertical Force Observations in which the amount of disturbance equalled or exceeded ·150 inch in the 6 years was 6957, being about 1 in 7 of the whole number of observations taken from the traces (51,843.)

Table I.—Aggregate Values of the disturbed observations.

Year ending 31st December 1859	540·235	inches.
,, ,, 1860	364·208	,,
,, ,, 1861	226·625	,,
,, ,, 1862 . . . ,	358·763	,,
,, ,, 1863	174·712	,,
,, ,, 1864	238·597	,,
Total in the 6 years . . .	1903·140	,,
Mean Annual Value . . $\frac{1903·140}{6}$ =	317·190	,,

TABLE II.—Ratios in each year to the mean annual value.

Year ending 31st December	1859 . . .	1·70
„ „	1860 . . .	1·15
„ „	1861 . . .	0·71
„ „	1862 . . .	1·13
„ „	1863 . . .	0·55
„ „	1864 . . .	0·75

TABLE III.—Aggregate Values in the different years divided into disturbances increasing and disturbances decreasing the Force.

	Increasing.	Decreasing.
Year ending 31st December 1859	402·525	137·710 inches.
1860	250·629	113·579 „
1861	147·988	78·637 „
1862	248·391	110·372 „
1863	89·371	85·341 „
1864	107·248	131·349 „
Total in the 6 years . . .	1246·152	656·988 „

The ratio of the values of the disturbances increasing the Force to those which decreased it was, on the average of the 6 years, as 1·9 to 1.

TABLE IV.—Aggregate Values of the disturbed observations distributed into the several months of their occurrence, with the ratios which the values in the preceding column bear to the mean monthly value or average of all the months.

Months.	Years ending December 31.						Sums in the 6 years.	Ratios.	Months.
	1859.	1860.	1861.	1862.	1863.	1864.			
	in.	in.	in	in.	in.	in.	in.		
Jan.	5·769	6·174	50·091	57·532	23·186	4·885	147·637	0·93	Jan.
Feb.	30·335	35·492	13·270	29·385	13·487	37·106	159·075	1·00	Feb.
Mar.	15·892	39·871	24·130	60·527	6·929	19·932	167·281	1·05	Mar.
April	53·598	36·032	12·550	21·844	11·326	21·197	156·547	0·99	April
May	12·813	27·417	7·636	12·023	4·812	17·412	82·113	0·52	May
June	33·154	16·388	23·825	1·236	8·537	19·621	102·761	0·65	June
July	91·814	32·996	3·176	22·071	17·327	27·676	195·060	1·23	July
Aug.	26·156	96·549	20·396	63·751	14·793	28·591	250·236	1·58	Aug.
Sept.	94·594	36·167	13·045	25·751	33·503	15·814	218·874	1·38	Sept.
Oct.	62·074	9·739	17·688	19·381	21·074	27·285	157·241	0·99	Oct.
Nov.	24·404	2·802	13·060	10·697	15·777	5·164	71·904	0·45	Nov.
Dec.	89·632	24·581	27·758	34·565	3·961	13·914	194·411	1·23	Dec.
Total in the 6 years..............							1903·140		
Mean monthly value $\frac{1903·140}{12}=$							158·595=1·00		

Tables V. and VI. exhibit the aggregate monthly values in the different years separated into disturbances increasing and disturbances decreasing the Force.

Table V.—Disturbances increasing the Force.

Months.	Years ending December 31.						Sums in the 6 years.	Ratios.	Months.
	1859.	1860.	1861.	1862.	1863.	1864.			
	in.	in.	in.	in.	in.	in.	in.		
Jan.	5·175	2·807	44·789	53·229	15·693	·618	122·311	1·18	Jan.
Feb.	28·749	33·134	8·324	9·137	7·430	17·725	104·499	1·01	Feb.
Mar.	8·625	21·598	15·464	59·548	4·827	11·995	122·057	1·17	Mar.
April	28·750	23·253	8·600	14·682	4·999	14·380	94·664	0·91	April
May	10·314	24·516	4·028	5·142	·787	4·099	48·886	0·47	May
June	26·253	11·052	20·166	·536	3·074	10·688	71·769	0·69	June
July	91·627	13·865	1·609	11·162	5·474	10·675	134·412	1·29	July
Aug.	13·208	79·502	8·510	34·432	6·875	6·516	149·043	1·44	Aug.
Sept.	49·395	19·488	5·190	16·810	20·233	5·359	116·475	1·12	Sept.
Oct.	44·824	7·236	8·178	9·791	11·404	10·600	92·033	0·89	Oct.
Nov.	15·536	2·132	9·561	8·340	6·584	4·334	46·487	0·45	Nov.
Dec.	80·069	12·046	13·569	25·582	1·991	10·259	143·516	1·38	Dec.
Total in the 6 years..............							1246·152		
Mean monthly value $\frac{1246 \cdot 152}{12}=$							103·846=1·00		

Table VI.—Disturbances decreasing the Force.

Months.	Years ending December 31.						Sums in the 6 years.	Ratios.	Months.
	1859.	1860.	1861.	1862.	1863.	1864.			
	in.	in.	in.	in.	in.	in.	in.		
Jan.	·594	3·367	5·302	4·303	7·493	4·267	25·326	0·46	Jan.
Feb.	1·586	2·358	4·946	20·248	6·057	19·381	54·576	1·00	Feb.
Mar.	7·267	18·273	8·666	·979	2·102	7·937	45·224	0·83	Mar.
April	24·848	12·779	3·950	7·162	6·327	6·817	61·883	1·13	April
May	2·499	2·901	3·608	6·881	4·025	13·313	33·227	0·61	May
June	6·901	5·336	3·659	·700	5·463	8·933	30·992	0·57	June
July	·187	19·131	1·567	10·909	11·853	17·001	60·648	1·11	July
Aug.	12·948	17·047	11·886	29·319	7·918	22·075	101·193	1·85	Aug.
Sept.	45·199	16·679	7·855	8·941	13·270	10·455	102·399	1·87	Sept.
Oct.	17·250	2·503	9·510	9·590	9·670	16·685	65·208	1·19	Oct.
Nov.	8·868	·670	3·499	2·357	9·193	·830	25·417	0·46	Nov.
Dec.	9·563	12·535	14·189	8·983	1·970	3·655	50·895	0·93	Dec.
Total in the 6 years..............							656·988		
Mean monthly value $\frac{656 \cdot 988}{12}=$							54·749=1·00		

TABLE VII.—Ratios of the Values of the Disturbances increasing the Vertical Force in the different *Months* to the Values of those which decrease it.

Months.	Ratios.
January	4·83
February	1·91
March	2·70
April	1·53
May	1·44
June	2·32
July	2·22
August	1·47
September	1·14
October	1·41
November	1·83
December	2·82

TABLE VIII.—Aggregate Values of the disturbed observations distributed into the several *hours* of their occurrence, together with the ratios of the values at the different hours to the mean hourly value, or average of all the hours.

Kew Astronomical Time	Years ending December 31.						Sums in the 6 years.	Ratios.	Kew Civil Time
	1859.	1860.	1861	1862	1863.	1864.			
h	in	in	in	in	in	in	in.		h
18	14·551	11·978	6·583	11·094	5·876	8·070	58·152	0·73	6 A.M.
19	16·009	8·444	5·071	9·645	4·320	7·515	51·004	0·64	7 „
20	13 101	8·325	3·499	9·714	2·311	6·235	43·185	0·54	8 „
21	14·957	8·977	3·975	10·247	2·563	5·016	45·735	0·58	9 „
22	15·045	10·278	5·175	9·981	2·249	3·970	46·698	0·59	10 „
23	16·800	7·892	6·645	11·849	2·925	4·322	50·433	0·64	11 „
0	19·440	9·785	6·940	11·117	2·487	4·713	54·482	0·69	Noon.
1	20·729	13·009	7·933	14·418	6·098	6·946	69·133	0·87	1 P.M.
2	31·517	19·449	11·005	16·148	8·546	9·820	96·485	1·22	2 „
3	32·969	25·802	14·167	20·785	12·787	12·538	119·048	1·50	3 „
4	36·217	32·044	15·142	24·823	13·913	15·888	138·027	1·74	4 „
5	45·483	31·387	16·138	25·744	14·056	17·958	150·766	1·90	5 „
6	34·717	28·297	18·006	24·216	11·704	15·393	132·333	1·68	6 „
7	31·013	22·343	16·719	17·181	9·966	11·246	108·468	1·37	7 „
8	23·175	15·562	13·302	18·555	7·017	7·319	84·930	1·07	8 „
9	18·584	9·533	10·972	13·430	4·150	4·788	61·457	0·77	9 „
10	15·579	6·106	5·173	10·491	5·493	6·773	49·615	0·63	10 „
11	17·209	7·469	6·915	13·952	6·137	11·551	63·233	0·80	11 „
12	20·943	13·942	7·986	12·890	6·758	14·557	77·076	0·97	Midnight.
13	23·557	12·339	7·568	14·570	7·970	16·748	82·752	1·04	1 A.M.
14	21·794	14·275	11·700	16·408	10·642	13·797	88·616	1·12	2 „
15	20·058	17·943	10·057	15·135	10·149	13·039	86·381	1·09	3 „
16	19·309	13·910	8·985	14·030	8·385	11·136	75·755	0·95	4 „
17	17·479	15·119	6·969	12·340	8·210	9·259	69·376	0·87	5 „
Total in the 6 years							1903·140		
Mean hourly value $\frac{1903\cdot140}{24}$ =							79·297=1·00		

Tables IX. and X. exhibit the aggregate values at the different hours, separated into

disturbances increasing and disturbances decreasing the Force, and the ratios of the values at each hour of both kinds of disturbances to their respective mean hourly values.

TABLE IX.—Disturbances increasing the Force.

Kew Astronomical Time.	Years ending December 31.						Sums in the 6 years.	Ratios.	Kew Civil Time.
	1859.	1860.	1861.	1862.	1863.	1864.			
h	in.	in.	in.	in.	in.	in.	in.		h
18	8·670	3·683	1·708	5·992	·374	1·184	21·611	0·42	6 A.M.
19	9·680	2·617	1·850	5·421	·543	1·534	21·645	0·42	7 ,,
20	9·202	3·726	2·634	5·547	·594	1·228	22·931	0·44	8 ,,
21	11·715	4·567	2·828	7·568	·960	1·416	29·054	0·56	9 ,,
22	11·269	6·148	3·631	7·316	·771	1·716	30·851	0·59	10 ,,
23	11·539	5·408	4·907	9·840	·757	2·387	34·838	0·67	11 ,,
0	15·708	6·920	5·224	7·403	1·178	2·590	39·023	0·75	Noon.
1	17·100	10·726	6·140	11·143	4·778	4·321	54·208	1·04	1 P.M.
2	28·780	18·341	9·796	14·091	7·407	7·585	86·000	1·66	2 ,,
3	29·778	24·962	12·914	19·214	11·859	9·772	108·499	2·09	3 ,,
4	33·912	31·484	13·970	23·517	12·190	13·079	128·152	2·47	4 ,,
5	43·464	31·063	15·105	24·939	12·481	14·910	141·962	2·73	5 ,,
6	31·059	27·547	16·974	23·750	10·019	12·529	121·878	2·35	6 ,,
7	28·088	21·830	15·060	16·523	8·724	9·428	99·653	1·92	7 ,,
8	20·543	14·901	12·255	16·943	5·554	5·613	75·809	1·46	8 ,,
9	16·163	8·940	8·483	10·998	2·780	2·649	50·013	0·96	9 ,,
10	11·937	3·914	3·225	7·078	2·676	2·100	30·930	0·59	10 ,,
11	10·265	3·116	1·778	5·528	2·114	2·374	25·175	0·49	11 ,,
12	9·239	2·551	1·458	4·431	·860	1·772	20·311	0·39	Midnight.
13	9·870	2·814	1·162	4·110	·651	2·091	20·698	0·40	1 A.M.
14	9·013	3·634	1·597	4·437	·407	1·351	20·439	0·39	2 ,,
15	9·537	4·620	1·684	3·830	·575	2·029	22·275	0·43	3 ,,
16	8·049	2·961	1·906	4·037	·627	2·042	19·622	0·38	4 ,,
17	7·945	4·156	1·699	4·735	·492	1·548	20·575	0·40	5 ,,
				Total in the 6 years			1246·152		
				Mean hourly value $\frac{1246·152}{24}=$			51·923=1·00		

TABLE X.—Disturbances decreasing the Force.

Kew Astronomical Time.	Years ending December 31.						Sums in the 6 years.	Ratios.	Kew Civil Time.
	1859.	1860.	1861.	1862.	1863.	1864.			
h	in.	in.	in.	in.	in.	in.	in.		h
18	5·881	8·295	4·875	5·102	5·502	6·886	36·541	1·33	6 A.M.
19	6·329	5·827	3·221	4·224	3·777	5·981	29·359	1·07	7 "
20	3·899	4·599	·865	4·167	1·717	5·007	20·254	0·74	8 "
21	3·242	4·410	1·147	2·679	1·603	3·600	16·681	0·61	9 "
22	3·776	4·130	1·544	2·665	1·478	2·254	15·847	0·58	10 "
23	5·261	2·484	1·738	2·009	2·168	1·935	15·595	0·57	11 "
0	3·732	2·865	1·716	3·714	1·309	2·123	15·459	0·57	Noon.
1	3·629	2·283	1·793	3·275	1·320	2·625	14·925	0·55	1 P.M.
2	2·737	1·108	1·209	2·057	1·139	2·235	10·485	0·38	2 "
3	3·191	·840	1·253	1·571	·928	2·766	10·549	0·38	3 "
4	2·305	·560	1·172	1·306	1·723	2·809	9·875	0·36	4 "
5	2·019	·324	1·033	·805	1·575	3·048	8·804	0·32	5 "
6	3·658	·750	1·032	·466	1·685	2·864	10·455	0·38	6 "
7	2·925	·513	1·659	·658	1·242	1·818	8·815	0·32	7 "
8	2·632	·661	1·047	1·612	1·463	1·706	9·121	0·33	8 "
9	2·421	·593	2·489	2·432	1·370	2·139	11·444	0·42	9 "
10	3·642	2·192	1·948	3·413	2·817	4·673	18·685	0·68	10 "
11	6·944	4·353	5·137	8·424	4·023	9·177	38·058	1·39	11 "
12	11·704	11·391	6·528	8·459	5·898	12·785	56·765	2·07	Midnight.
13	13·687	9·525	6·406	10·460	7·319	14·657	62·054	2·27	1 A.M.
14	12·781	10·641	10·103	11·971	10·235	12·446	68·177	2·49	2 "
15	10·521	13·323	8·373	11·305	9·574	11·010	64·106	2·34	3 "
16	11·260	10·949	7·079	9·993	7·758	9·094	56·133	2·05	4 "
17	9·534	10·963	5·270	7·605	7·718	7·711	48·801	1·78	5 "
				Total in the 6 years			656·988		
				Mean hourly value $\frac{656·988}{24}=$			27·374=1·00		

XII. *On Fluoride of Silver.*—Part II. *By* George Gore, *F.R.S.*

Received September 22, 1870,—Read January 12, 1871.

On account of the prospect of being able to isolate fluorine by means of the action of chlorine, bromine, or iodine on argentic fluoride, I have in this investigation very fully examined the behaviour of those substances with that compound.

Behaviour with Chlorine (continued).—To ascertain more completely the nature of the compound formed by fluoride of silver with chlorine and platinum at a red heat, I passed a very slow current of pure chlorine over 24·62 grains of argentic fluoride in a platinum boat within a 20-inch platinum tube during nine hours at a very low red heat. A very minute amount of a vapour which corroded glass was continually evolved; and traces of a sublimate occurred, consisting chiefly of argentic chloride. On heating the boat and tube to redness after the experiment, a boiling sound was heard, a little fume and much chlorine escaped, and a loss of weight of 11·70 grains took place*. On weighing the cooled apparatus a gain of 6·49 grains upon the original weight still remained, theory requiring 6·88 grains if the whole of the fluoride took up its equivalent of chlorine without any fluorine being expelled. The thick platinum boat was powerfully corroded, and immoveably fixed to the tube, even at a red heat†. The residuary salt was easily fusible, tasteless, of a dark red colour, and contained much combined platinum; 3·38 grains of it fused with alkaline carbonates yielded 2·56 grains of metal (theory requiring 2·51 grains), of which by analysis 1·82 grain was found to be silver, and ·74 grain was platinum: ·63 grain of chlorine was also found in it, showing that the chlorine in it was chemically equivalent in amount to the silver. The deficiency of ·19 grain required to make up the quantity taken I have set down as fluorine. The red salt corroded glass in the presence of damp air, and contained only a few very minute particles of platinum.

	Found.	Calculated.
Silver	1·82	1·723
Chlorine	·63	·566
Platinum	·74	·786
Fluorine	·19	·303
Total	3·38	3·378

The above calculation is made upon the assumption that the composition of the residue is in accordance with the equation $4AgF+4Cl+Pt=4AgCl, PtF_4$. The deficiency of

* Melted chloride of silver absorbs much chlorine gas during the act of cooling; the resulting compound may perhaps be used as a means of getting liquid chlorine.

† Platinum articles are very liable to adhere together in chlorine at a red heat, occasioning much injury of apparatus.

fluorine is accounted for by the continuous minute escape of that body during the heating process. Some of the excess of silver found is explained by the presence of a rather large amount of free silver in the original fluoride. The platinum boat and short tube had lost 24·94 grains by corrosion, and the outer tube had gained 15·33 grains of platinum by vapour of a salt of that metal having been transferred to it and decomposed by the heat; ·80 grain of loose platinum was also found; the amount of platinum therefore chemically combined in the red salt was about 8·81 grains (the analysis gave 9·046 grains), theory requiring 9·55 grains. The following explanation closely agrees with the results obtained. The 24·62 grains of fluoride of silver (including a little free silver, and containing about 3·683 grains of fluorine) gained about 6·882 grains of chlorine, forming therewith about 27·818 grains of argentic chloride. The 3·683 grains of fluorine united with 9·551 grains of platinum to form tetrafluoride, which united with the argentic chloride to form 41·052 grains of a double salt, which only very slowly evolved a small portion of its fluorine in a current of chlorine at a red heat*. The results of this experiment agree with those obtained with boats of platinum and gold in chlorine, in the retort-and-receiver apparatus (see Phil. Trans. Roy. Soc. 1870, p. 240).

I heated some boats of cryolite and of fluor-spar very carefully to redness; they all became full of minute cracks, and melted fluoride of silver passed freely through them. By melting argentic fluoride in boats of previously ignited alabaster it was completely decomposed.

More than forty mixtures of the fluorides of glucinum, cerium, magnesium, calcium, strontium, lithium, and sodium, also cryolite and colourless fluor-spar, were made into boats by pouring them in a melted white-hot state into platinum boats immersed in red-hot gypsum, and either immersing a smaller platinum boat in the liquid and removing it after the mixture had solidified, to form the hollow part, or grinding out the hollow by the methods employed by lapidaries. Many of the cooled products formed beautiful enamel-looking substances, and might probably be used to form vessels for containing fluorides and for other technical purposes. The best mixtures were, 1st, the fluorides of calcium and magnesium in the proportion of their equivalent weights; 2nd, 300 parts of fluoride of calcium, 6 of fluoride of magnesium, and 3 of fluoride of lithium. The mixtures fused at a strong red heat to a clear liquid like water. Some of the mixtures of the fluorides of magnesium and lithium, and of the fluoride of magnesium with cryolite or fluoride of strontium, yielded crystals (probably double salts) on solidifying. On melting argentic fluoride in any of these boats, it passed over their edges by capillary action. Boats were also cut out of pieces of caustic lime, and heated repeatedly to redness in a current of anhydrous hydrofluoric acid, but they did not become wholly converted into calcic fluoride.

More than fifty boats were also made by moulding various fluorides in a state of wet paste and baking them: the fluorides tried were those of bismuth, copper, nickel, cobalt, lead, cadmium, zinc, manganese, uranium, chromium, cerium, magnesium, calcium,

* The double salt absorbed about 11·70 grains of chlorine. See p. 321.

strontium, barium, lithium, and sodium; but, from various reasons, none of them could be used for the desired purpose.

To ascertain more definitely the behaviour of argentic fluoride with chlorine at 60° FAHR., I placed a platinum cup, containing 88·38 grains of the recently fused fluoride, in a dry glass bottle filled with 409·67 cub. centims. of perfectly dry and pure chlorine, the bottle being closed by a bung of vulcanized india-rubber coated with a mixture of paraffin and lampblack, and inverted in mercury during thirty-eight days. No mercury entered, and the bottle was not corroded. A partial diminution of colour of the chlorine was produced by absorption of one third of the gas by mercury around the bung; but the cup and silver-salt gained only ·17 grain in weight. The saline residue was not even superficially whitened, and was found to be almost entirely soluble in water. AIMÉ* passed chlorine over fluoride of silver at 60° Fahr. in glass vessels coated with caoutchouc; the caoutchouc was acted upon, and hydrofluoric acid produced.

To try the effect of a higher temperature, a stoppered glass bottle of 286·77 cub. centims. capacity (=13·38 grains of chlorine) was employed, its stopper being smeared with traces of very pure lampblack to make it more gas-tight. The platinum cup contained 70·77 grains of recently fused argentic fluoride, and was supported in the upper part of the bottle by a platinum wire. The mouth of the bottle was immersed in mercury, and the stopper secured. The bottle was kept at a temperature of 200° to 230° FAHR. during eleven days; the yellow colour of the gas was not then perceptible. It was further heated to the same temperature during four more days; no ingress of mercury or visible leakage took place, nor did the bottle become at all corroded. A crust of chloride of mercury formed around the stopper. After standing forty-eight hours at 60° Fahr., the neck of the bottle was broken under mercury, strong rarefaction was found, and the mercury rose rapidly until the bottle was about half full. The residuary gas was then yellow, and was wholly absorbable by mercury. The residue in the platinum cup was superficially white, and showed no appearance of containing chemically combined platinum. By heating the cup (loosely covered with a platinum lid) gradually to redness, no special odour or evolution of gas was perceivable; a trace of effervescence took place, such as always occurs with fluoride of silver on being fused in the air, and a loss of weight of ·39 grain occurred. The crust from the mouth of the bottle was found by analysis to contain 7·914 grains of chlorine, and no hydrofluoric acid. After dissolving the fluoride in the cup by hot water, and the metallic silver from the residue by dilute nitric acid, about ·80 grain of argentic chloride (=·2 grain of chlorine) was found. The cup after heating to redness had lost ·19 grain. From these results it is manifest that fluoride of silver heated to 230° Fahr. in a platinum vessel in dry chlorine during fifteen days, suffers only a minute superficial decomposition, and does not corrode the vessel as it does at a low red heat.

* GMELIN's Handbook of Chemistry, vol. ii. p. 359; Annales de Chimie et de Physique (1833), vol. lv. p. 443; POGGENDORFF's Annalen, Leipzig, vol. xxxii. p. 576; Journal für praktische Chemie, vol. ii. p. 469; LIEBIG's Annalen (1835), vol. xvi. p. 174.

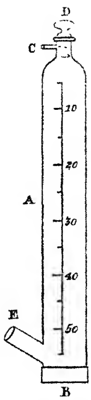
To ascertain if platinum vessels were corroded by chlorine in contact with argentic *chloride* at a red heat, a stream of the washed and dried gas was passed during half an hour over 166·67 grains of that substance in a melted state mixed with 10·30 grains of platinum filings, in a platinum boat within a roll of platinum foil in a glass tube. A gain of weight of 1·04 grain took place; a red-brown salt of platinum was formed; the boat was corroded, and had lost more than 1·58 grain in weight.

With regard to the behaviour of an aqueous solution of fluoride of silver with chlorine, PHIPSON states (Chemical News, vol. iv. p. 215), "When a solution of fluoride of silver is decomposed by a current of chlorine gas, the chloride of silver precipitated contains fluorine, and does not blacken under the influence of the solar rays; when exposed to the light it becomes of a light-brown tint, which is permanent even in sunlight. In this experiment a large portion of fluorine remains dissolved in the liquid; but it gradually decomposes the water, evolving large bubbles of oxygen gas, which appear to arise from the precipitate at the same time, and not before the glass becomes attacked by the hydrofluoric acid formed."

I passed an excess of chlorine through a dilute solution of argentic fluoride in a platinum cup; chloride of silver was freely precipitated; the mixture was then filtered in the dark, and boiled to expel the excess of chlorine; the clear liquid contained no silver, it was acid to test-paper, had no bleaching-power, contained a little hydrochloric acid and also hydrofluoric acid, the latter being shown by its giving a precipitate with a solution of chloride of barium. The silver chloride turned violet in daylight.

In some of these experiments I employed a graduated COOPER's receiver, A (fig. 8), 177·8 millims. high, 22·22 millims. diameter, and 60 cub. centims. total capacity, constructed of platinum, provided with a tightly fitting cap B to close its lower end, and a small exit-pipe C at its upper end, with an accurately fitting plug D, perforated in the direction of the dotted lines so as to act as a tap. The cap B having been cemented on air-tight by means of paraffin, the receiver was filled with washed chlorine. The tap D was then closed, and the lower end of the receiver immersed in an aqueous solution of argentic fluoride, with the arm E beneath the surface of the liquid. Rather rapid absorption of the gas took place, which ceased after about six hours. After standing twenty hours the lower opening of the vessel was closed, and the upper one momentarily opened; no odour of chlorine remained, and the vessel contained a gas which repeatedly and vigorously reinflamed a red-hot splint, it was therefore nearly pure oxygen. The bulk of this residuary gas was about 21·8 cub. centims. when corrected for difference of pressure*, the original volume of chlorine to the level of the arm E being about 52 cub. centims. A precipitate of chloride of silver was found freely in the vessel. In a second experiment I filled the same receiver with washed chlorine, introduced 20 grains of argentic fluoride, added 3·5 cub. centims. of distilled

Fig. 8.



* It is probable that some atmospheric air entered unobserved.

water, closed the receiver and shook it; much heat was evolved: in this experiment also air got in and vitiated the measurement. In a third experiment the receiver was filled with washed chlorine, 15 grains (=about 5 grains excess) of the brown fluoride put in, then 6 cub. centims. of distilled water added without agitation, the vessel then securely closed and immersed in a glass bottle which was quite filled with distilled water, and frequently shaken during several hours, and stood all night; no leakage of any kind took place. On opening the lower arm E under a measured amount of distilled water, strong rarefaction was found; the vessel was then closed and again shaken, and reopened under water; about 42 cub. centims. of water ran in altogether, and the volume of the gas, after correction for deficient pressure, was 11·89 cub. centims. The gas did not cloud a drop of solution of argentic nitrate, and was found to be pure oxygen. In this experiment, therefore, 53 cub. centims. or 2·455 grains of chlorine was absorbed, and 11·89 cub. centims. or ·248 grain of oxygen evolved. The following equation agrees substantially with the results obtained:—

$$8AgF+8Cl+4H_2O=5AgCl+3AgClO+8HF+O,$$

or

$$7AgCl+AgClO_3+8HF+O.$$

With Hydrochloric Acid.—Two platinum boats, Nos. 1 & 2, containing respectively 48·25 and 63·33 grains of recently fused fluoride of silver, were placed whilst still warm in a horizontal platinum tube, and a slow current of dried hydrochloric acid gas passed over them during $2\frac{1}{4}$ hours. No. 1 boat was placed near the exit end of the tube and kept cool, and the end of the tube nearly closed by a platinum stopper; whilst No. 2 was enclosed within a short tube of platinum placed in the middle part of the longer tube, and heated to incipient redness during the whole of the time. The acid gas was dried by passing through two bottles containing strong sulphuric acid. Vapour was evolved, which powerfully corroded glass. The contents of the cold boat soon became moist, and also white with superficial formation of argentic chloride; no sublimate appeared. After the experiment the residue in the cold boat weighed 50·43 grains, it should have been 54·52 grains if wholly converted into chloride; and that of the heated boat 71·91, theory requiring 71·55 grains. The residue in the cold boat contained soluble silver-salt, whilst that in the heated boat was quite insoluble in water The tubes and boats were not visibly corroded, and suffered only minute alterations in weight. Thinking that the liberated hydrofluoric acid might possibly by its liquefaction have influenced the effects produced in the cold boat, I placed a platinum boat containing 34·00 grains of recently fused fluoride of silver, within a similar long platinum tube, and passed dried hydrochloric acid gas similarly over it during one hour without application of heat. It then weighed 34·10 grains, and had acquired only a very thin white coating of argentic chloride. The residue was freely soluble in water, and the boat was unaltered in weight. From these experiments I conclude that fluoride of silver at a low red heat is wholly decomposed by anhydrous hydrochloric acid gas and converted into argentic chloride, and that in the massive state, as after fusion and cooling, it is

very slowly decomposed by that gas at 60° Fahr. with superficial formation of silver chloride.

With Chloric Acid.—A saturated aqueous solution of argentic fluoride manifested no signs of chemical change on the addition of aqueous chloric acid.

With Bromine.—The behaviour of argentic fluoride with bromine is generally similar to that with chlorine. 5·07 grains of earthy fluoride of silver in a closed glass bottle containing common liquid bromine, exposed during two days to sunlight, gained 1·15 grain in weight, theory requiring 2·43 grains if it had been wholly converted into bromide and its fluorine expelled.

A platinum cup containing 75·13 grains of recently fused argentic fluoride was placed in an inverted dry glass bottle; and within the cup was placed a small platinum crucible containing 42·16 grains of dry bromine which had been twice redistilled with fragments of fused chloride of calcium, and the last portions rejected. The stopper of the bottle having been firmly secured, the mouth of the bottle was immersed in mercury, and the whole set aside during thirty-six days at atmospheric temperature. Much free bromine still remained. The argentic salt had gained only ·38 grain in weight, and remained almost entirely soluble in water. In another similar experiment in which the liquid bromine was in contact with a larger surface of the silver-salt, similar results occurred. Bromine, therefore, has but little action on fused argentic fluoride at 60° Fahr.

To ascertain the effect of a higher temperature, two platinum cups were placed in a securely stoppered bottle inverted 3 inches deep in mercury with the mouths of the cups upwards. One cup contained 66·24 grains of the recently fused fluoride, and the other 28·64 grains of the redistilled and perfectly dry bromine. After standing six days at 60° Fahr., a gentle heat was applied; a small leakage of vapour occurred. After standing twenty-four more days at 60° Fahr., the bottle was again heated during two days to about 200° Fahr.; a small leakage again took place during the first few hours of heating. At the end of the twenty-four hours the colour of bromine had entirely disappeared, and the bottle was only slightly corroded. The bottle was again heated to 200° Fahr. during four days more and then cooled. Much bromide of mercury had formed around the outside of the stopper. The surface of the saline residue was yellow; and the cup and silver-salt had gained only 1·37 grain in weight. By digesting the residue in hot dilute nitric acid, about $3\frac{1}{4}$ grains of insoluble bromide was formed; the soluble portion yielded by analysis 71·83 grains of argentic fluoride=63·57 grains of fluoride of silver undecomposed. Bromine, therefore, acts slowly upon argentic fluoride at 200° Fahr., but less slowly than chlorine acts upon it.

Ordinary bromine, which had been rendered anhydrous, was passed during one hour over 51·88 grains of fluoride of silver at a low red heat in a platinum boat inside a short platinum tube within a longer one; a second platinum boat containing the bromine being placed near the first one, and the contiguous end of the long tube closed by a stopper of platinum, the heat extended to the bromine and produced a continuous supply of vapour. During the early part of the experiment a gas was evolved which corroded

damp glass freely; only traces of sublimate appeared. After the process the excess of bromine was expelled by heat. The short tube and its contents had gained 27·37 grains in weight. Neither of the tubes was visibly corroded: the long one, after being cleaned, had lost ·5 grain, and the short one only ·05 grain in weight. The boat containing the silver-salt was much corroded, but only where the saline matter touched it, as in similar experiments with chlorine; after removing nearly the whole (or 93·17 grains) of the saline residue, and cleaning the boat, 14·30 grains of platinum was found to have been removed by corrosion. The residuary salt was red, and evidently contained chemically combined platinum; it also visibly contained a little metallic platinum; soluble undecomposed fluoride was also present in it. The gain of weight of 27·37 grains may be accounted for thus:—22·97 grains of bromine decomposed 36·46 grains of argentic fluoride, uniting with 31·00 grains of silver and displacing 5·455 grains of fluorine, which by uniting with 14·2 grains of the boat, formed 19·655 grains of tetrafluoride of platinum, which further united with the 53·968 grains of bromide of silver to form 73·623 grains of a stable double salt; a further amount of 5·78 grains of bromine decomposed 9·17 grains of argentic fluoride, uniting with 7·80 grains of silver to form 13·58 grains of argentic bromide, and expelling 1·37 grain of fluorine, either in the free state or united chemically with some of the excess of bromine—most probably the latter. According to this, 45·63 grains of argentic fluoride was decomposed, and 6·35 grains remained unchanged. I did not ascertain whether a portion of the 1·37 grain of fluorine was expelled in the form of hydrofluoric acid, in consequence of the presence of a very small amount of organic matter, which I afterwards found the ordinary bromine contained.

In a second similar experiment with highly pure and anhydrous bromine, 26·68 grains of the fluoride was employed. A vapour was freely evolved, which corroded glass in the presence of damp air. The boat and its contents gained 11·30 grains in weight, theory requiring 12·81 grains if all the fluorine was expelled. The saline residue contained some red-brown platinum salt, and by analysis it yielded 30·67 grains of argentic bromide, 6·50 grains of undecomposed argentic fluoride, and 2·17 grains of metallic platinum. The boat and short tube lost 3·12 grains by corrosion; the remaining ·95 grain of platinum passed into solution, probably as 1·316 grain of tetrafluoride. I consider, from the results of these two experiments, that bromine gradually expels a portion of the fluorine (probably in the form of pentafluoride of bromine) from argentic fluoride in platinum vessels at a red heat, whilst the remainder of the fluorine corrodes and unites with the platinum of the vessels to form a red salt, as it does when chlorine is employed.

The vapour of ordinary bromine was also passed over fused argentic fluoride in a boat of Siberian graphite, which had been partially purified (see Phil. Trans. Roy. Soc. 1870, p. 242) but was not heated immediately before use. The silver-salt weighed after fusion 47·62 grains, and was slightly reduced to metal upon its surface in consequence of impurities, or of moisture, in the boat. The boat was heated to incipient redness in the vapour of bromine during 1½ hour, in the same manner as in the last

experiment, and the excess of bromine then expelled by heat; no sublimate occurred. The boat and its contents had increased in weight 22·39 grains, theory requiring 22·87 grains if all the salt was converted into argentic bromide and its fluorine expelled. The platinum tube was not corroded, nor altered in weight. The boat had lost 2·52 grains, apparently partly by corrosion; say, 2·25 by corrosion and ·27 by moisture (as found in a similar experiment). The saline residue was translucent, somewhat malleable, insoluble in water, and weighed 71·58 grains, theory requiring 70·49 grains of argentic bromide, the slight excess of weight being accounted for by the fused fluoride having contained a little reduced silver. By fusing 5·85 grains of it with an excess of alkaline carbonates, 3·36 grains of silver, entirely soluble in dilute nitric acid, was obtained, theory requiring exactly that amount. I consider that in this experiment the whole of the argentic fluoride was converted into bromide, the fluorine escaping in chemical union with the carbon of the boat as in similar experiments with chlorine (see Phil. Trans. Roy. Soc. 1870, pages 243, 244). The bromine used in this experiment was not quite pure.

I repeated the experiment with perfectly pure and anhydrous bromine, and a boat of Spanish graphite which had been perfectly purified by the process already referred to (page 327), the boat being heated nearly to redness immediately before the experiment. The boat after heating weighed 88·48 grains, and the fused fluoride 77·63 grains. The experiment lasted six hours, and much vapour, which corroded glass in damp air, was evolved. After expelling the excess of bromine, the boat was found slightly corroded, and, with its contents, weighed 201·82 grains. The saline residue adhered to the boat and could not be separately weighed; its weight, however, must have been about 115 grains, on account of the loss of weight of the boat; it contained scarcely perceptible traces of soluble silver-salt. The results of this experiment perfectly agree with those of the previous one.

It may be here remarked that gaseous fluoride of carbon does not corrode dry glass, and that the corrosive action of vaporous fluorine compounds upon glass is variable, and is caused in some cases by the compounds assuming the liquid state, and in others by the presence of traces of moisture.

On adding liquid bromine to a saturated aqueous solution of argentic fluoride, an abundant precipitate was produced, but no gas was evolved; on further adding fragments of the fluoride and stirring, effervescence occurred rather freely, and much heat and acid odour and a little oxygen were evolved.

The reaction may be represented by the following equations:—

$$8AgF + 8Br + 4H_2O = 5AgBr + 3AgBrO + 8HF + O,$$

or

$$7AgBr + AgBrO_3 + 8HF + O.$$

I now perfectly dried the yellow precipitate in a nearly covered platinum cup, and heated a portion of it with sulphuric acid in a glass test-tube; bromine was liberated freely, but the glass was not corroded. With dilute hydrochloric acid the yellow powder

effervesced freely and evolved much chlorine. The dried powder changed to a dirty grey colour in sunlight. I also heated 26·56 grains of the dry powder gradually to gentle redness in a deep platinum cup; much effervescence occurred, and an odourless gas, proved to be oxygen, was evolved; the residue was argentic bromide. The loss of weight was ·80 grain; theory requires a loss of 1·095 grain if the reaction took place according to the following equation:—

$$5AgBr + 3AgBrO, \text{ or } 7AgBr + AgBrO_3 = 8AgBr + 3O.$$

The deficiency of loss is accounted for by the oxygen previously expelled by the heat of the reaction, and that expelled by the heat applied in drying the mixture.

26·76 grains of earthy fluoride of silver was dissolved in a small amount of water in a deep platinum cup, an excess of bromine mixed with it, and repeatedly evaporated to dryness with water and an excess of bromine each time, and finally fused at a low red heat; the weight of argentic bromide found was 39·25 grains, theory requiring 39·39 grains; the deficiency was due to moisture in the fluoride.

With Hydrobromic Acid.—An aqueous solution of fluoride of silver was instantly and completely precipitated by an excess of aqueous hydrobromic acid.

With Bromic Acid.—Aqueous bromic acid produced a copious white precipitate in a dilute aqueous solution of argentic fluoride.

With Iodine.—KÄMMERER* has already made an experiment of heating fluoride of silver with iodine. He introduced iodine into a perfectly dry tube of glass along with a small stoppered glass tube filled with an excess of the fluoride†.

After expelling all the air by vapour of iodine, he broke the inner tube, and heated the apparatus to about 70° to 80° C. (=158° to 176° Fahr.) for twenty-four hours. The contents of the glass tube were then colourless, the iodine had disappeared, and the glass was transparent. The tube was then opened under mercury, the gas transferred to a eudiometer, and rapidly absorbed by a fragment of potash. After this absorption no trace of silica or iodine could be found in the potash; the oxygen (of the potash) displaced by the fluorine had combined either with potash or water to form peroxide of potassium or of hydrogen. The tube was not at all attacked. He considered he had isolated fluorine in this experiment, and mentioned Sir HUMPHRY DAVY'S statement that fluorine does not attack glass, and may be transferred over mercury. He also proposed to try bromine instead of iodine.

To ascertain the effect of iodine upon argentic fluoride at moderately elevated temperatures I made several experiments.

A platinum cup containing 32·75 grains of pure and recently fused iodine was inverted within a second platinum cup, containing 34·26 grains of recently fused and still hot

* Phil. Mag. 1863, vol. xxv. p. 213; Chemisches Centralblatt, August 1862, p. 523; MILLER'S 'Chemistry,' 3rd edition, vol. ii. p. 159.

† In all those of my preliminary experiments in which fluoride of silver was heated in contact with glass, serious interferences occurred; and if the silver-salt was not thoroughly fused it contained moisture, which acted upon glass, and also greatly promoted the absorption of the iodine.

argentic fluoride*, and the two cups placed in a glass bottle, the air of which had been dried by a stick of caustic potash†. The bottle, with its stopper well secured, was inverted in mercury, with the mouth of the larger cup upwards, and kept at a temperature of 150° to 200° Fahr. during eight days. A small leakage of air took place during the first hour. The bottle was further heated during three weeks to a temperature between 200° and 300° Fahr., and was then somewhat corroded. After a third heating to between 300° and 450° Fahr. during two more weeks, until the colour of iodine had entirely disappeared, and during one day more, the corrosion was rather considerable‡, and no mercury had entered the bottle. On opening the bottle under mercury, no contraction or rarefaction was found, and the bottle contained some fluoride of silicon; 3·53 grains of alkaline silico-fluoride was scraped off the bottle. The two cups (and their contents) had lost ·58 grain in weight and were not corroded. On heating them to low redness in a long platinum tube retort provided with a long exit-tube, the outer end of the retort being kept cold, a boiling sound occurred and iodine sublimed freely; tetrafluoride of silicon also escaped and deposited silica around the orifice; the loss of weight, including only a small portion of the free iodine, was 3·44 grains. No signs were observed of a condensed liquid. By further careful heating of the whole retort, 3·30 grains of free iodine (apparently containing some fuming vapour) was expelled, and a small amount of argentic iodide sublimed. I rubbed 22·89 grains of the very easily fusible residue to fine powder in plenty of boiling water§ until all soluble silver-salt was extracted, and precipitated the filtered solution; 3·00 grains of argentic chloride was obtained $=6·909$ grains of argentic fluoride‖, in the total 59·70 grains of saline residue freed from excess of iodine. If we assume 6·0 grains of unchanged fluoride of silver to have been present in the fused residue after expulsion of the free iodine, and the original fused silver fluoride to have contained 6 per cent. of free silver, the results may be approximately explained thus:—2·41 grains of iodine united with the 2·05 grains of free silver to form 4·46 grains of AgI; 26·21 grains of iodine decomposed 26·21 grains of AgF, forming 48·5 grains of AgI, and took the place of 3·92 grains of fluorine, only a small portion of which was expelled by the heat employed to effect the reaction.

* The fluoride probably contained about 6 per cent. of free silver.

† It was absolutely necessary in these experiments to exclude the least trace of moisture; otherwise the iodine was rapidly absorbed.

‡ It is astonishing how much corrosion of glass the most minute amount of watery vapour will under such circumstances produce; the moisture probably acts in such a way that a series of chemical reactions take place, attended by continued reproduction of a portion of water.

§ A mixture of argentic iodide and fluoride melts to a thin black oily looking liquid under a *small* quantity of boiling water.

‖ This number is too great; I have found by experiment that iodide of silver dissolves in a strong aqueous solution of argentic fluoride, and is reprecipitated on diluting the liquid.

Approximate Composition of the Saline Residue.

Before expulsion of Fluorine and Iodine.

Loosely united fluorine	3·92	} =6·74
Loosely united iodine	2·82	
AgF	6·00	
AgI	53·70	
Loss	·57	
Total	67·01	

After expulsion of Fluorine and Iodine.

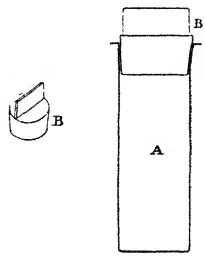
AgF	6·00
AgI	53·70
Total	59·70

These numbers show some disagreements; but more accurate ones could not be obtained.

In a second similar experiment with 35·98 grains of the fluoride, the results were generally similar: the vapour expelled by fusing the residue extinguished red-hot charcoal. The residue, after expulsion of loosely united fluorine and iodine, weighed 63·62 grains, and doubtless consisted of 60·23 grains of iodide of silver and 3·48 grains of argentic fluoride. In a third experiment 57·55 grains of iodine and 80·85 grains of fluoride were employed; the results were again similar. The yellow residue weighed 137·28 grains. Crystals of silicon heated to redness in the vapour expelled by melting the saline residue, were quickly corroded, but did not visibly incandesce. The sublimed iodine fumed strongly in the air. After expulsion of all the free iodine, the residue had lost 14·12 grains and weighed 123·16 grains = a gain of weight of 42·31 grains by the silver-salt =49·753 grains of fluoride decomposed = 49·753 grains of iodine combined =7·443 grains of fluorine liberated. In a fourth experiment the argentic fluoride was in a thick layer at the bottom of the cup; and after heating the substances to about 250° Fahr. during ten weeks, more than half the iodine and fluoride remained unchanged.

From the results of these experiments, I conclude that iodine slowly displaces fluorine from argentic fluoride at temperatures between 200° and 500° Fahr., without corroding platinum vessels, and forms a loosely combined compound of fluorine, argentic iodide and iodine, from which the loosely combined fluorine and ineffective iodine are expelled at a red heat.

Fig. 9.



As the results were somewhat interfered with by the formation of fluoride of silicon, I employed instead of a glass bottle a platinum one of the form A, fig. 9, 76·2 millims. deep, and 25·4 millims. diameter, provided with a hollow stopper (B) ground into it as air-tight as possible.

Two platinum cups were placed inside the bottle, one within the other—the outer and deeper one containing 64·80 grains of pure and fused iodine, and the other 80·46 grains of recently fused fluoride. The bottle, with its stopper secured, was inverted its whole depth in mercury at about 200° to 250° Fahr. during thirty-six days and then cooled. On opening, its gaseous contents exhibited no tension; it was full of a vapour of an

intensely dusty odour, which fumed very strongly in the air, was heavy, and might be poured. The iodine cup was empty, the other contained the usual yellow residue, most of which had been fused by heat applied in extracting the stopper. On heating the cups gradually to redness, the fuming vapour and free iodine were expelled. The saline residue after fusion weighed 126·73 grains =8·14 grains of fluorine expelled. The cups were not corroded.

To try the effect of a higher temperature, I employed a platinum retort of the form A, fig. 10, 153 millims. long and 22 millims. diameter, with an air-tight stopper B, and a thick

Fig. 10.

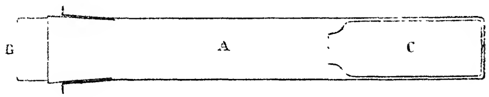
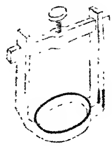


Fig. 11.



platinum bottle C with a wide mouth, to contain the fluoride, and an arrangement (fig. 11) to secure the stopper. 29·83 grains of fluoride and 35·33 grains of iodine were taken; and nearly half the retort was placed horizontally in a hole in a block of cast iron, and kept at 600° Fahr. during ten hours, with frequent rotation. The temperature was partly determined by means of a bit of cadmium placed upon the block. No leakage occurred; and no tension of vapour was found on opening the cooled retort. The results were similar to those of the last experiment. By fusing the saline residue it lost 18·36 grains, and then weighed 46·80 grains =2·98 grains of fluorine expelled. The bottle was slightly corroded and had lost ·35 grain in weight.

In a second experiment with this retort, 46·73 grains of fluoride and 34·98 grains of iodine were taken, and the retort heated to about 720° Fahr. during fifteen hours. The results were again similar. All the free iodine had been absorbed by the saline residue. The silver-salt after fusion weighed 67·81 grains =3·708 grains of fluorine expelled. The bottle lost 1·20 grain in weight. After a third experiment, with much larger quantities of materials at a still higher temperature, much fuming iodine remained, and a platinum boat containing the fluoride had lost 9·63 grains in weight.

From these experiments it is evident that platinum is corroded by contact with iodine and argentic fluoride at a temperature of 700° Fahr.

I also passed vapour of recently fused iodine during two hours over 135·29 grains of previously melted fluoride in the same apparatus and manner as in the experiments with bromine (see page 326). An acid odour was evolved, and a vapour which corroded glass quickly, but no sublimate except iodine appeared; the sublimed iodine fumed strongly in the air. After heating the platinum apparatus to redness the gain of weight was 89·62 grains, =104·568 grains of the fluoride decomposed and its fluorine expelled, if we allow 4·65 grains of iodine taken up by 3·96 grains of free silver in the original fused fluoride (see analysis of fluoride of silver, Phil. Trans. Roy. Soc. 1870, p. 229).

The platinum boat was corroded and had lost 6·08 grains in weight. The total weight of the residue, including platinum in it, was 230·55 grains; it showed no visible colour of chemically combined platinum. 5·17 grains of it, digested with a strong solution of iodide of potassium, left ·03 grain of platinum. A second portion, weighing 49·06 grains, digested with a hot solution of mercuric nitrate, left 2·99 grains of platinum in large fragments. A third portion, of 8·60 grains, exhausted of soluble matter by hot water, lost 1·53 grain (including a small amount of dissolved argentic iodide, see note ||, page 330); and the filtered liquid gave by precipitation 1·37 grain of argentic chloride, =31·80 grains of undecomposed fluoride (including a little iodide) in the whole of the residue: ·04 grain of platinum was found in the clear filtered liquid. The insoluble part of the 8·60 grains was a pale-yellow powder, easily fusible. and weighed 7·07 grains; by twice fusing it with an excess of pure alkaline carbonates, 3·23 grains of metallic silver, entirely soluble in dilute nitric acid (except ·01 grain of platinum), was obtained, pure argentic iodide requiring 3·23 grains. The composition of the total saline residue was about as follows:—

	grains.
Iodide of silver	195·00
Fluoride of silver	30·72
Metallic platinum	5·01
Combined platinum	1·07
Total	231·80

I repeated this experiment with perfectly pure iodine, passing the vapour of the iodine over 29·83 grains of the fluoride during four hours at a low red heat: similar results occurred. A colourless vapour was evolved. The saline residue appeared to contain traces of a red platinum salt. The gain of weight of the boat and its contents was 23·14 grains =27·21 grains of the fluoride converted into iodide. By analysis, as in the last experiment, 4·22 grains of argentic chloride was obtained, =3·73 grains of undecomposed fluoride (this number is too high, for the reason already stated); the other ingredients were also similarly determined. The boat was slightly corroded and had lost ·43 grain.

Composition of Residue.

	grains.	
Iodide of silver, about . . .	50·35	$\Big\}=52·97$
Fluoride of silver, about . .	2·62	
Platinum	·43	
Total	53·40	

It is evident that, in these experiments at a red heat, iodine expelled fluorine freely from argentic fluoride, either in a free state or united with some of the excess of iodine.

Several other kinds of platinum apparatus besides those described were devised and employed for effecting the reaction of iodine with argentic fluoride, but were found less effective.

To ascertain the effect of a carbon vessel, a boat of Siberian graphite weighing 48·36 grains was employed; it had been partially purified in the manner already described (Phil. Trans. Roy. Soc. 1870, p. 242), and was heated to incipient redness in a nearly closed vessel immediately before use. It contained 48·91 grains of recently fused argentic fluoride slightly reduced to silver by impurities in the graphite. A gentle current of pure iodine vapour was maintained over the red-hot salt during two hours in the same apparatus and manner as in the experiment with bromine (see p. 327); a vapour apparently heavy was evolved, which fumed in the air, and was strongly acid to litmus paper without first bleaching it. After expelling the excess of iodine by a red heat, the tube and its contents had gained 40·00 grains, theory requiring 41·55 grains if all the fluorine was expelled and the boat had suffered no loss. The sublimed iodine was dry, and did not fume in the air. The boat and its contents weighed 137·18 grains =a gain of 39·91 grains; and one end of the boat was a little corroded; the salt adhered to the boat, and could not be separately weighed. The platinum articles were not corroded, but ·38 grain of the silver-salt had been transferred to them. By melting the salt out of the boat, the latter was found corroded beneath. The salt was evidently argentic iodide. 9·30 grains of it fused with an excess of alkaline carbonates yielded 4·295 grains of metallic silver, =9·345 grains of the iodide. A second portion, weighing 36·50 grains, digested with boiling water until all soluble salt was extracted, and the filtered solution precipitated, gave ·24 grain of argentic chloride =·212 grain of argentic fluoride, or ·523 grain of undecomposed fluoride in the total 90 grains of residue. I consider that in this experiment, as in similar ones with a graphite boat in bromine and chlorine, the fluorine of the salt was expelled in chemical union with the carbon of the boat. It did not appear to me necessary to repeat this experiment with a perfectly purified boat.

On adding iodine to a saturated aqueous solution of argentic fluoride the mixture become yellow, and evolved some heat, but no visible bubbles of gas were liberated; by adding fragments of the fluoride to the mixture and stirring, a strongly acid odour was ultimately evolved. The reaction was probably similar to those with chlorine and bromine, and according to the following equation:—

$$8AgF + 8I + 4H_2O = 5AgI + 3AgIO + 8HF + O, \text{ or } 7AgI + AgIO_3 + 8HF + O.$$

With Hydriodic Acid.—An aqueous solution of argentic fluoride was instantly and completely precipitated by a brown solution of hydriodic acid, with evolution of heat.

With Iodic Acid.—Aqueous iodic acid produced a copious white precipitate with a dilute solution of argentic fluoride.

I am now examining the substance produced by the reaction of iodine and argentic fluoride under the influence of heat.

Added June 3rd, 1871.—I have since determined the chemical composition of the volatile substance evolved by the action of argentic fluoride and iodine at red heat.

XIII. *On the Constitution of the Solid Crust of the Earth.*

By Archdeacon Pratt, *M.A., F.R.S.*

Received September 19,—Read December 22, 1870.

Contents.

Introduction.

A few years ago I proposed the following hypothesis regarding the Constitution of the Earth's Solid Crust, viz.:—that the variety we see in the elevation and depression of the earth's surface, in mountains and plains and ocean-beds, has arisen from the mass having contracted unequally in becoming solid from a fluid or semifluid condition*: and that

* I first proposed this hypothesis in a paper printed in the Proceedings of the Royal Society, No. 64, 1864; see pp. 270–276 of that paper; and afterwards in the third edition of my 'Figure of the Earth,' pp. 134–137.

Mr. Airy was the first to suggest, in Phil. Trans. 1855, p. 101, a deficiency of matter below mountain-regions; and he there pointed out that such a deficiency would counteract in great measure the effect of the Himalayas themselves on the plumb-line, the attraction of which, I had shown in a previous paper in the same volume, by direct calculation, would be considerable and would introduce new anomalies. The reasoning, however, by which he proceeded to show that this deficiency must exist involved conditions which appeared to me inadmissible—viz. (1) that the solid crust is comparatively thin, and (2) that the density of the solid crust is less than that of the lava on which it was supposed to float. See my remarks on Mr. Airy's paper at pp. 51, 52, Phil. Trans. 1856; in which also I give reasons for not admitting, what his data require, that the present form of the surface has arisen solely or mainly from hydrostatic principles. This hypothesis of deficiency of matter, as there advanced, does not appear to rest on any true physical basis.

In the Phil. Trans. 1858, p. 745, following up Mr. Airy's suggestion of deficiency of matter—but not as he conceived it to exist, in a thin crust, immediately below the mountain-mass, and by buoyancy supporting the crust by the principle of floatation—I proposed the hypothesis of the mountain-mass having been formed by upheaval, by a slight expansion of the solid crust, and a corresponding attenuation of its density, from a great depth below (par. 4, p. 747); and I showed by calculation, in that paper, that the resulting effect of such attenuation on the plumb-line would be considerable, and quite comparable with the effect which calculation showed would be produced by the mountains themselves.

Immediately after this another source of disturbance of the plumb-line suggested itself to me, viz. deficiency

below the sea-level under mountains and plains there is a deficiency of matter, approximately equal in amount to the mass above the sea-level; and that below ocean-beds there is an excess of matter, approximately equal to the deficiency in the ocean when compared with rock; so that the amount of matter in any vertical column drawn from the surface to a level surface below the crust is now, and ever has been, approximately the same in every part of the earth.

2. The process by which I arrived at this hypothesis I will explain. In the Philosophical Transactions for 1855 and 1858 I showed that the Himalayas and the Ocean must have a considerable influence in producing deflection of the plumb-line in India. But by a calculation of the mean figure of the earth, taking into account the effect of local attraction, it appeared that nowhere on the Indian Arc of meridian through Cape Comorin is the resultant local attraction, arising from all causes, of great importance*. This result at once indicated that in the crust below there must be such variations of density as nearly to compensate for the large effects which would have resulted from the attraction of the mountains on the north of India and the vast ocean on the south, if they were the sole causes of disturbance,—and that, as this near compensation takes place all down the arc, nearly 1500 miles in length, the simplest hypothesis is, that beneath the mountains and plains there is a deficiency of matter nearly equal to the mass above the sea-level, and beneath ocean-beds an excess of matter nearly equal to the deficiency in the ocean itself.

3. The compensation, should the hypothesis be true, is not complete, but approximate;

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of matter in the ocean; and in a paper in the same volume of the Phil. Trans. (p. 779) I showed by calculation that the vast ocean stretching down to the south pole would produce considerable effects in the southern parts of India, such as the Survey altogether failed to detect. This seemed to imply that, as beneath mountain-regions there is a deficiency of matter, so beneath ocean-beds there must be an excess, in order to account for the deficiency in the effect of the ocean (which of itself would be large) not being discernible. The thought of an excess of matter below the ocean-bed accords with the remark which Sir John Herschel once made, that the ocean-bed of the Pacific must be more dense than the average surface of the solid crust, otherwise the protuberant ocean would be drawn away and would flow to other parts of the surface.

My calculations had shown, then, that a considerable effect on the plumb-line must result from each of the following causes taken separately:—(1) the mountain-region, (2) the ocean, (3) any widespread, though slight, deficiency or excess of matter in the solid crust. As there is reason to believe that resultant local attraction is nowhere in India very great, and is generally small, it must follow that, generally speaking, below high ground there is a deficiency of matter, and below ocean-beds an excess.

But there had not been at that time any physical hypothesis proposed to account for these two conditions of the crust, and to connect them together as results of one and the same cause. An hypothesis was, however, suggested in 1864, in my paper in the 'Proceedings of the Royal Society' alluded to at the beginning of this note, and referred to in the text, viz. that all the varieties we see in the earth's surface (in mountains, plains, and ocean-beds) have arisen from the earth's mass having contracted unequally in a vertical direction, in passing from a fluid to a solid state—a necessary result of its fluid origin being that the amount of matter in any vertical column down to a level surface is the same and always has been the same, whatever the changes in length it may have undergone.—Calcutta, April 6, 1871.

* See Proc. Royal Soc. No. 64, 1864; but especially Phil. Mag. January and February, 1867.

for deflections of the plumb-line do exist—such, for instance, as that near Moscow. Thus also in India a comparison of the amplitudes of arcs obtained by observation and by calculation shows the same. I have constructed the following Table to bring this to view. The data in the 1st, 2nd, and 3rd columns of numbers are derived from the Chapter on the Figure of the Earth in the volume of the British Ordnance Survey.

TABLE I.

Stations.	Observed latitudes.	Measured distances between successive stations, in feet.	Amplitudes between successive stations.		Difference, or relative deflection of plumb-line.	Deflections of plumb-line relative to Punnae.	Equivalent horizontal meridian force. + means south.
			Observed or astronomical.	Calculated: $a=20926184$, $b=20855301$.			
	° ′ ″		° ′ ″	° ′ ″	″	″	
Punnae	8 9 31 132	....	... .	....	....	... .	....
Retchapolliam ..	10 59 42 276	1029173·7	2 50 11·14	2 50 10·32	+0 82	+0·82	+0·0000040 g
Dodagoontah.. ..	12 59 52·165	727386·3	2 0 9 89	2 0 15·19	−5·30	−4·48	−0·0000217 „
Namthabad	15 5 53 562	761813 4	2 6 1·40	2 5 55·48	+5·92	+1·44	+0·0000070 „
Damargida . .	18 3 15·292	1073410 9	2 57 21·73	2 57 23·41	−1·68	−0·24	−0·0000012 „
Takal Khera ...	21 5 51 532	1105539·8	3 2 36 24	3 2 38·56	−2·32	−2·56	−0·0000124 „
Kalianpur	24 7 11·262	1097364·9	3 1 19·73	3 1 13 62	+6·11	+3·55	+0·0000172 „
Kahana	29 30 48 322	1961138·0	5 23 37·06	5 23 41·65	−4·59	−1·04	−0 0000050 „

I have calculated the amplitudes by means of the formula

$$\lambda=\frac{2s}{a+b}\left\{1+\tfrac{3}{2}\varepsilon\cos 2m\right\}.$$

For finding such very small angles as the deflections, which are the differences of very much larger angles, no doubt the introduction of the square of the ellipticity would slightly modify the results in the last three columns; but not so as to affect the use I shall make of them. It is seen from this Table (last column but one) that the deflections of the plumb-line, though small, are yet sensible quantities; and they do not correspond with the heights of the neighbourhood of the several stations.

The hypothesis, therefore, is not exact, but only approximately true, when applied generally, on a large scale. This, indeed, we might anticipate for other reasons. For example, if the crust below the ocean-beds has contracted or expanded at all (which no doubt it has) since it became too thick* to be able to adjust itself as it floated upon the

* The late Mr. HOPKINS pointed out that the amount of precession in the earth's axis, caused by the disturbing force of the sun and moon, would be very different in amount as the solid crust was thin or thick; and he made a calculation (Philosophical Transactions, 1839, 1840, 1842) based upon this idea, and showed that the crust must now be at least 1000 miles thick. M. DELAUNAY has lately read a paper before the Academy of Sciences controverting Mr. HOPKINS's idea—saying that the interior fluid must long ago have conformed to the motion of the crust in consequence of friction and viscidity, and be now moving with it as if the whole were solid. If the crust moved round a steady axis, this might be true. But this is not the case. The force which causes precession is continually tending to draw the earth's pole towards the pole of the ecliptic—but does not move it in that direction, but, combining with the rotatory motion, causes it to shift through a small angle at right angles to the line joining the two poles. The extent of the angle must depend upon the force, the length of the infinitesimal portion of time, and the moment of inertia of the crust; for the fluid, during this infinitesimal portion of time, will not have been able to acquire the new motion; the crust, having no solid connexion with the fluid, will slip over it, with a twist. Suppose even that at the present instant the fluid were

fluid below, and if this contraction or expansion was different from that of the dry land, water would flow in or out of the ocean and disturb the exact equality of matter in any two vertical columns drawn down from the surface of the land and the water. Also, as the crust contracted and brought into play the prodigious force of compression, which would inevitably cause the crust to give way at the weakest part and produce anticlinal lines, crushing, sliding, and interpenetration, there would be a slight increase of mass in some parts on this account. From these and similar causes it is readily seen that, since the epoch when the crust ceased to be thin enough to adjust its own position according to varying circumstances, changes must have occurred which would modify the previous state of things. Still the result of these modifying causes must be but slight, compared with those large effects of the mountains and ocean and crust, which nearly compensate each other, and which suggest the hypothesis. Another cause arises from the calculation itself. It is necessary to assume some law of distribution of the mass, that the calculation may be possible. I assume that the deficiency or excess of matter is distributed *uniformly* to a depth bearing a fixed ratio to the height of the land or the depth of the ocean. The actual distribution most likely differs from this. But this is taken as an average. We must expect, for these reasons, to find that the hypothesis is not satisfied with exact precision.

4. Colonel J. T. Walker, R.E., Superintendent of the Great Trigonometrical Survey of India, to whom I had communicated the formulæ developed in this paper, has lately supplied me with information showing the results of the Pendulum Observations recently made along the Great Indian Arc of meridian and at other places, and has obligingly allowed me to make use of the data. While these observations have been going on I have looked forward with great interest to the results, as I felt persuaded that the observations would furnish me with the means of testing, in a new and independent way, the truth of my hypothesis regarding the constitution of the earth's crust. It is the object of the present communication to show with what measure of success the test has been applied*.

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moving exactly as the crust, the force producing precession would from this instant give the crust a new motion, which the fluid has not, and which it has not time to acquire before, in the next small portion of time, the crust has shifted again with the same twist. The amount of precession must therefore depend upon the moment of inertia during the time it is generated, and therefore upon the thickness of the crust and not at all upon the fluid.

* Since the above was written, Colonel Walker has sent me a copy of a printed letter and Note on the Pendulum Observations, from which I extract the following remarks. "The observations at the five northernmost "stations indicate that there is much probability that the density of the strata of the earth's crust under and "in the vicinity of the Himalayan mountains is less than that under the plains to the south, the deficiency "increasing as the stations approach the Himalayas, and being greatest when they are north of the Siwaliks. "On the other hand, the observations of the five southernmost stations show an increase of density in proceeding "from the interior of the peninsula to the coast at Cape Comorin. Thus both groups of observations tend to "confirm the hypothesis that there is a diminution of density in the strata of the earth's crust under moun-"tains and continents, and an increase of density under the bed of the ocean." This is the hypothesis I published in 1864: see Proceedings, No. 64.

§ 1. *Data regarding Pendulum Observations in India.*

5. From the information furnished me by Colonel WALKER I complete Table II. I have selected five stations on the Arc of Meridian at nearly the same distances from one another in succession, averaging about 5° 15′ apart. I have also taken stations on the coast and on an island (Minicoy). Punnae is close to Cape Comorin.

TABLE II.					TABLE III.			
Stations.	Geodetic coordinates. North latitude.	Geodetic coordinates. East longitude.	Geodetic coordinates. Heights, in feet.	Observed numbers of vibrations at stations, reduced for thermometer and barometer.	Relative gravity.	Reduction for height above sea.	Allowance for latitude.	Relative gravity freed from effects of height and latitude.
Indian Arc Stations.								
Punnae	8° 10′	77° 41′	44	85978·18	1−0·0005074	+ 42	− 1062	1−0·0006094
Bangalore	13 4	77 37	3007	74 63	1−0·0005900	+2879	− 2689	1−0 0005710
Damargida	18 3	77 43	1934	86·16	1−0 0003219	+1852	− 5050	1−0·0006417
Kalianpur	24 7	77 42	1765	86005·76	1+0 0001340	+1690	− 8783	1−0 0007753
Kaliana	29 31	77 42	826	22·26	1+0 0005177	+ 791	−12769	1−0 0006801
Coast Stations.								
Punnae . . .	8 10	77 41	44	85978·18	1−0·0005074	+ 42	− 1062	1−0·0006094
Alleppy .	9 30	76 20	6	81·23	1−0·0004365	+ 6	− 1433	1−0·0005792
Mangalore	12 52	74 49	7	84·27	1−0 0003658	+ 7	− 2609	1−0 0006260
Madras .. .	13 4	80 17	27	84·40	1−0 0003628	+ 26	− 2689	1−0·0006291
Cocanada . ..	16 56	82 18	9	93·56	1−0·0001498	+ 9	− 4463	1−0·0005952
Ocean Station.								
Minicoy . .	8 17	73 2	6	82 31	1−0·0004114	+ 6	− 1092	1−0·0005200

6. The numbers of vibrations of the pendulum at the several stations differ from each each other for three reasons. The stations differ (1) in latitude, (2) in height above the sea, (3) in local attraction. Were all these allowed for, the numbers would come out the same for the several stations. In Table III. I allow for the influence of the first and second of these causes; and the last column shows by its variations what we have to attribute to the third cause, viz. local attraction.

The numbers in the first column of Table III. are obtained from those of the last column of Table II. by dividing them all by 86000, and then doubling the small (that is, the decimal) part, because gravity varies as the square of the number of vibrations. In the second and third columns of Table III. the numbers are the last of seven places of decimals, the ciphers and the decimal point being omitted for convenience. The numbers in the third column are obtained by means of CLAIRAUT'S Theorem, which shows that gravity varies as $1+(\frac{5}{2}m-\varepsilon)\sin^2 l$; the ellipticity being $\frac{1}{295}$.

§ 2. *Data regarding the contour of the Continent of India.*

7. Viewing the continent of India generally, and allowing for ridges of hills and hollows, it may be said to lie more or less evenly, as far as this problem is concerned, up to the foot of the Himalayas. With regard to the Himalayas, by a careful examination of published documents and maps of the Great Trigonometrical Survey, and of Colonel R. STRACHEY'S map referred to in the Philosophical Transactions for 1858, p. 774, and

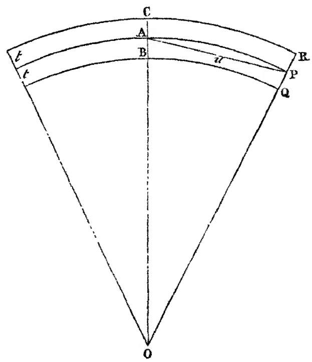
by laying down the heights on a plan, I come to the conclusion that the mass of the Himalayas may, for the purpose of this problem, be represented as a vast tableland, 15,000 feet or 2·48091 miles above the sea-level. If a number of zones are drawn around the three stations Kaliana, Kalianpur, and Damargida, their width being about 50 miles (49·45 exactly, as will be seen further on), then the tableland will begin on the 2nd zone from Kaliana, the 8th from Kalianpur, and the 17th from Damargida; and the horizontal extent of the tableland lying on these and the following zones, as determined from the plan, is shown by the values of β in the following Table IV. In some cases β is made up of two or even of three portions added together, when the zones cross the tableland in two or more places, owing to its irregular outline.

TABLE IV.

Kaliana.		Kalianpur.		Damargida.	
Zones.	β.	Zones.	β.	Zones.	β.
2	74	8	45	17	22
3	110	9	62	18	41
4	127	10	79	19	50
5	137	11	88	20	62
6	144	12	92	21	68
7	149	13	97	22	73
8	113	14	76	23	58
9	88	15	58	24	46
10	79	16	47	25	32
11	66	17	46	26	9
12	56	18	35	27	12
13	109	19	76	28	12
14	87	20	46	29	12
15	79	21	6		
16	61	22	6		
17	37				

§ 3. *Formulæ for the Vertical Attraction of a Spherical Cap of matter on the earth's surface on the mid points of its upper and lower surfaces, and of its divisions into a Central Portion and Zones.*

8. By a spherical cap is meant such a part of a spherical shell as would be generated by the revolution of the figure A P Q B round the vertical A B.



A is the station attracted. The chord A P$=u$ miles; the thickness of each cap, above and below the station-level, is t miles. Let c and r be the distances of the attracted point A and of any particle of the cap from O; θ the angle between c and r; $z=c-r$; $u=c$ chord θ; $v=c$ vers θ, which $=u^2\div 2c$.

FIRST. Suppose the cap immediately *below* the station-level. The attraction of an elementary ring

of matter round A, reckoned positive downwards,

$$=2\pi r\sin\theta\,.\,dz\,.\,rd\theta\,.\,\varrho\,\frac{c-r\cos\theta}{(c^2+r^2-2cr\cos\theta)^{\frac{3}{2}}}=\frac{2\pi\varrho r^2}{c^2}\,\frac{d}{d\theta}\left(\frac{r-c\cos\theta}{\sqrt{c^2+r^2-2cr\cos\theta}}\right)d\theta dz$$

$$=\frac{2\pi\varrho}{c^2}\,(c-z)^2\frac{d}{d\theta}\left(\frac{2c\sin^2\frac{1}{2}\theta-z}{\sqrt{z^2+4c(c-z)\sin^2\frac{1}{2}\theta}}\right)d\theta dz.$$

Integrating from $\theta=0$ to $\theta=\theta$, and then putting $2c\sin\frac{1}{2}\theta=u$ and $u^2=2cv$,
Total Attraction of the Cap

$$=\frac{2\pi\varrho}{c^2}\int_0^t\left\{(c-z)^2+\frac{(v-z)(c-z)^2}{\sqrt{z^2+2(c-z)v}}\right\}dz$$

$$=\frac{2\pi\varrho}{c^2}\left\{\frac{c^3-(c-t)^3}{3}+\int_0^t\frac{(v-z)(c-z)^2dz}{\sqrt{z^2-2vz+2cv}}\right\}.$$

Integrating by parts, the integral becomes

$$=-(c-z)^2\sqrt{z^2-2vz+2cv}-2\int(c-z)\sqrt{z^2-2vz+2cv}\,.\,dz$$

$$=-(c-z)^2\sqrt{z^2-2vz+2cv}+\tfrac{2}{3}(z^2-2vz+2cv)^{\frac{3}{2}}+2(v-c)\int\sqrt{z^2-2vz+2cv}\,.\,dz$$

$$=-(c-z)^2\sqrt{z^2-2vz+2cv}+\tfrac{2}{3}(z^2-2vz+2cv)^{\frac{3}{2}}$$
$$-(v-c)\{(v-z)\sqrt{z^2-2vz+2cv}-(v^2-2cv)\log_e(v-z+\sqrt{z^2-2vz+2cv})\}$$

$$=-\{\tfrac{1}{3}z^2-(c-\tfrac{1}{3}v)z-c^2-\tfrac{7}{3}cv+v^2\}\sqrt{z^2-2vz+2cv}$$
$$+(v-c)(v^2-2cv)\log_e(v-z+\sqrt{z^2-2vz+2cv}).$$

Putting this for the integral, replacing $2cv$ by u^2, and taking the limits,
Vertical Attraction of the Cap below the station-level

$$=\frac{2\pi\varrho c}{3}\left\{1-\left(1-\frac{t}{c}\right)^3+\frac{3u}{c}-\frac{7u^3}{2c^3}+\frac{3u^5}{4c^5}\right.$$
$$-\left[\frac{3u}{c}-\frac{7u^3}{2c^3}+\frac{3u^5}{4c^5}-\left(\frac{3u}{c}-\frac{u^3}{2c^3}\right)\frac{t}{c}+\frac{u}{c}\frac{t^2}{c^2}\right]\sqrt{\frac{t^2+v^2}{u^2}-\frac{t}{c}}$$
$$\left.+3\left(\frac{u^2}{2c^2}-1\right)\left(\frac{u^4}{4c^4}-\frac{u^2}{c^2}\right)\log_e\frac{\frac{v}{2c}-\frac{t}{u}+\sqrt{\frac{t^2+v^2}{u^2}-\frac{t}{c}}}{\frac{u}{2c}+1}\right\}.$$

This is the exact expression.

9. Secondly. Suppose the Cap is immediately *above* the station-level. The above formula requires in this case some modification. In the first place, when the limits of θ are taken and θ is put $=0$, the radical in the denominator is now $-z$, and not z as before. This will change the sign of the first term $(c-z)^2$ in the integral with regard to θ, and will change the signs of the first and second terms within the brackets of the final integral. Again, the limits of integration with regard to z must be taken from $z=-t$ to $z=0$, which is the same as putting $-t$ for t and also changing the sign of every term of the

final integral. Making these changes, and still estimating, as I always shall do, the attraction positive downwards,

Vertical Attraction of the Cap above the station-level

$$=\frac{2\pi\rho c}{3}\left\{1-\left(1+\frac{t}{c}\right)^3-\frac{3u}{c}+\frac{7u^3}{2c^3}-\frac{3u^5}{4c^5}\right.$$

$$+\left[\frac{3u}{c}-\frac{7u^3}{2c^3}+\frac{3u^5}{4c^5}+\left(\frac{3u}{c}-\frac{u^3}{2c^3}\right)\frac{t}{c}+\frac{u}{c}\frac{t^2}{c^2}\right]\sqrt{\frac{t^2+u^2}{u^2}+\frac{t}{c}}$$

$$\left.-3\left(\frac{u^2}{2c^2}-1\right)\left(\frac{u^4}{4c^4}-\frac{u^2}{c^2}\right)\log_e\frac{\frac{u}{2c}+\frac{t}{u}+\sqrt{\frac{t^2+u^2}{u^2}+\frac{t}{c}}}{\frac{u}{2c}+1}\right\}.$$

As these formulæ are to be applied to find the vertical attraction of the superficial portions of the earth, it may be here stated that, as the attractions will be always small quantities, the earth may be regarded as a sphere, and c taken equal to the mean radius 3956 miles, as the height or depth of any Cap above or below the sea-level will be comparatively small.

10. These formulæ may be much reduced for use by approximation. The square of $t\div c$ will be neglected; for the greatest value $t\div c$ will have in this paper will be $1\div 13$; and therefore its square will be $1\div 169$. Expanding, then, in powers of $t\div c$ and neglecting its square, and observing that as c^2 occurs in the denominator of every term of the coefficient of the log., we may neglect t^2 everywhere in the log. itself, we have

Vertical Attraction of the Cap below the station-level

$$=2\pi\rho\left\{t+u-\frac{7u^3}{6c^3}+\frac{u^5}{4c^5}-\sqrt{t^2+u^2}+\frac{7u^3}{6c^3}-\frac{u^5}{4c^5}+\frac{ut}{c}-\frac{u^3t}{6c^3}\right.$$

$$\left.+\frac{ut}{2c}-\frac{7u^3t}{12c^3}+\frac{u^5t}{8c^5}+\left(\frac{u^2}{2c^2}-1\right)\left(\frac{u^4}{4c^4}-\frac{u^2}{c^2}\right)\log_e\left(1-\frac{t}{u}\right)\right\}$$

$$=2\pi\rho\left(u+t-\sqrt{u^2+t^2}+\frac{ut}{2c}\right).$$

If we take the density of the surface to be half the mean density of the earth, and g be gravity, then

Vertical Attraction of a Cap below the station-level

$$=\frac{3g}{4c}\left(u+t-\sqrt{u^2+t^2}+\frac{ut}{2c}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The second formula in like manner gives

Vertical Attraction of a Cap above the station-level

$$=-\frac{3g}{4c}\left(u+t-\sqrt{u^2+t^2}-\frac{ut}{2c}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

11. The cap may be divided into Zones and a Central Portion in the following way. Let u and w be the chords of the angular distances from the station of the bounding

circles of any zone, drawn around the station on the sphere which represents the sea-level. For the zones beyond the central portion $t \div u$ is frequently so small that its fourth power may be neglected. The case where it is not so small will be considered afterwards. Then formula (1) gives for a cap of thickness t below the station-level

$$\text{Vertical Attraction } = \frac{3g}{4c}\left(t+\frac{ut}{2c}-\frac{t^2}{2u}\right);$$

and therefore for the part of the cap over the zone

$$\text{Vertical Attraction } = \frac{3g}{4c}\,\frac{w-u}{2c}\left(t+\frac{t^2c}{uw}\right).$$

Hence, if h be the height of the station above the sea-level, the Vertical Attraction of a mass on the zone, up to the station-level

$$=\frac{3g}{4c}\,\frac{w-u}{2c}\left(h+\frac{h^2c}{uw}\right).$$

Taking the difference of these, and putting the height of the superficial mass on the zone above the sea-level (that is, $h-t$) $=k$,

Vertical Attraction of the mass on the zone at the sea-level

$$=\frac{3g}{4c}\,\frac{w-u}{2c}\,k\left(1+\frac{2h-k}{c}\,\frac{c^2}{uw}\right).$$

If the superficial mass rises above the station-level, we use the formula (2), which gives

$$\text{Vertical Attraction } = \frac{3g}{4c}\,\frac{w-u}{2c}\left(t-\frac{t^2c}{uw}\right);$$

and this, added to the attraction of the mass between the sea-level and the station-level, gives, observing that in this case $k=h+t$,

Vertical Attraction of the mass on the zone at the sea-level

$$=\frac{3g}{4c}\,\frac{w-u}{2c}\left(h+t+\frac{(h^2-t^2)c}{uw}\right)$$

$$=\frac{3g}{4c}\,\frac{w-u}{2c}\,k\left(1+\frac{2h-k}{c}\,\frac{c^2}{uw}\right),$$

precisely the same formula as before.

The formula is also true when applied to parts covered by the ocean. Let, as before, h be the height of the station above the sea-level, but k the depth of the ocean (supposed uniform under the zone). The ratio of the density of sea-water to that of rock, which equals 2·78 (half the mean density of the earth), $=0\cdot363$. Then $h+k$ and h are heights of the station and of the surface of the attracting ocean above the level of the ocean-bed; and therefore, by means of the formula above proved,

Vertical Attraction of the ocean under the zone on the sea-level

$$=\frac{3g}{4c}\,\frac{w-u}{2c}\,0\cdot363\,k\left(1+\frac{2(k+h)-k}{c}\cdot\frac{c^2}{uw}\right);$$

and therefore the effect of the *deficiency* of density of the ocean below that of rock

$$=-\frac{3g}{4c}\,\frac{w-u}{2c}\,0\cdot637\,k\left(1+\frac{2h+k}{2c}\,\frac{c^2}{uw}\right),$$

which is precisely the same formula as before, $-k$ being put for k because it is measured down below the sea-level, and the density being that of the deficiency of attracting matter.

Hence for all cases in which the thickness of the mass on the zone is such that the fourth power of its ratio to the mid-distance of the zone may be neglected,

Vertical Attraction of the mass on the zone

$$=\frac{3g}{4c}\frac{w-u}{2c}k\left(1+\frac{2h-k}{c}\frac{c^2}{uw}\right).$$

12. In order to simplify this as much as possible, I shall so divide the sea-level into zones that $w-u$ is the same for the central portion and each zone, equal d. Suppose n is the number of divisions—that is, a central portion, $n-2$ zones, and a central portion at the antipodes, all following the above law. Let $u_1, u_2, u_3, \ldots u_n$ be the chords to the successive bounding circles. Then

$$u_1=d,\ u_2=u_1+d=2d,\ \ldots\ldots\ldots\ u_n=nd, \text{ also } =2c\,;$$

$$\therefore \frac{u_1}{2c}=\frac{1}{n},\ \frac{u_2}{2c}=\frac{2}{n},\ \ldots\ \frac{u_r}{2c}=\frac{r}{n},\ \ldots\ \frac{u_n}{2c}=\frac{n}{n}=1,$$

and the formula becomes

Vertical Attraction of the mass on the rth zone after the central portion

$$-\frac{3g}{4c}\frac{k}{n}\left(1+\frac{2h-k}{4c}\frac{n^2}{r(r+1)}\right). \qquad (3)$$

In the calculations of this paper I shall take $n=160$, which makes the radius of the central portion and the width of each zone $=49\cdot45$ miles, nearly $=50$.

I would here observe that matter may be always transferred in imagination *in azimuth* round the station without altering its effect on the vertical attraction. An application of this principle to an actual case in the earth may often assist in getting a better average for the mass. Any zone may be subdivided into smaller zones, if necessary, according to the same law. Also any zone may be divided into four-sided compartments by great circles so drawn through the station as to divide it into portions, the average heights of which may represent the mass, if it be very irregular, better than the mean height of the whole would.

If the fourth power of the thickness may not be neglected, as is done above, the formulæ (1) and (2) must be used without expanding the radical. This I shall revert to in the latter part of the next Section.

§ 4. *Formulæ for the "Resultant Vertical Attraction" of the Central Portion of the Cap, and of the Zones.*

13. By the expression "*Resultant* Vertical Attraction" of a mass I mean the vertical attraction of the mass, *diminished* by the effect of the attenuation spread uniformly below the sea-level according to the hypothesis. In the case of the ocean, the resultant vertical attraction will be the (negative) attraction of the deficiency of matter in the ocean,

increased by the attraction of the addition of matter spread uniformly through the crust below the sea-bed according to the hypothesis.

14. If the Central Portion is distributed through a depth mh below the sea-level, the Vertical Attraction of this distributed mass would equal the difference of the attraction of two caps running down to depths $(m+1)h$ and h, which, by means of the first formula in paragraph 11,

$$=\frac{3g}{4cm}\left\{\left((m+1)h+\frac{(m+1)ah}{2c}-\frac{(m+1)^2h^2}{2a}\right)-\left(h+\frac{ah}{2c}-\frac{h^2}{2a}\right)\right\};$$

and when this is subtracted from the same formula,

$$\text{Resultant Vertical Attraction of Central Portion}=\frac{3g}{4c}\frac{(m+1)h^2}{2a}.$$

Put $a=2c\div n=49{\cdot}45$, $c=3956$, and this

$$=0{\cdot}0000968h^2g \text{ or } 0{\cdot}0001917h^2g,$$

according to whether $m=50$ or 100.

If the mass on the rth zone is distributed down through a depth mh below the sea-level (that is, between the depths h and $(m+1)h$), the second formula in paragraph 11, after substituting for u and w in terms of r, gives for its effect

$$\frac{3g}{4cmn}\left\{\left((1+m)h+\frac{(1+m)^2h^2}{4r(r+1)}\frac{n^2}{c}\right)-\left(h+\frac{h^2n^2}{4r(r+1)c}\right)\right\}=\frac{3g}{4cn}\left(h+\frac{(m+2)h^2}{4r(r+1)}\frac{n^2}{c}\right).$$

Subtracting this from the same formula, h being first put for t,

$$\text{Resultant Vertical Attraction for the zone}=-\frac{3gn}{16c^2}\frac{(m+1)h^2}{r(r+1)}.$$

Put $n=160$, $c=3956$, and this

$$=0{\cdot}0000978\frac{h^2g}{r(r+1)} \text{ or } 0{\cdot}0001937\frac{h^2g}{r(r+1)},$$

according to whether $m=50$ or 100.

15. These formulæ I now tabulate for the five stations—observing that the surrounding land up to the sea stretches over radii of 0, 3, 5, 9, 14 zones respectively.

TABLE V. ($m=50$).

	Punnae.	Bangalore.	Damargida.	Kalianpur.	Kaliana.
$h^2=$	0·0000094	0·3242313	0·1341757	0·1117431	0·0244735
Central Part	......	0·0000314 g	0·0000130 g	0·0000120 g	0·0000024 g
Zone 1	......	0·0000159 ,,	0·0000066 ,,	0·0000055 ,,	0·0000012 ,,
,, 2	......	0·0000053 ,,	0·0000022 ,,	0·0000018 ,,	0·0000004 ,,
,, 3	......	0·0000026 ,,	0·0000011 ,,	0·0000009 ,,	0·0000002 ,,
,, 4	......	......	0·0000007 ,,	0·0000005 ,,	0·0000001 ,,
,, 5	......	......	0·0000005 ,,	0·0000003 ,,	0·0000001 ,,
,, 6	......	......	......	0·0000002 ,,	0·0000001 ,,
,, 7	......	......	......	0·0000001 ,,	......
,, 8	......	......	......	0·0000001 ,,	......
,, 9	......	......	......	0·0000001 ,,	......
,, 10	......	......	......	......	......
,, 11	......	......	......	......	......
,, 12	......	......	......	......	......
,, 13	......	......	......	......	......
,, 14	......	......	......	......	......
Totals, $m=$ 50	......	0·0000552 g	0·0000241 g	0·0000215 g	0·0000045 g

For $m=100$ it will be quite near enough to double these; viz.

Totals, $m=100$	...	0·0001104 g	0·0000482 g	0·0000430 g	0·0000090 g

16. Suppose on a zone of any width only a comparatively small portion of its whole circuit has a mass standing on it. Then, as the distance from the centre of the mass increases, the angular width varies nearly as the distance inversely. By paragraph 11 the Vertical Attraction of the mass on the whole zone, t being its thickness,

$$=\frac{3g}{4c}\frac{w-u}{2c}\left(t+\frac{t^2c}{uw}\right).$$

From which it is easy to deduce that, for a whole zone,

$$\text{Resultant Vertical Attraction}=-\frac{3g}{4}\frac{w-u}{2c}\frac{(m+1)t^2}{uw}.$$

This varies very nearly inversely as the square of the distance from the centre of the mass. Hence, if the mass stands on only a comparatively small portion of the zone measured horizontally at the station, the Resultant Vertical Attraction varies nearly inversely as the cube of the distance.

17. I will now consider the case of a zone the height of the mass upon it being such that we must not neglect any power of the depth through which the corresponding attenuation reaches.

We must revert to formula (1). The effect of the attenuation below the zone equals the difference of the effects of two masses, each 1-mth of the density of rock, running down to depths h and $h+mk$ below the station-level. Call u and w, as before, the bounding chords of the zone. The effect of the attenuation

$$=-\frac{3g}{4cm}\Big(w-u-\sqrt{w^2+(h+mk)^2}+\sqrt{u^2+(h+mk)^2}$$

$$+\frac{w-u}{2c}(h+mk)-w+u+\sqrt{w^2+h^2}-\sqrt{u^2+h^2}-\frac{w-u}{2c}h\Big).$$

Suppose that the zone is the rth, then

$$w=\frac{2c(r+1)}{n},\quad u=\frac{2cr}{n}.$$

Substituting these, neglecting the fourth power of $h\div u$, and introducing a subsidiary angle ϕ, such that

$$\frac{n(h+mk)}{c(2r+1)}=\tan\phi,$$

the effect of the attenuation

$$=\frac{3g}{4nm}\left\{-2+\sqrt{4(r+1)^2+(2r+1)^2\tan^2\phi}-\sqrt{4r^2+(2r+1)^2\tan^2\phi}-\frac{mk}{c}+\frac{n^2h^2}{4r(r+1)c^2}\right\}.$$

The pair of radicals in this expression

$$=\sqrt{1+(2r+1)^2\sec^2\phi+2(2r+1)}-\sqrt{1+(2r+1)^2\sec^2\phi-2(2r+1)}$$

$$=\frac{2(2r+1)}{(1+(2r+1)^2\sec^2\phi)^{\frac{1}{2}}}+\frac{(2r+1)^3}{(1+(2r+1)^2\sec^2\phi)^{\frac{5}{2}}}+\ldots \text{ by expansion}$$

$$=2\cos\phi\left(1-\frac{\cos^2\phi}{2(2r+1)^2}\right)+\frac{\cos^5\phi}{(2r+1)^2}+\ldots$$

$$=2\cos\phi-\frac{\cos^3\phi-\cos^5\phi}{(2r+1)^2}\text{ nearly }=2\cos\phi-\frac{2\cos\phi-\cos3\phi-\cos5\phi}{16(2r+1)^2}.$$

substituting this in the expression for the effect of the attenuation, and adding it to the vertical attraction of the mass above the sea-level given in formula (3),

$$\text{Resultant Vertical Attraction for the zone}=\frac{3g}{2mn}\,\frac{\beta}{360}\,\mathrm{R},$$

where β is the angular extent, at the station, of the part of the zone on which attracting matter stands, at the height k; and R is given by the following formula:—

$$\mathrm{R}=-1+\cos\phi-\frac{2\cos\phi-\cos3\phi-\cos5\phi}{32(2r+1)^2}-\frac{n^2}{8c^2}\,\frac{m(k^2-2hk)-h^2}{r(r+1)},\text{ and }\tan\phi=\frac{h+mk}{2r+1}\,\frac{n}{c}.$$

The expression for R may be somewhat simplified for zones beyond a certain distance. For when ϕ is sufficiently small to allow of its fourth power being neglected,

$$\phi^2=\frac{n^2}{c^2}\left(\frac{h+mk}{2r+1}\right)^2,$$

and the part of R depending on ϕ becomes by expansion

$$-\frac{\phi^2}{2}\left(1+\frac{1}{(2r+1)^2}\right)+\frac{\phi^4}{24}\left(1+\frac{22}{(2r+1)^2}\right).$$

Neglecting ϕ^4, and substituting for ϕ^2, this becomes

$$-\frac{n^2}{2c^2}\left(\frac{h+mk}{2r+1}\right)^2\left(1+\frac{1}{(2r+1)^2}\right),$$

and

$$\mathrm{R}=-\frac{n^2}{2c^2}\left\{\frac{(h+mk)^2}{(2r+1)^2}\left(1+\frac{1}{(2r+1)^2}\right)+\frac{m(k^2-2kh)-h^2}{4r(r+1)}\right\}.$$

In order to ascertain for what zones this simpler formula for R may be used, I observe that in the final result decimals are to be retained to the 7th place in the ratio of vertical attraction to gravity. Hence $3\beta R \div 720nm$ must be calculated to seven places of decimals. The largest value β will have is 149°. Hence in $447R \div 720nm$ a quantity as small as 0·0000001 must be retained, or a quantity in R as small as $0·000000161nm$, or say $0·00000016nm$, must be retained. Hence the neglected term

$$\frac{\phi^4}{24}\left(1+\frac{22}{(2r+1)^2}\right) \text{ or } \frac{n^4}{24c^4}\left(\frac{h+mk}{2r+1}\right)^4\left(1+\frac{22}{(2r+1)^2}\right) \text{ must be } < 0·00000016nm,$$

or

$$2r+1 > \frac{n(h+mk)}{174}\sqrt[4]{\frac{1}{nm}\left(1+\frac{22}{(2r+1)^2}\right)}.$$

When numerical values are given to the quantities involved it will be easy to find the least integral value of r which satisfies this condition; that value of r shows the first zone for which the second form of R can be used.

18. I purpose making, as I have already said, $n=160$. The several formulæ now calculated in the last paragraph I gather together and write down here, n being put $=160$.

$$\text{Resultant Vertical Attraction for zone} = \frac{g\beta}{38400}\frac{R}{m}. \quad \quad (4)$$

$$R=-1+\cos\phi-\frac{2\cos\phi-\cos 3\phi-\cos 5\phi}{32(2r+1)^2}-0·00020\,\frac{m(k^2-2kh)-h^2}{r(r+1)}, \quad \quad (5)$$

$$\tan\phi=0·04045\,\frac{h+mk}{2r+1},$$

or

$$R=-0·00082\left(\frac{(h+mk)^2}{(2r+1)^2}+\frac{(h+mk)^2}{(2r+1)^4}+\frac{m(k^2-2kh)-h^2}{4r(r+1)}\right), \quad \quad (6)$$

when

$$2r+1 \text{ is} > \frac{h+mk}{3·8}\left[\frac{1}{m}\left(1+\frac{22}{(2r+1)^2}\right)\right]^{\frac{1}{4}}. \quad \quad (7)$$

§ 5. *Numerical application of the formulæ to find the "Resultant Vertical Attraction" of the Himalayas upon Stations of the Indian Arc of Meridian through Cape Comorin.*

19. The formulæ of the last paragraph I shall now apply to find the resultant vertical attraction of the Himalayas at the three nearest of the stations I have entered in the Table in § 1, viz. Kaliana, Kalianpur, Damargida. The attraction at the rest can be found more simply. I shall take two cases of m, viz. $m=50$ and $m=100$.

FIRST, $m=50$.

Station Kaliana.

$$h=0\cdot1564,\quad k=2\cdot8409,\quad h+mk=142\cdot202.$$
$$h^2=0\cdot02446,\quad k^2-2kh=7\cdot18208,\quad (h+mk)^2=20220\cdot84.$$

By (7) we have $2r+1=15$, $r=7$. Hence formula (5) must be used up to the 6th zone; after that formula (6) up to the 17th or last.

$$\tan\phi=0\cdot04045\frac{h+mk}{2r+1}=\frac{5\cdot75207}{2r+1}.$$

r.	$\tan\phi$.	ϕ.	$\cos\phi$.	$2\cos\phi-\cos3\phi-\cos5\phi=$	P.
2	1·15041	49° 0′	0·65606	1·31212+0·83867+0·42262	2·57341
3	0·82172	39 25	0·77255	1·54510+0·47332+0·55757	2·57599
4	0·63912	32 35	0·84261	1·68522+0·13485+0·95588	2·77595
5	0·52273	27 36	0·88620	1·77240−0·12533+0·74314	2·39021
6	0·44247	23 52	0·91449	1·82898−0·31565+0·48989	2·00322

By (4) and (5) $R=-1+\cos\phi-\dfrac{P}{32(2r+1)^2}-\dfrac{0\cdot07182}{r(r+1)}$, Resultant$=\dfrac{\beta}{1920000}Rg$.

r.	Values of R.	β.	Resultants.
2	−0·34394−0·00322−0·01197=−0·35913	74°	−0·0000138 g
3	−0·22745−0·00189−0·00598=−0·23532	110	−0·0000135 „
4	−0·15739−0·00108−0·00359=−0·16206	127	−0·0000107 „
5	−0·11380−0·00063−0·00239=−0·11682	137	−0·0000083 „
6	−0·08551−0·00037−0·00171=−0·08759	144	−0·0000066 „

$$\text{By (6) } R=\frac{-16\cdot58107}{(2r+1)^2}-\frac{16\cdot58107}{(2r+1)^4}-\frac{0\cdot07362}{r(r+1)}.$$

7	−0·07369−0·00033−0·00131=−0·07533	149	−0·0000058 g
8	−0·05737−0·00020−0·00102=−0·05859	113	−0·0000034 „
9	−0·04593−0·00013−0·00082=−0·04688	88	−0·0000021 „
10	−0·03760−0·00009−0·00067=−0·03836	79	−0·0000016 „
11	−0·03134−0·00006−0·00056=−0·03196	66	−0·0000011 „
12	−0·02653−0·00004−0·00047=−0·02704	56	−0·0000008 „
13	−0·02274−0·00003−0·00040=−0·02317	109	−0·0000013 „
14	−0·01972−0·00002−0·00035=−0·02009	87	−0·0000009 „
15	−0·01725−0·00002−0·00031=−0·01758	79	−0·0000007 „
16	−0·01523−0·00002−0·00027=−0·01552	61	−0·0000005 „
17	−0·01354−0·00001−0·00024=−0·01379	37	−0·0000003 „
	Resultant at Kaliana for the Himalayas		−0·0000714 g

Station Kalianpur.

$$h=0·3363,\ k=2·84091,\ h+mk=142·3818.$$
$$h^2=0·11310,\ k^2-2kh=6·15992,\ (h+mk)^2=20272·58.$$

By (7) we have $2r+1=15$, and $r=7$. Hence only formula (6) need be used, because the first zone in this case is the 8th.

$$R=-\frac{16·62352}{(2r+1)^2}-\frac{16·62352}{(2r+1)^4}-\frac{0·06312}{r(r+1)}.$$

r.	Values of R.	β.	Resultants.
8	−0·05752 − 0·00020 − 0·00088 = −0·05860	45°	−0·0000014 g
9	−0·04605 − 0·00013 − 0·00063 = −0·04681	62	−0·0000015 „
10	−0·03769 − 0·00009 − 0·00057 = −0·03835	79	−0·0000016 „
11	−0·03141 − 0·00006 − 0·00048 = −0·03195	88	−0·0000015 „
12	−0·02660 − 0·00004 − 0·00040 = −0·02704	92	−0·0000013 „
13	−0·02280 − 0·00003 − 0·00035 = −0·02318	97	−0·0000012 „
14	−0·01976 − 0·00002 − 0·00030 = −0·02008	76	−0·0000008 „
15	−0·01730 − 0·00002 − 0·00026 = −0·01758	58	−0·0000005 „
16	−0·01526 − 0·00001 − 0·00023 = −0·01550	47	−0·0000004 „
17	−0·01357 − 0·00001 − 0·00021 = −0·01379	46	−0·0000003 „
18	−0·01214 − 0·00001 − 0·00018 = −0·01233	35	−0·0000002 „
19	−0·01093 − 0·00001 − 0·00017 = −0·01111	76	−0·0000004 „
20	−0·00989 − 0·00001 − 0·00015 = −0·01005	46	−0·0000002 „
21	−0·00899 .... − 0·00014 = −0·00913	6	....
22	−0·00821 .... − 0·00012 = −0·00833	6	. .
	Resultant at Kalianpur for the Himalayas		−0·0000113 g

Station Damargida.

$$h=0·3663,\ k=2·84091,\ h+mk=142·4118.$$
$$h^2=0·13418,\ k^2-2kh=5·98947,\ (h+mk)^2=20281·13.$$

By (7) we have $2r+1=15$, $r=7$. Hence we need use only the formula (6).

$$R=-\frac{16·63052}{(2r+1)^2}-\frac{0·06136}{r(r+1)}.$$

r.	Values of R.	β.	Resultants.
17	− 0·01358 − 0·00020 = − 0·01378	22°	−0·0000002 g
18	− 0·01215 − 0·00018 = − 0·01233	41	−0·0000003 „
19	− 0·01093 − 0·00016 = − 0·01109	50	−0·0000003 „
20	− 0·00989 − 0·00015 = − 0·01004	62	−0·0000003 „
21	− 0·00899 − 0·00013 = − 0·00912	68	−0·0000003 „
22	− 0·00865 − 0·00012 = − 0·00877	73	−0·0000003 „
23	− 0·00753 − 0·00011 = − 0·00764	58	−0·0000002 „
24	− 0·00693 − 0·00010 = − 0·00703	46	−0·0000002 „
25	− 0·00639 − 0·00009 = − 0·00648	32	−0·0000001 „
26	− 0·00592 − 0·00008 = − 0·00600	9	....
27	− 0·00550 − 0·00008 = − 0·00558	12	....
28	− 0·00512 − 0·00008 = − 0·00520	12	....
29	− 0·00477 − 0·00007 = − 0·00484	12	....
	Resultant at Damargida for the Himalayas...		−0·0000022 g

SECONDLY, $m=100$.

Station Kaliana.

$$h=0{\cdot}1564,\ k=2{\cdot}84091,\ h+mk=284{\cdot}247.$$

$$k^2=0{\cdot}02446,\ k^2-2kh=7{\cdot}18208,\ (h+mk)^2=80790{\cdot}68.$$

By (7) we have $2r+1=25$, $r=12$. Hence up to the 11th zone we must use formula (5) and after that (6).

$$\tan\phi=0{\cdot}04045\,\frac{h+mk}{2r+1}=\frac{11{\cdot}49779}{2r+1}.$$

r.	$\tan\phi$.	ϕ.	$\cos\phi$.	$2\cos\phi-\cos 3\phi-\cos 5\phi=$	P.
2	2·29956	66° 30′	0·39875	0·79750+0·94264−0·88701	0·85313
3	1·64254	58 40	0·52002	1·04004+0·99756−0·39608	1·64152
4	1·27753	51 53	0·61726	1·23452+0·91104+0·18367	2·32923
5	1·04525	46 16	0·69130	1·38260+0·75241+0·62479	2·75980
6	0·88444	41 29	0·74915	1·49830+0·56569+0·88768	2·95167
7	0·76652	37 28	0·79371	1·58742+0·33107+0·99182	2·91031
8	0·67634	34 4	0·82839	1·65678+0·21132+0·98580	2·85390
9	0·60515	31 11	0·85551	1·71102+0·06192+0·91295	2·68589
10	0·54751	28 42	0·87715	1·75430−0·06802+0·80386	2·49014
11	0·49990	26 34	0·89441	1·78882−0·17880+0·67987	2·28989

By (4) and (5) $R=-1+\cos\phi-\frac{P}{32(2r+1)^2}-\frac{0{\cdot}14364}{r(r+1)}$, and Resultant $=\frac{\beta}{3840000}R$.

r.	Values of R.	β.	Resultants.
2	−0·60125−0·00106−0·02394=−0·62625	74°	−0·0000120 g
3	−0·47998−0·00105−0·01193=−0·49296	110	−0·0000141 ,,
4	−0·38274−0·00095−0·00728=−0·39097	127	−0·0000130 ,,
5	−0·30870−0·00072−0·00479=−0·31421	137	−0·0000112 ,,
6	−0·25085−0·00054−0·00342=−0·25481	144	−0·0000096 ,,
7	−0·20629−0·00045−0·00256=−0·20930	149	−0·0000081 ,,
8	−0·17161−0·00031−0·00199=−0·17391	113	−0·0000051 ,,
9	−0·14449−0·00025−0·00160=−0·14634	88	−0·0000034 ,,
10	−0·12285−0·00019−0·00131=−0·12435	79	−0·0000026 ,,
11	−0·10559−0·00014−0·00109=−0·10682	66	−0·0000018 ,,

$$\text{By (6) } R=-\frac{66{\cdot}24836}{(2r+1)^2}-\frac{66{\cdot}24836}{(2r+1)^4}-\frac{0{\cdot}14723}{r(r+1)}.$$

12	−0·10600−0·00017−0·00094=−0·10711	56	−0·0000016 g
13	−0·09087−0·00013−0·00081=−0·09181	109	−0·0000030 ,,
14	−0·07877−0·00010−0·00070=−0·07957	87	−0·0000020 ,,
15	−0·06894−0·00010−0·00061=−0·06965	79	−0·0000014 ,,
16	−0·06083−0·00005−0·00054=−0·06142	61	−0·0000010 ,,
17	−0·05509−0·00004−0·00048=−0·05561	37	−0·0000005 ,,
	Resultant at Kaliana for the Himalayas		−0·0000904 g

Station Kalianpur.

$$h=0\cdot3363,\ k=2\cdot84091,\ h+mk=284\cdot427.$$
$$h^2=0\cdot11310,\ k^2-2kh=6\cdot15992,\ (h+mk)^2=80809\cdot42.$$

By (7) we have, as before, $r=12$; and we must use formula (5) to the 11th zone, and after that formula (6).

$$\tan\phi=0\cdot04045\,\frac{h+mk}{2r+1}=\frac{11\cdot50587}{2r+1}.$$

r.	$\tan\phi$.	ϕ.	$\cos\phi$.	$2\cos\phi-\cos3\phi-\cos5\phi=$	P.
8	0·67681	34° 6′	0 82806	1·65612+0·21303+0·98629=	2·85544
9	0 60556	31 12	0·85536	1·71072+0·06279+0·91355=	2·68706
10	0·54790	28 43	0 87701	1·75402−0·06714+0 80472=	2·49160
11	0·50026	26 35	0·89428	1·79856−0·17794+0·68093=	2·30155

$$\text{By (5) } R=-1+\cos\phi-\frac{P}{32(2r+1)^2}-\frac{0\cdot12318}{r(r+1)}.$$

r.	Values of R.	β.	Resultants.
8	−0·17194−0·00031−0·00171=−0·17396	45°	−0·0000020 g
9	−0·14464−0·00024−0·00137=−0·14625	62	−0·0000024 ,,
10	−0·12299−0·00017−0·00112=−0·12428	79	−0·0000026 ,,
11	−0 10572−0 00014−0 00093=−0 10679	88	−0 0000024 ,,

$$\text{By (6) } R=-\frac{66\cdot26372}{(2r+1)^2}-\frac{66\cdot26372}{(2r+1)^4}-\frac{0\cdot12626}{r(r+1)}.$$

12	−0·10602−0·00017−0·00081=−0·10700	92	−0·0000026 g
13	−0·09090−0·00013−0·00069=−0 09172	97	−0 0000023 ,,
14	−0·07879−0 00009−0·00060=−0·07948	76	−0 0000016 ,,
15	−0·06895−0 00007−0·00053=−0·06955	53	−0·0000010 ,,
16	−0 06085−0·00005−0·00046=−0·06136	47	−0·0000008 ,,
17	−0 05409−0·00004−0·00041=−0 05454	46	−0·0000007 ,,
18	−0·04849−0 00004−0·00037=−0·04881	35	−0·0000005 ,,
19	−0 04357−0·00003−0·00033=−0·04393	76	−0 0000009 ,,
20	−0·03941−0 00002−0·00030=−0 03973	46	−0·0000005 ,,
21	−0 03584−0 00002−0 00027=−0·03613	6	−0·0000001 ,,
22	−0·03272−0 00002−0·00023=−0·03297	6	−0·0000001 ,,
	Resultant at Kalianpur for the Himalayas	....	−0·0000205 g.

Station Damargida.

$$h = 0{\cdot}3663,\ k = 2{\cdot}84091,\ h+mk = 284{\cdot}457.$$
$$h^2 = 0{\cdot}13418,\ k^2-2kh = 5{\cdot}98947,\ (h+mk)^2 = 80915{\cdot}78.$$

By (7) we have, as before, $r=12$; and therefore formula (6) may be used throughout.

$$\text{By (6) } R = -\frac{66{\cdot}35093}{(2r+1)^2} - \frac{0{\cdot}12278}{r(r+1)}.$$

r.	Values of R.	β.	Resultants.
17	$-$ 0·05416 $-$ 0·00040 $=$ $-$ 0·05456	22°	$-$0 0000003 g
18	$-$ 0·04847 $-$ 0 00036 $=$ $-$ 0·04883	41	$-$0·0000005 „
19	$-$ 0·04208 $-$ 0 00032 $=$ $-$ 0 04240	50	$-$0·0000006 „
20	$-$ 0·03947 $-$ 0·00029 $=$ $-$ 0·03976	62	$-$0·0000006 „
21	$-$ 0·03588 $-$ 0·00026 $=$ $-$ 0·03614	68	$-$0 0000006 „
22	$-$ 0·03276 $-$ 0·00024 $=$ $-$ 0 03300	73	$-$0·0000006 „
23	$-$ 0 03004 $-$ 0·00022 $=$ $-$ 0·03022	58	$-$0·0000005 „
24	$-$ 0·02763 $-$ 0 00020 $=$ $-$ 0 02783	46	$-$0·0000003 „
25	$-$ 0 02669 $-$ 0 00019 $=$ $-$ 0·02688	32	$-$0·0000002 „
26	$-$ 0·02362 $-$ 0 00018 $=$ $-$ 0·02380	9	$-$0 0000001 „
27	$-$ 0·02184 $-$ 0 00017 $=$ $-$ 0·02201	12	$-$0·0000001 „
28	$-$ 0·02042 $-$ 0 00016 $=$ $-$ 0·02058	12	$-$0·0000001 „
29	$-$ 0·01906 $-$ 0·00015 $=$ $-$ 0·01921	12	$-$0 0000001 „
	Resultant at Damargida for the Himalayas		$-$0·0000046 g

20. As the Resultant Vertical Attraction at Damargida, which is far from the Himalayas, is small, we may find that at the still further stations in the list, viz. Bangalore and Punnae and the others, by the law of the inverse cube, which I have proved in paragraph 17. The distance of Damargida from the centre of the tableland which I have taken to represent the Himalayas in this problem, is about 16°. Hence the Resultant Vertical Attraction caused by the Himalayas will be as follows:—

	$m=50$.	$m=100$		$m=50$.	$m=100$.
At Bangalore	$-$0·0000010 g	20 g	At Mangalore	$-$0·0000009 g	18 g
„ Punnae	$-$0 0000005 ..	11 „	.. Alleppy	$-$0·0000006 „	13 „
„ Cocanada	$-$0·0000019 „	38 ..	„ Minicoy	$-$0·0000005 „	11 „
„ Madras......	$-$0·0000010 „	20 „			

§ 6. *Calculation of the "Resultant Vertical Attraction" of the Sea at Stations on a Continent, on a Coast, or on an Island.*

21. I will suppose a straight coast-line, the sea-bottom shelving down and then rising again, so as practically to be equivalent to a uniform descent to a depth H at a distance U from the shore, and beyond that to have no sensible effect on the Resultant Vertical Attraction.

Let u be the horizontal distance from the shore, and z the depth of an elementary horizontal prism of sea-water, of indefinite length, running parallel to the coast-line; a the distance on the sea-level of a station in the interior of the continent on which the effect of the sea is to be found. Suppose the horizontal prism to consist of two parts of indefinite length, divided at the point opposite the station on the coast. The attraction

of each of these towards the point of division equals the mass of each divided by the product of the distances of its extremities from the station. Hence the deficiency of vertical attraction of the sea on the station in question

$$=-\frac{3g}{4c}\,0\cdot637\int_0^{\mathrm{U}}\!\!\int_0^z\frac{zdudz}{z^2+(u+a)^2}=\frac{-1\cdot9g}{8c}\int_0^{\mathrm{U}}\log_e\left(\frac{z^2+(u+a)^2}{(u+a)^2}\right)du.$$

Suppose matter equal in amount to the deficiency in the ocean is spread down uniformly through a depth $m\,.\,z$. Then

$$\text{Resultant Vertical Attraction}=\frac{1\cdot9g}{8c}\int_0^{\mathrm{U}}\left(\frac{1}{m}\log_e\frac{m^2z^2+(u+a)^2}{(u+a)^2}-\left(1+\frac{1}{m}\right)\log_e\frac{z^2+(u+a)^2}{(u+a)^2}\right)du$$

$$=\frac{1\cdot9g}{8c}\int_0^{\mathrm{U}}\frac{(m-1)z^2du}{(u+a)^2}\text{ nearly, }=\frac{1\cdot9g}{8c}\frac{\mathrm{H}^2}{\mathrm{U}^2}(m-1)\int_0^{\mathrm{U}}\frac{u^2}{(u+a)^2}\,du$$

$$=\frac{1\cdot9g}{8c}\frac{\mathrm{H}^2}{\mathrm{U}^2}(m-1)\mathrm{U}\left\{1-\frac{2a}{\mathrm{U}}\log_e\frac{\mathrm{U}+a}{a}+\frac{a}{\mathrm{U}+a}\right\}.$$

This formula I will apply to find the effect of the sea upon our stations. I will suppose that the bottom shelves down at the same angle on an average on the east and the west coasts, so as to make $\mathrm{H}\div\mathrm{U}=1\div600$, but that on the east coast $\mathrm{U}=600$, and on the west 900 miles.

The distances of the five stations from the east coast are about 0, 180, 330, 560, 840 miles; and from the west coast about 0, 190, 300, 480, 700 miles. The distances of Alleppy and Mangalore from the east coast are about 120 and 350 miles; and of Madras and Cocanada from the west coast about 350 and 560. For these the formula gives as follows:—

TABLE VI.

Stations.	East Coast, $m=50$.	East Coast, $m=100$.	West Coast, $m=50$.	West Coast, $m=100$.	Totals, $m=50$.	Totals, $m=100$.
Punnae	0·0000050 g	0·0000100 g	0·0000077 g	0·0000154 g	0·0000127 g	0·0000254 g
Bangalore	0·0000018 „	0·0000036 „	0·0000032 „	0·0000064 „	0·0000050 „	0·0000100 „
Damargida	0·0000010 „	0·0000020 „	0·0000024 „	0·0000048 „	0·0000034 „	0·0000068 „
Kalianpur	0·0000006 „	0·0000012 „	0·0000016 „	0·0000032 „	0·0000022 „	0·0000044 „
Kaliana	0·0000003 „	0·0000006 „	0·0000012 „	0·0000024 „	0·0000015 „	0·0000030 „
Punnae	0·0000050 „	0·0000100 „	0·0000100 „	0·0000200 „	0·0000127 „	0·0000254 „
Alleppy	0·0000022 „	0·0000044 „	0·0000044 „	0·0000088 „	0·0000099 „	0·0000198 „
Mangalore	0·0000010 „	0·0000020 „	0·0000020 „	0·0000040 „	0·0000087 „	0·0000174 „
Madras	0·0000050 „	0·0000100 „	0·0000100 „	0·0000200 „	0·0000072 „	0·0000144 „
Cocanada	0·0000050 „	0·0000100 „	0·0000100 „	0·0000200 „	0·0000066 „	0·0000132 „

22. I will now take the case of an Island, and suppose it to be in the form of a cylinder of radius a in a sea of uniform depth h. Then by formula (1) we easily obtain the following result:—

Resultant Vertical Attraction at the middle of the Island

$$=\frac{1\cdot9g}{4c}\left\{\frac{u+(1+m)h-\sqrt{u^2+(1+m)^2h^2}}{m}-\frac{1+m}{m}(u+h-\sqrt{u^2+h^2})\right.$$

$$\left.-\frac{a+(1+m)h-\sqrt{a^2+(1+m)^2h^2}}{m}+\frac{1+m}{m}(a+h-\sqrt{a^2+h^2})\right\}$$

$$=\frac{1\cdot9g}{4c}\left\{(a+h-\sqrt{a^2+h^2})\frac{1+m}{m}-\left(a-\frac{a^2}{2(m+1)h}\right)\frac{1}{m}\right\},\ u\text{ being large.}$$

Minicoy Island is about 250 miles west of Punnae. Its average radius is about 2·5 miles; and three miles from the shore the sea is about 300 fathoms, or about one-third of a mile. If, then, to make our cylinder accord with this case, right angles being cut down, we make $a=5$ and $h=1$, we have

	$m=50.$	$m=100.$
Minicoy Island	$0{\cdot}0000988\,g$	$0{\cdot}0001046\,g$

§ 7. *Application of these results to test the truth of the author's hypothesis regarding the Constitution of the Earth's Crust.*

23. In the following Table I bring together the several data and results of this paper in order to compare them.

TABLE VII.

Stations.	Relative gravity freed from effects of height and latitude.	Correction for Local Attraction. By Dr. Young's formula.	By the present hypothesis. $m=50$. Plains.	Himalayas.	Sea.	$m=100$. Plains.	Himalayas.	Sea.	Differences of Gravity. Actual effects of local attraction.	Corrections of local attraction. By Dr Young.	By this hypothesis. $m=50$.	$m=100$
										A.	B.	C.
Indian Arc Stations.												
Punnae ..	1−0·0006094	−14	..	+5	−127	......	+11	−254	......	......	......	....
Bangalore .	1−0·0005710	−960	−552	+10	−50	−1104	+20	−100	+384	−946	−471	−941
Damargida .	1−0·0006417	−617	−241	+22	−34	−482	+46	−68	−323	−603	−132	−261
Kalianpur . .	1−0·0005753	−563	−215	+113	−22	−430	+205	−44	+341	−549	−3	−26
Kaliana	1−0·0006801	−264	−45	+714	−15	−90	+904	−30	−707	−250	+776	+1027
Coast Stations.												
Punnae	1−0·0006094	−14		+5	−127	....	+11	−254	.. .. ..	......	....	....
Alleppy	1−0·0005792	−2	. . .	+6	−99	..	+13	−198	+302	+12	+29	+58
Mangalore	1−0·0006260	−2	.	+9	−87	..	+18	−174	−166	+12	+44	+87
Madras	1−0·0006291	−9	. ..	+10	−72	.. ..	+20	−144	−197	+5	+59	+119
Cocanada	1−0·0005952	−3	. . .	+19	−66	.	+38	−132	+142	+11	+74	+149
Ocean Station.												
Minicoy	1−0·0005200	−2	. ..	+5	−988	. ...	+11	−1046	+894	+12	−863	−792

Under the heading "Correction for Local Attraction" Dr. YOUNG'S is found by taking one third for the "Reduction for Height" in Table III. The three columns for Plains, Himalayas, and Sea are taken from paragraphs 15, 19, 21, 22. These three columns, in each case of m, are added together, and after the values for Punnae are subtracted the results are recorded in columns B and C.

In order the better to compare these results with the quantities which are to be accounted for, I compile the following Table from the columns A, B, C of Table VII. The sign + indicates a force acting downwards. In all the columns the numbers are the last figures of seven places of decimals in the ratio to gravity, the decimal point and ciphers being omitted for convenience.

TABLE VIII.

Stations.	Differences of Gravity.			
	Relative effects of local attraction deduced from Pendulum Observations.	Residual errors after correction by the methods of		
		Dr. YOUNG.	This hypothesis.	
			$m=50$.	$m=100$.
Indian Arc Stations.				
Punnae	....	....	....	....
Bangalore	+384	−562	− 78	−557
Damargida	−323	−926	−455	−584
Kalianpur	+341	−208	+338	+315
Kaliana	−707	−957	+ 69	+320
Coast Stations.				
Punnae	....	....	....	....
Alleppy	+302	+314	+331	+360
Mangalore	−166	−154	−122	− 79
Madras	−197	−192	−138	− 78
Cocanada	+142	+153	+216	+291
Ocean Station.				
Minicoy	+894	+ 906	+ 31	+102

24. This Table contains all the final results necessary to enable us to judge of the truth of the hypothesis which I advocate in this paper, and which I will now discuss.

From the first column of numbers we learn that, according to Pendulum Observations, gravity at the four stations I have chosen north of Punnae, when every cause of variation is eliminated except Local Attraction, is alternately in excess and defect of that at Punnae, the first (at Bangalore) being in excess. In the whole range of stations in my list, the effect on gravity at Kaliana and Minicoy is the most important—the first in defect, the last in excess.

I will now consider how far the effects exhibited in this Table are accounted for on Dr. YOUNG's (or the usually received) method, and my own.

The Coast stations shall first be taken. In these neither method has much success in accounting for the local attraction. A survey of the form of the land and sea-bottom near those places would very likely change this result. The local vertical attraction at Alleppy is the greatest, and is in excess. This may be accounted for probably in part by the sea between it and Minicoy being *deeper* than the general slope which I have assumed in the calculation, as will be understood when I come to refer to Minicoy. None of the local vertical attractions at the coast stations are *very* large.

Next let us take the Indian Arc and Ocean Stations. The second column of numbers shows that Dr. YOUNG's correction, so far from improving matters, introduces very large residual errors, and those on the Arc are all in the same direction. And I may add that, if his method of allowing for the effect of all superficial causes of disturbance is fully carried out, a negative quantity must be added to correct the effect of the Himalayas at Kaliana, as they are below its horizon, and a positive quantity at Punnae for the sea. And when all are referred to Punnae the whole series of numbers for this mode of correction, which disregards the state of the interior, is even greater than before. The third and fourth columns show the effect of the method of this paper, in the two cases of the

compensation below running down 50 and 100 times the heights above the sea-level and depths of the sea below that level. The first of these gives the best results. This, then, I shall discuss. The only residual errors which are large are those at Damargida, showing a defect in gravity at that place compared with Punnae, and at Kalianpur in excess. On examining the numbers in the last column of Table I., paragraph 3, it will be seen that there is a very small horizontal force at Damargida, and at the stations next to it, north and south, the horizontal force is directed *from* Damargida in both cases. This indicates a deficiency of matter in the neighbourhood of that station, which accords with the residual error in my last Table VIII. Also in Table I. we see that there is a horizontal force from Kalianpur towards Takal Khera, and also from Takal Khera towards Kalianpur. This indicates an abnormal excess of matter between those stations, and nearer to Kalianpur than to Takal Khera, as the force at the former is the larger of the two. This accords with an excess of matter near to Kalianpur, which is indicated by an excess of gravity shown in my Table VIII. At Bangalore the Table indicates a slight abnormal deficiency of matter. But Table I. shows a north horizontal force at Dodagoontah. more than four miles south of Bangalore. This would seem to imply that the slight deficiency of matter which causes the defect of gravity at Bangalore runs further south of Dodagoontah than it does north. It is generally difficult to compare the horizontal and vertical effects of a hypothetical excess or defect of matter, as all depends upon its situation relatively to the stations. Thus an excess or defect immediately below a station will not affect the plumb-line, whereas a defect or excess near the surface, and between stations and far from both, will not affect the vertical force materially. On the whole the peculiarities at Damargida, Kalianpur, and Bangalore seem to be sufficiently accounted for. The other residual errors in Table VIII., at Kaliana and Minicoy, are so small that they may be considered evanescent; and seeing that the local attractions at those places, as shown in the first column of Table VIII., are very large, this result speaks decidedly in favour of the hypothesis. The fact that the anomalous circumstance is accounted for by the hypothesis, that a station out at sea exhibits a considerable increase in gravity, although surrounded by the ocean, which has a deficiency of attracting matter, is a very strong argument in favour of the hypothesis. Were the exact contour of the continent and the neighbouring sea-bed better known, the application of this method might be carried out more completely. As it is, however, what remains unexplained is not important.'

When we remember, then, that the calculations have been conducted in the particular case of the distribution of matter, in excess and defect, being *uniform*, whereas in the contracting of the mass this is not at all likely to have strictly been the case, and observe the way in which the hypothesis nearly explains the errors, while the usually received method does not do so at all, but indeed aggravates them considerably, I think the hypothesis may be regarded as receiving support from the Pendulum Observations recently made on the extensive continent, coast, and (in one important instance) the neighbouring sea of India.

XIV. *On the Mineral Constituents of Meteorites.* By Nevil Story-Maskelyne, *M.A., F.R.S., Professor of Mineralogy, Oxford, and Keeper of the Mineral Department, British Museum.*

Received November 3, 1870,—Read January 26, 1871.

XII. *The Breitenbach Meteorite.*

The Siderolite of Breitenbach was acquired for the British Museum in the year 1863. It was found (in 1861) at Breitenbach in Bohemia, at a spot not very far distant from the Saxon frontier, or indeed from Rittersgrün, in Saxony, a place in which a very fine mass, that bears a close resemblance to the Siderolite of Breitenbach, was almost contemporaneously found. A little way to the west of the centre of the line joining Rittersgrün and Breitenbach lies Steinbach, a village in the environs of Johanngeorgenstadt. near Schwartzenberg; and here in 1751 was also found a mixed meteoric mass in which, as in the two already mentioned, iron, sponge-like in its structure, encloses siliceous minerals that do not present a familiar aspect. The three meteorites are, in fact, so similar to one another and so dissimilar to any others in European collections, that there can be little doubt they belonged originally to the same meteoric fall.

Stromeyer* in the year 1825 examined a siderolite in which he found as much as 61·88 per cent. of silica. This remarkable result, together with the numbers of his analysis, he interpreted as indicating the presence of a magnesian trisilicate, probably meaning thereby a sesquisilicate (magnesium epideutosilicate). The specimen which he analyzed he described as coming from Grimma, in Saxony. This specimen was, in fact, a portion of a mass preserved in the collection of the Duke of Gotha, and doubtless believed by Stromeyer to be a portion of a stone which was known to have fallen in the middle of the sixteenth century in a wood near Naunhof in the neighbourhood of Grimma. Chladni†, however, held this view to be untenable, grounding his opinion on the completeness of the meteorite preserved at Gotha, both as regards its form and its crust, while he adds that the Naunhof mass must have been far too great to allow of its being transported, and, indeed, that it had never been rediscovered. It is in every way probable that the material Stromeyer really had to work upon was from a Saxon locality, and in fact a specimen from a fall, to which the Rittersgrün and Breitenbach siderolites belong. Breithaupt‡ believes the fall in question to have been the "*Eisenregen*" which occurred at Whitsuntide, 1164, in Saxony, when a mass of iron fell in the town of Meissen§.

An inspection of a polished surface of either of these masses reveals the iron in patches of irregular form, which exhibit the characteristic crystalline structure of meteoric irons

* Pogg. Ann. iv. p. 195.

† Feuer-Meteore, pages 326 & 212.

‡ Berg. und Hütt. Zeitung, xxi. p. 322.

§ Feuer-Meteore, p. 198.

when etched. The interspaces are partly filled by meteoric pyrites (troilite) in small patches, recognizable by its pinchbeck brown colour, the rest of the surface being occupied by a greenish and greyish-brown crystalline magma. It is of the ingredients of the last-mentioned portion of the meteorite that I shall first speak. On treating the whole mass with mercuric chloride at 100° for some hours the iron and the troilite are dissolved, and the magma before alluded to remains unattacked. But it has now lost its compound structure, and is found to consist of three substances:—1, highly crystalline, bright green, or else greenish-yellow grains; 2, rusty brown, sometimes nearly black, sometimes also nearly colourless grains of a mineral that presents crystalline features, but on which definite crystalline planes are of great rarity; and 3, crystalline grains of chromite.

The first of these three minerals proved to be a ferriferous enstatite, or bronzite, the second is a mineral to which I do not at present assign a name, for it corresponds in all respects, except its crystalline form, with the tridymite of Professor VOM RATH. In respect of their forms, however, it is difficult to suppose that the two minerals are identical.

XIII. *Bronzite of the Breitenbach Siderolite.*

My friend Professor VON LANG measured crystals of the bronzite of the Breitenbach meteorite at the British Museum so long ago as 1863, and during last year he published his results*,—results that were mineralogically important as affording for the first time satisfactory and complete data for the crystallography of a rhombic mineral with the formula of an enstatite. This investigation was made exceptionally difficult by the very partially developed or merohedral character of the crystals on which Professor VON LANG had to experiment. A similar difficulty attended the crystallography of the silica of this meteorite. I need only recapitulate of Dr. Von LANG'S results the elements and some of the important angles of the crystal.

$$\textit{Elements}:\!-\!a:b:c=0{\cdot}89568:0{\cdot}84960:1,$$

which give the following angles by calculation:—

$$\begin{aligned} 110.010 &= 44^\circ\ 8' \\ 101.100 &= 41\ 11 \\ 011.010 &= 40\ 16 \end{aligned}$$

The mineral often presents itself in little spherules, invariably green in their tint and crystalline in their structure, as revealed by their optical characters, and sometimes, but very rarely, carrying here and there a crystal face. In fact the faces thus presenting themselves seem to do so almost fortuitously, and on the grains, which present a nearer approach to a true crystalline superficies, the faces that are developed exhibit very little of the symmetrical correspondence with other faces, or of the prevalence of those of any special forms, such as is ordinarily met with in crystals.

The specific gravity of this mineral is 3·238, that of the silicates in the Steinbach siderolite, as determined by STROMEYER, having been 3·276, and as estimated by RUMLER 3·23. The hardness is 6.

* Bericht der Akad. Wiss. Wien, Bd. 59, ii. p. 848.

The blackened aspect of some of the bronzite was due to a mere superficial coating of iron oxide, arising doubtless from the oxidation of a portion of the nickeliferous iron. It was invariably found that this film was easily removed by hydrogen chloride, leaving the bronzite of a bright green colour, and that the action of the acid on the mineral extended no further.

Two analyses of this mineral were made, the one by the hydrogen fluoride method of distillation*, the other by fusion with mixed alkaline carbonates, and the results were as follow:—

	I.	II.	Mean.	Oxygen.
Silicic acid	56·101	56·002	56·051	29·89
Magnesium oxide . .	30·215	31·479	30·847	12·34
Iron protoxide . . .	13·583	13·295	13·439	2·97
	99·899	100·776	100·337	

These numbers correspond very closely with the formula $(Mg_{\frac{4}{5}}\,Fe_{\frac{1}{5}})\,Si\,O_3$.

XIV. *Silica crystallized in the Rhombic System, as a Constituent of the Breitenbach Siderolite.*

It has already been stated that the second mineral associated with the bronzite in this meteorite is free silica, possessing the lighter specific gravity presented by quartz after fusion, and crystallized in forms that belong to the orthorhombic system. To this mineral, which is distinct in its system and forms from the tridymite of VOM RATH, I propose to give the name Asmanite A'sman, being the Sanscrit term (corresponding to the Greek ἄκμων) for the thunderbolt of Indra. In bulk it forms about one-third of the mass of mixed siliceous minerals. The grains of this mineral are found mixed with those of the bronzite after the iron, the troilite, and the chromite have been removed. They are very minute and much rounded, and, though entirely crystalline, they very rarely indeed present faces that offer any chance for a result with the goniometer; indeed out of the several thousand of these little grains comprised in some two grammes that were isolated of the mineral, it was only possible to find with a lens about a dozen specimens with sufficiently distinct crystallographic features; and of these only four or five proved to be available for examination and comparison. In several, however, the optic axes were plainly to be distinguished when properly examined with a NÖRRENBERG's polarizing microscope; and by this means the angles given by planes belonging to zones otherwise too incomplete for a reliable result were brought into comparison on different crystals.

Fortunately one minute crystal was met with in which the consecutive planes in half of the zone [1 0 0, 0 0 1] were complete enough to give reliable data for two of the parametral ratios, while the planes of the form (1 1 0) in the zone [0 0 1, 0 1 0] were all present, and one of them sufficiently brilliant to give an image by reflection.

This crystal, designated by the letter C in the subjoined Table (p. 363), enabled me to make use with confidence of the approximate measurements obtained on the other crystals, and the more so as in the polariscope it was easy to recognize in the normal to the plane 1 0 0 the first mean line of the optic axes.

* Philosophical Transactions, 1870, p. 189.

reflecting plane $\bar{1}\,1\,0$ on the crystal C. In fact the faces of the form (1 1 0), though parallel to what appears to be a somewhat difficult cleavage, never present a good even reflecting surface, being generally more or less conchoidal when cleavage-surfaces, and dull when natural faces.

The cleavage-plane 0 0 1 has a vitreous lustre; the lustre on the planes of the forms 1 0 0 and 1 0 1, as also of the rounded surface in the zone with them, is usually of a resinous character, strongly recalling that of opal.

It has already been observed that the faces of the octaid forms are almost invariably rounded. Fair approximate measurements, however, of three of the faces in the zone [0 0 1, 1 1 0] on the crystal C were obtained; and one octaid plane on that crystal, repeated in two octants, gave measurements that accord fairly with the somewhat complex symbol 5 4 8. This face, though not giving a reflected image, is the best octaid plane upon the crystal.

That the mineral belongs to the rhombic and not to a uniaxal system is emphatically evidenced, independently of these measurements, by its optical characters, as shown in its very distinct and widely separated optic axes. As has been said, the first mean line is the normal of the face 1 0 0, that to face 0 0 1 is the second mean line. The first mean line is parallel to the axis of least optical elasticity, so that the crystal is positive in its optical character. The apparent angle, as measured in air, of the optic axes was approximately determined as 107° to 107° 30′. The axes for the red rays are slightly more dispersed than those for the blue.

The crystalline grains which constitute this ingredient of the meteorite, when first obtained, are of a rusty brown and sometimes even black colour; treatment for a short time with dilute hydrogen chloride, however, entirely removes this iron stain and leaves the granules in a state of colourless purity, in which state they are readily distinguished from the grains of the accompanying bronzite.

The specific gravity of the mineral gave the number 2·245. Its hardness is 5·5.

Two analyses were made by different methods, and the results are given below.

I. 0·3114 substance, distilled with pure hydrogen fluoride, gave 1·1136 gramme of potassium fluosilicate, 0·0035 gramme iron oxide, 0·0018 calcium oxide, and 0·0132 gramme magnesium phosphate.

These determinations denote the following percentages:—

Silicic acid	97·43
Iron oxide	1·124
Calcium oxide	0·578
Magnesium oxide	1·509
	100·641

II. 0·2653 gramme of carefully selected substance, evaporated with an excess of ammonium fluoride, left 0·0021 gramme residue, chiefly iron oxide.

This determination denotes the following percentage composition:—

Silicic acid	[99·21]
Iron oxide &c.	0·79
	100·00

Besides the distinct cleavage parallel to the plane 001 already alluded to, and the other, less distinct, parallel to the planes of the form (1 0 1), there seem also to be divisional planes or, rather, surfaces along which the crystals break up with the greatest facility; even drying them on blotting-paper proving often sufficient to destroy the integrity of specimens that might otherwise seem to promise good results to the goniometer.

Partsch*, in his description of the Vienna Collection of Meteorites, identifies as a specimen of the Steinbach siderolite a fragment with a label, "Native iron, jagged and hackly, with quartz in grains and a yellow fluor-spar" (*gediegenes, zahnicht und zackicht gewachsenes Eisen mit körnichtem Quarz und gelblichtem Flussspath*).

Breithaupt†, in his paper describing the Rittersgrün Siderolite, makes its chief silicate to be peridot. It is doubtless bronzite. In addition to troilite and schreibersite, he records the presence of "another mineral the composition of which is not yet determined."

It should be mentioned that, with a view to test the relative solvent action of alkaline carbonates on quartz and the meteoric silica, weighed portions of each were digested with a ten per cent. solution of sodium carbonate for ten hours at 100° C. under precisely similar conditions. Of the quartz 7·843 per cent. had dissolved, of the Breitenbach silica 9·437 per cent.

XV. *Iron of the Breitenbach Siderolite.*

Of the other minerals forming the mass of this meteorite, namely troilite, some little schreibersite, chromite, present only in minute quantities, a crystal of which, however, was measured and gave angles corresponding to a regular octahedron, and, finally, the nickeliferous iron, which forms the sort of sponge-like skeleton that unites the whole, the last alone demands detailed investigation.

Two analyses, the former by the lead process, the latter by the barium method, gave the following results:—

	I.	II.	Mean.	Equivalent ratios.
Iron . . .	89·975	90·878	90·426	3·229
Nickel . .	9·642	8·927	9·284	0·314
Cobalt . .	0·383	0·195	0·29	0·01
	100·000	100·000	100·000	

The equivalent ratios, it will be seen, differ but slightly from Fe : (Ni, Co)=10 : 1. Some small amount of the above iron will have been present as troilite, which dissolved with the nickeliferous iron in the mercuric chloride. Copper occurs in the Breitenbach iron, but only as a trace.

Rube‡, who analyzed the Rittersgrün iron, found iron 87·31, nickel 9·63, and cobalt 0·58 per cent. The material appears not to have been entirely free from silicate.

* Die Meteoriten im k. k. Hof-Mineralien-Kabinette. Wien, 1843, p. 95.

† Berg. und Hütt. Zeitung, xxi. p. 321.

‡ Ibid. p. 72.

XVI. *The Shalka Aërolite.*

In the year 1860 the Ritter von HAIDINGER* first gave an account of the very remarkable meteorite that fell at Shalka, in Bancoorah, Bengal, on the 30th of November, 1850. The ash-like dark grey substance which forms the mass of this meteorite was described by the illustrious Viennese authority as a new silicate, to which he gave the name of Piddingtonite. He was, in fact, led to do this in consequence of an analysis of the mineral by K. VON HAUER, which yielded oxygen ratios corresponding to "a compound of bisilicate and trisilicate of iron and magnesium."

This assumed sesquisilicate, however, which has haunted the mineralogy of meteorites under the names of Chladnite, Shepardite, and, subsequently, of Piddingtonite, was surmised by GUSTAV ROSE† to be in this case a mixture of more than one silicate, while in the similar instance of the Bishopsville meteorite Dr. LAWRENCE SMITH‡ had already proved the supposed Chladnite to have been no other than an augitic mineral, in fact one with the composition and characters of enstatite. The mineral, however, which forms the mass of Shalka does not seem to be so easily disposed of; for Professor RAMMELSBERG§ has recently published an analysis of the mineral or minerals under discussion, and asserts the Shalka meteorite to contain something like 12 per cent. of olivine of the composition $2Fe_2 Si O_4 + 3Mg_2 Si O_4$‖, the remainder being bronzite.

This meteorite had been examined some time back in the British Museum Laboratory with a very different result, and the discrepancies between this result and those as well of RAMMELSBERG as of VON HAUER induced me to have the analysis confirmed by further investigation. The conclusion, however, to which these experiments have led still leave the discrepancy where it was. In fact the selection out of the débris of the meteorite of the different ingredient minerals, or what seemed to be such, had at first led to the belief that it might indeed consist, first, of a grey silicate; secondly, a more mottled grey and possibly mingled mineral; and thirdly, chromite, which is present in considerable quantity, and often in very perfect crystals. An analysis, however, of the mixed silicates gave a result so nearly in accordance with that of a definite enstatite, that the view seemed hardly tenable. Furthermore, the analysis of the mottled variety gave as its result that this mineral is no other than bronzite.

The analysis of a very small amount of the débris of the meteorite gave the following numbers:—

		Oxygen.
Silicic acid	45·37	24·197
Iron protoxide	19·06	4·236
Calcium oxide	2·214	0·632
Magnesium oxide . . .	15·636	6·254
Chromite	17·717	
	99·997	

* Ber. Akad. Wiss. Wien, xli. p. 251.

† Beschreibung und Eintheilung der Meteoriten, p. 125.

‡ SILLIMAN'S Am. Journ. Sc. xxxviii. p. 225.

§ Ber. der Deutsch. Chem. Gesellschaft. Berlin, iii. p. 522.

‖ POGG. Ann. cxl. p. 312.

Two analyses of the mottled variety of silicate furnished the results given below.

	I.	Oxygen.	II.	Oxygen.
Silicic acid	52·831	28·176	52·725	28·12
Iron protoxide . . .	21·863	4·859	22·992	5·109
Calcium oxide . . .	0·502	0·143	——	
Magnesium oxide . .	24·266	9·706	24·085	9·63
Chromite	0·643		——	
	100·105		99·802	

These numbers correspond with the formula $(Mg_{\frac{3}{4}} Fe_{\frac{1}{4}})Si\,O_3$, which is identical with the bronzite of the Manegaum meteorite*.

In fact the olivine found by RAMMELSBERG does not exist in the sample of the meteorite analyzed at the British Museum. This is probably due to the portions of the meteorite examined in his laboratory and mine being from different parts of the mass.

To check these determinations, a portion of the mottled variety was submitted to the action of acid in the cold, and subsequent treatment with alkali to remove the liberated silicic acid. The results now given show this action to have been confined to that of a solvent.

I. By treatment with a mixture of one part of strong hydrogen chloride and two of water for sixty-six hours in the cold, and subsequently with soda, there were removed the following percentages of

		Oxygen.
Silicic acid	1·507	0·804
Iron protoxide . . .	0·974	0·216
Magnesium oxide . .	1·058	0·423
	3·539	

II. A corresponding treatment of another portion with a mixture of one part of strong hydrogen sulphate and two of water, for 240 hours, gave the numbers:—

		Oxygen.
Silicic acid	3·900	2·08
Iron oxide	1·799	0·399
Magnesium oxide . .	1·877	0·75
	7·576	

The slight excess of iron found in both cases was doubtless the result of a little unseparated meteoric iron. It certainly would not justify my assigning any appreciable portion of the silica to the constitution of an olivinous ingredient of the meteorite.

In recording the results detailed in the analyses here given, I have to express my obligations to Dr. WALTER FLIGHT, Assistant in my Department at the British Museum, for his skilful and zealous cooperation.

* Philosophical Transactions, 1870, p. 189.

XV. *On the Problem of the In-and-Circumscribed Triangle.* *By* A. Cayley, *F.R.S.*

Received December 30, 1870,—Read February 9, 1871.

The problem of the In-and-Circumscribed Triangle is a particular case of that of the In-and-Circumscribed Polygon: the last-mentioned problem may be thus stated—to find a polygon such that the angles are situate in and the sides touch a given curve or curves. And we may in the first instance inquire as to the number of such polygons. In the case where the curves containing the angles and touched by the sides respectively are all of them distinct curves, the number of polygons is obtained very easily and has a simple expression: it is equal to twice the product of the *orders* of the curves containing the several angles respectively into the product of the *classes* of the curves touched by the several sides respectively; or, say, it is equal to twice the product of the orders of the angle-curves into the product of the classes of the side-curves. But when several of the curves become one and the same curve, and in particular when the angles are all of them situate in and the sides all touch one and the same curve, it is a much more difficult problem to find the number of polygons. The solution of this problem when the polygon is a triangle, and for all the different relations of identity between the different curves, is the object of the present memoir, which is accordingly entitled "On the Problem of the In-and-Circumscribed Triangle;" the methods and principles, however, are applicable to the case of a polygon of any number of sides, the method chiefly made use of being that furnished by the theory of correspondence, as will be explained. The results (for the triangle) are given in the following Table; for the explanation of which I remark that the triangle is taken to be $a\mathrm{B}c\mathrm{D}e\mathrm{F}$; viz. a, c, e are the angles, B, D, F the sides; that is, B, D, F are the sides ac, ce, ea respectively, and a, c, e are the angles FB, BD, DF respectively. And I use the same letters a, c, e, B, D, F to denote the curves containing the angles and touched by the sides respectively; viz. the angle a is situate in the curve a, the side B touches the curve B, and so for the other angles and sides respectively. An equation such as $a=c$ or $a=\mathrm{B}$ denotes that the curves a, c or, as the case may be, the curves a, B are one and the same curve: it is in general convenient to use a new letter for denoting these identical curves; viz. I write, for instance, $a=c=x$ or $a=\mathrm{B}=x$, to denote that the curves a, c or, as the case may be, the curves a, B are one and the same curve x; the new letters thus introduced are x, y, z, there being in regard to them no distinction of small letters and capitals. The expression "no identities" denotes that the curves are all distinct. But I use also the letters a, c, e, b, d, f, x, y, z, and A, C, E, B, D, F, X, Y, Z quantitatively, to denote the orders and classes of the curves a, c, e, B, D, F, x, y, z respectively; thus, in the Table, for the case 1 "no identities" the number of triangles is given as $=2ace\mathrm{BDF}$, which agrees with

the before-mentioned result for the polygon: for the case 2 the several separate identities $a=c$, $a=e$, $c=e$ are of course equivalent to each other; and selecting one of them, $a=c=x$, the number of triangles is given as $=2x(x-1)e\text{BDF}$. There is a convenience in thus writing down the several forms $a=c$, $a=e$, $c=e$ of the identity or identities which constitute the 52 distinct cases of the Table; and I have accordingly done so throughout the Table, the expression for the number of triangles being however in each case given under one form only. It only remains to mention that for the curve x the Greek letter ξ denotes what may be termed the "stativity" of the curve, viz. this is $=$ number of cusps $+3$ times the class, or, what is the same thing, $=$ number of inflections $+3$ times the order; viz. the curve is determined by its order x, class X, and ξ; and similarly for η and ζ.

Observe that, in the column "Specification," each line is to be read separately from the others, and, where the word "or" occurs, the two parts of the line are to be read separately; thus case 5, the six forms are $a=\text{B}$, $a=\text{F}$, $c=\text{D}$, $c=\text{B}$, $e=\text{F}$, $e=\text{D}$: the letter x (or, as the case may be, x, y, or x, y, z) accompanies the first of the given forms; in the present instance $a=\text{B}=x$, and it is to this first form that the number of triangles, here $2(\text{X}x-\text{X}-x)\,ce\text{DF}$, applies.

I remark that what is primarily determined is the number of positions of a particular angle of the triangle, and that in some cases, on account of the symmetry of the figure, the number of triangles is a submultiple of this number; viz. the number of positions of the angle is to be divided by 2 or 6; this is expressly shown, by means of a separate column, in the Table.

No. of Case.	Specification.	No. of forms.	Totals.	No. of triangles.	Divided by.
1	No identities	1	1	$2aceBDF$	
2	$a=c=x$ $c=e$ $e=a$	3		$2x(x-1)eBDF$	
3	$D=F=x$ $F=B$ $B=D$	3		$2X(X-1)Bace$	
4	$a=D=x$ $c=F$ $e=B$	3		$2XxceBF$	
5	$a=B=x$, or $a=F$ $c=D$,, $c=B$ $e=F$,, $e=D$	6	15	$2(Xx-X-x)ceDF$	
6	$a=c=e=x$	1		$\{2x(x-1)(x-2)+X\}BDF$	
7	$B=D=F=x$	1		$\{2X(X-1)(X-2)+x\}ace$	
8	$a=c=B=x$ $c=e=D$ $e=a=F$	3		$2x(x-3)(X-2)eDF$	
	Carried over.	(5)	(16)		

TABLE (continued).

No. of Case.	Specification.	No. of forms	Totals.	No. of triangles.	Divided by.
	Brought over.	(5)	(16)		
9	$D=F=e=x$ $F=B=a$ $B=D=c$	3		$2X(X-3)(x-2)acB$	
10	$a=c=D=x$, or $a=c=F$ $c=e=F$ „ $c=e=B$ $e=a=B$ „ $e=a=D$	6		$2(x-1)(Xx-X-x)eBF$	
11	$D=F=a=x$, or $D=F=c$ $F=B=c$ „ $F=B=e$ $B=D=e$ „ $B=D=a$	6	20	$2(X-1)(Xx-X-x)ccB$	
12	$c=e=x$, $a=D=y$ $e=a$ $c=F$ $a=c$ $e=B$	3		$2x(x-1)yYBF$	
13	$F=B=x$, $a=D=y$ $B=D$ $c=F$ $D=F$ $e=B$	3		$2X(X-1)Yyce$	
14	$c=e=x$, $a=B=y$, or $c=e$, $a=F$ $e=a$ $c=D$ „ $e=a$, $c=B$ $a=c$ $e=F$ „ $a=c$, $e=D$	6		$2x(x-1)(Yy-Y-y)DF$	
15	$F=B=x$, $D=e=y$, or $F=B$, $D=c$ $B=D$ $F=a$ „ $B=D$, $F=e$ $D=F$ $B=c$ „ $D=F$, $B=a$	6		$2X(X-1)(Yy-Y-y)ac$	

16	$c=e=x,\ D=F=y$, or $c=e,\ D=B$ $e=a\quad F=B\quad$ „ $e=a,\ F=D$ $a=c\quad B=D\quad$ „ $a=c,\ B=F$	6		$2x(x-1)Y(Y-1)aB$	
17	$c=e=x,\ B=F=y$ $e=a\quad D=B$ $a=c\quad F=D$	3		$2x(x-1)Y(Y-1)aD$	2
18	$a=D=x,\ c=B=y$, or $a=D,\ e=F$ $c=F\quad e=D\quad$ „ $c=F,\ a=B$ $e=B\quad a=F\quad$ „ $e=B,\ c=D$	6		$2xX(Yy-Y-y)eF$	
19	$c=F=x,\ e=B=y$ $e=B\quad a=D$ $a=D\quad c=F$	3		$2xyXYaD$	
20	$c=D=x,\ e=F=y$, or $c=B,\ e=D$ $e=F\quad a=B\quad$ „ $e=D,\ a=F$ $a=B\quad c=D\quad$ „ $a=F,\ c=B$	6		$2\{xyXY-xy(X+Y)-XY(x+y)+2xy+2XY\}aB$	
21	$c=B=x,\ e=F=y$ $e=D\quad a=B$ $a=F\quad c=D$	3	45	$2\{xyXY-xy(X+Y)-XY(x+y)+2xY+2yX\}aD$	
22	$a=D=x,\ c=F=y,\ e=B=z$	1		$2xyzXYZ$	
23	$a=B=x,\ c=D=y,\ e=F=z$ $a=F\quad c=B\quad e=D$	2		$2\{(xyzXYZ-xyz(YZ+ZX+XY)-XYZ(yz+zx+xy)$ $+2xyz(X+Y+Z)+2XYZ(x+y+z)-4xyz-4XYZ\}$	
	Carried over.	(3)	(81)		

TABLE (continued).

No. of Case.	Specification.	No. of forms.	Totals.	No. of triangles.	Divided by.
	Brought over.	(3)	(81)		
24	$a=D=x,\ c=B=y,\ e=F=z$ $c=F \quad e=D \quad a=B$ $e=B \quad a=F \quad c=D$	3		$2xX\{yzYZ-yz(Y+Z)-YZ(y+z)+2yz+2YZ\}$	
25	$a=c=x,\ D=F=y,\ e=B=z$ $c=e \quad F=B \quad a=D$ $e=a \quad B=D \quad c=F$	3		$2x(x-1)Y(Y-1)zZ$	2
26	$a=c=x, B=D=y, e=F=z$ or $a=c, B=F, e=D$ $c=e \quad D=F \quad a=B$,, $c=e, D=B, a=F$ $e=a \quad F=B \quad c=D$,, $e=a, F=D, c=B$	6		$2x(x-1)Y(Y-1)(zZ-z-Z)$	
27	$a=c=e=x,\ B=F=y$ $a=c=e \quad D=B$ $a=c=e \quad F=D$	3	15	$\{2x(x-1)(x-2)+X\}Y(Y-1)D$	2
28	$B=D=F=x,\ c=e=y$ $B=D=F \quad e=a$ $B=D=F \quad a=c$	3		$\{2X(X-1)(X-2)+x\}y(y-1)a$	2
29	$a=c=B=x,\ D=F=y$ $c=e=D \quad F=B$ $e=a=F \quad B=D$	3		$2x(x-3)(X-2)Y(Y-1)e$	2
30	$e=D=F=x,\ a=c=y$ $a=F=B \quad c=e$ $c=B=D \quad e=a$	3		$2X(X-3)(x-2)y(y-1)B$	2

31	$c=e=D=x,\ a=B=y$, or $c=e=D,\ a=F$ $e=a=F \quad c=D$,, $e=a=F,\ c=B$ $a=c=B \quad e=F$,, $a=c=B,\ e=D$	6		$2x(x-3)(X-2)(yY-y-Y)F$	
32	$F=B=a=x,\ D=e=y$, or $F=B=a,\ D=c$ $B=D=c \quad F=a$,, $B=D=c,\ F=e$ $D-F=e \quad B=c$,, $D=F=e,\ B=a$	6		$2X(X-3)(x-2)(yY-y-Y)c$	
33	$B=F=y,\ a=e=D=x$, or $B=F.\ a=c=D$ $D=B \quad c=a=F$,, $D=B,\ c=e=F$ $F=D \quad e=c=B$,, $F=D,\ e=a=B$	6		$2(x-1)(xX-x-X)Y(Y-1)c$	
34	$c=e=y,\ B=D=e=x$, or $c=e,\ D=F=a$ $e=a \quad B=F=a$,, $e=a,\ F=B=c$ $a=c \quad F=B=c$,, $a=c,\ B=D=e$	6		$2(X-1)(xX-x-X)y(y-1)F$	
35	$a=D=y,\ c=e=B=x$, or $a=D,\ c=e=F$ $c=F \quad e=a=D$,, $c=F,\ e=a=B$ $c=B \quad a=c=F$,, $e=B,\ a=c=D$	6		$2(x-1)(xX-x-X)yYF$	
36	$a=D=y,\ B=F=e=x$, or $a=D,\ B=F=c$ $c=F \quad D=B=a$,, $c=F,\ D=B=e$ $e=B \quad F=D=c$,, $e=B,\ F=D=a$	6		$2(X-1)(xX-x-X)yYc$	
37	$a=e=D=x,\ c=B=y$, or $a=e=B,\ c=D$ $c=a=F \quad e=D$,, $c=a=D,\ e=F$ $e=c=B \quad a=F$,, $e=c=F,\ a=B$	6		$2(x-1)\{xyXY-xy(X+Y)-XY(x+y)+2xy+2XY\}F$	
	Carried over.	(54)	(96)		

TABLE (continued).

No. of Case.	Specification.	No. of forms.	Totals.	No. of triangles.	Divided by.
	Brought over.	(54)	(96)		
38	$B=D=a=x,\ F=e=y$, or $B=D=e,\ F=a$ $D=F=c \quad B=a$ „ $D=F=a,\ B=c$ $F=B=e \quad D=c$ „ $F=B=c,\ D=e$	6		$2(X-1)\{xyXY-xy(X+Y)-XY(x+y)+2xy+2XY\}c$	
			60		
39	$a=c=e=B=x$ „ $=D$ „ $=F$	3		$\{X^2+X(2x^3-10x^2+12x-1)-4x^3+20x^2-16x-3\xi\}DF$	
40	$B=D=F=e=x$ „ $=a$ „ $=c$	3		$\{x^2+x(2X^3-10X^2+12X-1)-4X^3+20X^2-16X-3\xi\}ac$	
41	$c=e=D=F=x$, or $c=e=D=B$ $e=a=B=B$ „ $e=a=F=D$ $a=c=F=D$ „ $a=c=B=F$	6		$2(x-3)(X-3)(xX-x-X)aB$	
42	$a=c=D=F=x$ $c=e=F=B$ $e=a=B=D$	3		$\{X^2(2x^2-6x+4)+X(-6x^2+18x-4)+4x^2-4x-4\xi\}eB$	2
			15		
43	$a=c=e=x,\ B=D=F=y$	1		$2x(x-1)(x-2)Y(Y-1)(Y-2)+yx(x-1)(x-2)+XY(Y-1)(Y-2)$	6
44	$e=D=F=x,\ a=c=B=y$ $a=F=B \quad c=e=D$ $c=B=D \quad e=a=F$	3		$2(x-2)X(X-3)(Y-2)y(y-3)$	2

45	$a=D=B=x$, $c=e=F=y$, or $a=D=F$, $c=e=B$ $c=F=D$ $e=a=B$,, $c=F=B$, $e=a=D$ $e=B=F$ $a=c=D$,, $e=B=D$, $a=c=F$	6	10	$2(X-1)(y-1)\{XYxy-XY(x+y)-xy(X+Y)+2xy+2XY\}$	
46	$a=c=y$, $B=D=F=e=x$ $c=e$ $B=D=F=a$ $e=a$ $B=D=F=c$	3		$y(y-1)\{x^2+x(2X^3-10X^2+12X-1)-4X^3+20X^2-16X-3\xi\}$	2
47	$D=F=y$, $a=c=e=B=x$ $F=B$ $a=c=e=D$ $B=D$ $a=c=e=F$	3		$Y(Y-1)\{X^2+X(2x^3-10x^2+12x-1)-4x^3+20x^2-16x-3\xi\}$	2
48	$a=c=D=F=x$, $e=B=y$ $c=e=F=B$ $a=D$ $e=a=B=D$ $c=F$	3		$\{X^2(2x^2-6x+4)+X(-6x^2+18x-4)+4x^2-4x-4\xi\}yY$	2
49	$a=B=y$, $c=e=D=F=x$, or $a=F$, $c=e=B=D$ $c=D$ $e=a=F=B$,, $c=B$, $c=a=D=F$ $e=F$ $a=c=B=D$,, $e=D$, $a=c=F=B$	6	15	$2(x-3)(X-3)\{xyXY-(x+y)XY-(X+Y)xy+2xy+2XY\}$	
50	$c=e=B=D=F$ $e=a=B=D=F$ $a=c=B=D=F$	3		a into $x^3(\quad + 1)$ $+x^2(2X^3-14X^2+ 28X-11)$ $+x(-10X^3+70X^2-116X- 8)$ $+12X^3-76X^2+ 64X$ $+\xi(- 6x - 4X+ 42)$	2
	Carried over.	(3)	(196)		

TABLE (continued).

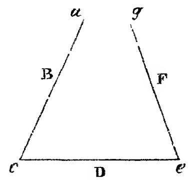
No. of Case.	Specification.	No. of forms.	Totals.	No. of triangles.	Divided by.
	Brought over.	(3)	(196)		
51	$F=B=a=c=e$ $B=D=a=c=e$ $D=F=a=c=e$	3	6	D into $X^3(\qquad\qquad +1)$ $+X(2x^3-14x^2+28x-11)$ $+X(-10x^3+70x^2-116x-8)$ $+12x^3-76x^2+64x$ $+\xi(-6X-4x+42)$	2
52	$a=c=e=B=D=F$	1	1	$X^4(\qquad\qquad +1)$ $+X^3(2x^3-18x^2+52x-46)$ $+X^2(-18x^3+162x^2-420x+221)$ $+X(52x^3-420x^2+704x+172)$ $+1x^4-46x^3+221x^2+172x$ $+\xi\left\{\begin{array}{l} X^2(\qquad -9) \\ +X(\quad -12x+135) \\ \quad -9x^2+135x-600 \end{array}\right\}$	6
			203		

The foregoing results are chiefly obtained by means of the theory of correspondence; viz. if instead of the triangle aBcDeF we consider the unclosed trilateral aBcDeFg, where the points a and g are situate on one and the same curve, say the curve $a=g$, then the points a and g have a certain correspondence, say a (χ, χ') correspondence with each other; and when a, g are a "united point" of the correspondence, the trilateral in question becomes an in-and-circumscribed triangle aBcDeF; that is, the number of triangles is equal to that of the united points of the correspondence, subject however (in many of the cases) to a reduction on account of special solutions. It may be remarked that by the theory of correspondence the number of the united points is, in several of the cases, but not in all of them, $=\chi+\chi'$. But in some instances I employ a functional method, by assuming that the identical curves are each of them the aggregate of the two curves x, x': we here obtain for the number ϕx of the triangles belonging to the curve x a functional equation $\phi(x+x')-\phi x-\phi x'=$ given function; viz. the expression on the right-hand side depends on the solution of the preceding cases, wherein the number of identities between the several curves is less than in the case under consideration; and taking it to be known, the functional equation gives $\phi x=$ particular solution + linear function of (x, X, ξ). The particular solution is always easily obtainable, and the constants of the linear function can be determined by means of particular forms of the curve x.

The Principle of Correspondence as applied to the present Problem.—Article Nos. 1 to 6.

1. Consider the unclosed trilateral aBcDeFg, where the points a and g are on one and the same curve, $a=g$. Starting from an arbitrary point a on the curve a, we have aBc any one of the tangents from a to the curve B, touching this curve, say at the point B, and intersecting the curve c in a point c; viz. c is any one of the intersections of aBc with the curve c; we have then similarly cDe any one of the tangents from c to the curve D, touching it, say at D, and intersecting the curve e in a point e; viz. the point e is any one of the intersections in question; and then in like manner we have eFg any one of the tangents from e to the curve F, touching it, say at F, and intersecting the curve g $(=a)$ in a point g; viz. g is any one of the intersections in question. Suppose that to a given position of a there correspond χ positions of g; it is easy to find the value of χ; viz. if (as above tacitly supposed) the curves a, B, c, D, e, F are all of them distinct curves, then the number of the tangents aBc is $=$B; there are on each of them c points c; through each of these we have D tangents cDe; on each of these e points e; through each of these F tangents cFg; and on each of these a points g; that is, $\chi=$BcDeFa. But if some of the curves become one and the same curve—if, for instance, $a=$B$=c$, the line aBc is here a tangent from a point a on the curve, we exclude the tangent at the point a, and the number of the remaining tangents is $=(\mathrm{A}-2)$; each tangent meets the curve in the point a counting once, the point B counting twice, and in $(a-3)$ other

Fig. 1.



points; that is, the number of the points c is $=(\mathrm{A}-2)(a-3)$, and so in other cases; the calculation is always immediate, and the only difference is that, instead of a factor a or A, we have such factor in its original form or diminished by 1, 2, or 3, as the case may be. Similarly starting from g, considered as a given point on the curve $g(=a)$, we find χ' the number of the corresponding points a; thus in the case where the curves are all distinct curves, we have $\chi'=\mathrm{F}e\mathrm{D}c\mathrm{B}a\ (=\chi)$; and so in other cases we find the value of χ'. The points (a, g) have thus a (χ, χ') correspondence, where the values of χ, χ' are found as above.

2. There will be occasion to consider the case where in the triangle $a\mathrm{B}c\mathrm{D}e\mathrm{F}$ (or say the triangle $a\mathrm{B}c\mathrm{D}e\mathrm{F}a$) the point a is not subjected to any condition whatever, but is a free point. There is in this case a "locus of a," which is at once constructed as follows: viz. starting with an arbitrary tangent $a\mathrm{B}c$ of the curve B, touching it at B and intersecting the curve c in a point c; through c we draw to the curve D the tangent $c\mathrm{D}e$, touching it at D and intersecting the curve e in a point e; and finally from e to the curve F the tangent $e\mathrm{F}a$, touching it at F and intersecting the original arbitrary tangent $a\mathrm{B}c$ in a point a, which is a point on the locus in question. We can, it is clear, at once determine how many points of the locus lie on an arbitrary tangent of the curve B (or of the curve F).

3. The general form of the equation of correspondence is

$$p(\mathrm{a}-\alpha-\alpha')+q(\mathrm{b}-\beta-\beta')+\ldots\ldots=k\Delta^*;$$

viz. if on a curve for which twice the deficiency is $=\Delta$ we have a point P corresponding to certain other points P′, Q′, ... in such wise that P, P′ have an (α, α') correspondence, P, Q′ a (β, β') correspondence, &c.; and if (a) be the number of the united points (P, P′), (b) the number of the united points (P, Q′), &c.; and if moreover for a given position of P on the curve the points P′, Q′ ... are obtained as the intersections of the curve with a curve Θ (depending on the point P) which meets the curve k times at P, p times at each of the points P′, q times at each of the points Q′, &c.; then the relation between the several quantities is as stated above: see my "Second Memoir on the Curves which satisfy given conditions," Philosophical Transactions, vol. 159 (1868), pp. 145–172. I omit for the present purpose the term "Supp.," treating it as included in the other terms.

4. In the present case we consider, as already mentioned, the unclosed trilateral $a\mathrm{B}c\mathrm{D}e\mathrm{F}g$, where the angles a, g are on one and the same curve $a(=g)$ (the curve in the general theorem); and the curve Θ is the system of lines $e\mathrm{F}g$ which by their intersec-

* To avoid confusion with the notation of the present memoir, I abstain in the text from the use of D as denoting the deficiency, and there is a convenience in the use of a single symbol for twice the deficiency; but writing for the moment D to denote the deficiency, I remark, in passing, that perhaps the true theoretical form of the equation is

$$k(0-\mathrm{D}-\mathrm{D})+p(\mathrm{a}-\alpha-\alpha')+q(\mathrm{b}-\beta-\beta')+\ldots=0;$$

viz. the point P is here considered as having with itself a (D, D) correspondence, the number of the united points therein being $=0$.

tion with the curve a determine the points g. Considering these as the points (P, P′) of the general theorem we have $p=1$: I change the notation, and instead of $\mathrm{a}-\alpha-\alpha'$ write $\mathrm{g}-\chi-\chi'$; viz. I take (g) for the number of the united points (a, g), and suppose that the points (a, g) have a (χ, χ') correspondence. The most simple case is when the curve a is distinct from each of the curves e, F; here all the intersections of the line-system $e\mathrm{F}g$ with the curve a are points g, that is we have *only* the correspondence (a, g); and since the line-system $e\mathrm{F}g$ does not pass through the point a, we have simply

$$\mathrm{g}-\chi-\chi'=0.$$

5. But suppose that the curves a, e, F are one and the same curve, say that $a=e=\mathrm{F}$; understanding by the point F the point of contact of a line $e\mathrm{F}g$ with the curve a, then the intersections of the line-system $e\mathrm{F}g$ with the curve a are the points g each once, the points F each twice, and the points e each as many times as there are lines $e\mathrm{F}g$ through the point e, say each M times. (In the present case, where the curves e, F are identical, we have $\mathrm{M}=\mathrm{F}-2$ or $\mathrm{F}-3$ according as the curve D is or is not distinct from the curve F; in the cases afterwards referred to, the values may be F or $\mathrm{F}-1$; that is, we have always $\mathrm{M}=\mathrm{F}, \mathrm{F}-1, \mathrm{F}-2, \mathrm{F}-3$, as the case may be.) We have to consider the several correspondences (a, g), (a, F), (a, e); k is as before$=0$; and the form of the theorem is

$$(\mathrm{g}-\chi-\chi')+2(\mathrm{f}-\phi-\phi')+\mathrm{M}(\mathrm{e}-\epsilon-\epsilon')=0,$$

where the symbols denote as follows, viz.

(a, g)	have a	(χ, χ')	correspondence	and No. of	united points	$=\mathrm{g}$,
(a, F)	„	(ϕ, ϕ')	„	„	„	$=\mathrm{f}$,
(a, e)	„	(ϵ, ϵ')	„	„	„	$=\mathrm{e}$,

so that the determination of g here depends upon that of $\mathrm{f}-\phi-\phi'$ and $\mathrm{e}-\epsilon-\epsilon'$.

6. The curve a might however have been identical with only one of the curves e, F; viz. if $a=\mathrm{F}$, but e is a distinct curve, then the equation will contain the term $2(\mathrm{f}-\phi-\phi')$, but not the term $\mathrm{M}(\mathrm{e}-\epsilon-\epsilon')$; and so if $a=e$, but F is a distinct curve, then the equation will not contain $2(\mathrm{f}-\phi-\phi')$, but will contain $\mathrm{M}(\mathrm{e}-\epsilon-\epsilon')$: it is to be noticed that in this last case we have $\mathrm{M}=\mathrm{F}$ or $\mathrm{M}=\mathrm{F}-1$, according as the curve D is not, or is, one and the same curve with F. The determination of (g) here depends upon that of $\mathrm{f}-\phi-\phi'$ or $\mathrm{e}-\epsilon-\epsilon'$, as the case may be. These subsidiary values $\mathrm{f}-\phi-\phi'$ and $\mathrm{e}-\epsilon-\epsilon'$ are obtained by means of a more simple application of the principle of correspondence, as will appear in the sequel*, but for the moment I do not pursue the question.

Locus of a free angle (a).—Art. Nos. 7 to 14.

7. I consider the case where a is a distinct curve $\neq e$, $\neq \mathrm{F}$, and where, as was seen, the equation is simply

$$\mathrm{g}-\chi-\chi'=0.$$

* See *post*, Nos. 24 *et seq.*

I suppose further that a is distinct from all the other curves, or say, *simpliciter*, that a is a distinct curve. The values of χ, χ' will here each of them contain the factor a, say we have $\chi=a\omega$, $\chi'=a\omega'$; and therefore the equation gives $g=a(\omega+\omega')$. It is obvious that ω, ω' are the values assumed by χ, χ' respectively in the particular case where the curve a is an arbitrary line $(a=1)$; and $\omega+\omega'$ is the number of the united points on this line.

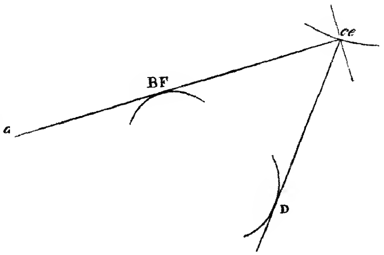
8. Suppose now that in the triangle $a\mathrm{B}c\mathrm{D}e\mathrm{F}a$ the point a is a free point, we have, as above mentioned, a locus of a, and the united points on the arbitrary line are the intersections of the line with this locus; that is, the locus meets the arbitrary line in $\omega+\omega'$ points; or, what is the same thing, the order of the locus is $=\omega+\omega'$.

9. I stop for a moment to remark that in the particular case where the curve B is a point (B=1), then in the construction of the locus of a the arbitrary tangent $a\mathrm{B}c$ is an arbitrary line through B, and the construction gives on this line ω positions of the point a. But drawing from B a tangent to the curve F, and thus constructing in order the points F, e, D, c, a, the construction shows that B is an ω'-tuple point on the locus; and (by what precedes) an arbitrary line through B meets the locus in ω other points; that is, in the particular case where the curve B is a point, the order of the locus of a is $=\omega+\omega'$, which agrees with the foregoing result.

10. The construction for the locus of a may be presented in the following form: viz. drawing to the curve D a tangent $c\mathrm{D}e$, meeting the curves c, e in the points c, e respectively; then if from any point c we draw to the curve B a tangent $c\mathrm{B}a$, and from any point e to the curve F a tangent $e\mathrm{F}a$, the tangents $c\mathrm{B}a$, $e\mathrm{F}a$ intersect in a point on the required locus. Hence if in any particular case (that is for any particular position of the tangent $c\mathrm{D}e$) the lines $c\mathrm{B}a$, $e\mathrm{F}a$ become one and the same line, the point a will be an indeterminate point on this line; that is, the line in question will be part of the locus of a.

11. The case cannot in general arise so long as the curves B, F are distinct from each other; but when these are one and the same curve, say when B=F, it will arise, and that in two distinct ways. To show how this is, suppose, to fix the ideas, that the curves c, D, e are distinct from each other and from the curve B=F. Then the first mode is that shown in the annexed "first-mode figure," viz. we have here a tangent at D passing through a point ce of the intersection of the curves c, e, and from this point a tangent drawn to the curve B=F. For the position in question of the tangent of D, the points c, e coincide with each other, and we have thus the coincident tangents $c\mathrm{B}a$ and $e\mathrm{F}a$ to the identical curves B=F. It is further to be remarked that the number of the points of intersection is $=ce$; from each of these there are B tangents to the curve B=F (in

Fig. 2. First-mode figure.



all ce . B tangents), and each of these counts once in respect of each of the D tangents to the curve D, that is, it counts D times. We have thus, as part of the locus of a, ce . B lines each D times, or, say, first-mode reduction$=ce$.B.D.

12. The second mode is that shown in the annexed "second-mode figure." The

Fig. 3. Second-mode figure.

tangent from D is here a common tangent of the curves D, and B=F. This meets the curve c in c points, and the curve e in e points; and attending to any pair of points c, e, these give the tangents cBa, eFa, coinciding with the common tangent in question, and forming part of the locus of a. The number of the common tangents is =BD; but each of these counts once in respect of each combination of the points c, e, that is in all ce times. And we have thus as part of the locus BD lines each $c.e$ times, or, say, second-mode reduction=BD.$c.e$. This is (as it happens) the same number as for the first mode; but to distinguish the different origins I have written as above ce.B.D and BD.$c.e$ respectively.

13. It is important to remark that each of the two modes arises whatever relations of identity subsist between the curves c, e, D, and B=F, but with considerable modification of form. Thus if the curves c, e are identical ($c=e$) but distinct from D, then in the first-mode figure ce may be a node or a cusp of the curve $c=e$, or it may be a point of contact of a common tangent of the curves D, and $c=e$. As regards the node, remark that if we consider a tangent of D meeting the curve $c=e$ in the neighbourhood of the node, then of the two points of intersection each in succession may be taken for the point c, and the other of them will be the point e; so that the node counts twice. It requires more consideration to perceive, but it will be readily accepted that the cusp counts three times. Hence if for the curve $c=e$ the number of nodes be $=\delta$ and that of cusps $=\varkappa$, the value of the first-mode reduction is $=(2\delta+3\varkappa+\mathrm{C})$BD, or, what is the same thing, it is $=(c^2-c)$BD.

As regards the second-mode figure, the only difference is that c, e will be here any pair of intersections (each pair twice) of the tangent with the curve $c=e$; the value is thus $=(c^2-c)$BD.

It would be by no means uninteresting to enumerate the different cases, and indeed there might be a propriety in doing so here; but I have (instead of this) considered the several cases when and as they arise in connexion with any of the cases of the in-and-circumscribed triangle.

14. Observe that the general result is, that in the case B=F of the identity of the curves B and F, but not otherwise, the locus of a includes as part of itself a system of lines; or, say, that it is made up of these lines, and of a residual curve of the order $\omega+\omega'-$Red., which is the proper locus.

Application of the foregoing Theory as to the locus of (a). Art. Nos. 15 to 17.

15. Reverting now to the case where the angle a is not a free angle but is situate on a given curve a, then if the curve a is distinct from the curves e, F, the number of positions of a is, as was seen, $\mathrm{g}=\chi+\chi'$. But the points in question are the intersections of the curve a with the locus of a considered as a free angle; and hence in the case B=F, but not otherwise, they are made up of the intersections of the curve a with the system of lines, and of its intersections with the proper locus of a. But the intersections with the system of lines are improper solutions of the problem (or, to use a locution which may be convenient, they are "heterotypic" solutions): the true solutions are the intersections with the proper locus of a; and the number of these is not $\chi+\chi'$, $=a(\omega+\omega')$, but it is $=a(\omega+\omega'-\text{Red.})$; say it is $=\chi+\chi'-\text{Red.}$, where the symbol "Red." is now used to signify a times the number of lines, or reduction in the expression $\omega+\omega'-\text{Red.}$ of the order of the proper locus of a.

16. It is however to be noticed that if the curve a, being as is assumed distinct from the curves e, and F=B, is identical with one or both of the remaining curves c, D, the foregoing expression $\chi+\chi'-\text{Red.}$ *may* include positions which are not true solutions of the problem, viz. the curve a may pass through special points on the proper locus of a, giving intersections which are a new kind of heterotypic solutions*.

17. But this cannot happen if the curve a is distinct also from the curves c, D; or, say, simply when a is a distinct curve. The conclusion is, that in the case where a is a distinct curve we have

$$\mathrm{g}=\chi+\chi'-\text{Red.},$$

where the term "Red." vanishes except in the case of the identity B=F of the curves B, F; and that when this identity subsists it is $=a$ times the reduction in the order of the locus of a considered as a free angle; viz. this consists of a first-mode and a second-mode reduction as above explained.

Remarks in regard to the Solutions for the 52 *Cases.* Art. Nos. 18 to 23.

18. Before going further I remark that the principle of correspondence applies to corresponding and united tangents in like manner as to corresponding and united points, and that all the investigations in regard to the in-and-circumscribed triangle might thus be presented in the reciprocal form, where, instead of points and lines, we have lines and points respectively. But there is no occasion to employ any such reciprocal process; the result to which it would lead is the reciprocal of a result given by the original process, and as such it can always be obtained by reciprocation of the original result, without any performance of the reciprocal process.

* More generally, if the curve a be a curve identical with any of the other curves, then if treating in the first instance the angle a as free we find in any manner the locus of a, the required positions of the angle a are the intersections of this locus and of the curve a; but these intersections will in general include intersections which give heterotypic solutions. The determination of these is a matter of some delicacy, and I have in general treated the problems in such manner that the question does not arise; but as an example see *post*, Case 43.

19. It is hardly necessary to remark that although reciprocal results would, by the employment of the two processes respectively, be obtained in a precisely similar manner, yet that this is not so when only one of the reciprocal processes is made use of; so that, using one process only, it may be and in general is easier and more convenient to obtain directly one than the other of two reciprocal results; for instance, to consider the case $B=D=F$ rather than $a=c=e$, or *vice versâ*; and that it is sufficient to do this, and having obtained the one result, directly to deduce from it the other by reciprocity; but that it may nevertheless be interesting to obtain each of the two results directly.

20. It is moreover obvious that although the several forms of the same case, for instance Case 2, $a=c$, $a=e$, or $c=e$, are absolutely equivalent to each other, yet that, when as above we select a vertex a, and seek for the number of the united points (a, g), the process of obtaining the result will be altogether different according to the different form which we employ. For instance, in the case just referred to, if the form is taken to be $a=c$ or $c=e$, then the equation $\mathrm{g}=\chi+\chi'$ is applicable to it; but not so if the form is taken to be $a=e$. It would be by no means uninteresting in every case to consider the several forms successively and get out the result from each of them; I shall not, however, do this, but only consider two or more forms of the same case when for comparison, illustration, verification, or otherwise it appears proper so to do. The translation of a result, for instance, of a form $a=e$ or $c=e$ into that for the form $a=c=x$ is so easy and obvious, that it is not even necessary formally to make it.

21. I do not at present further consider the general theory, but proceed to consider in order the 52 cases, interpolating in regard to the general theory such further discussion or explanation as may appear necessary. In the several instances in which the equation $\mathrm{g}=\chi+\chi'$ is applicable, it is sufficient to write down the values of χ, χ', the mode of obtaining these being already explained.

The 52 cases for the in-and-circumscribed triangles.

Case 1. No identities.

$$\chi=\mathrm{B}c\mathrm{D}e\mathrm{F}a,\ \chi'=\mathrm{F}e\mathrm{D}c\mathrm{B}a(=\chi),$$
$$\mathrm{g}=2ace\mathrm{BDF}.$$

Case 2. $a=c=x$.

$$\chi=\mathrm{B}(x-1)\mathrm{D}e\mathrm{F}x,\ \chi'=\mathrm{F}e\mathrm{D}x\mathrm{B}(x-1)(=\chi),$$
$$\mathrm{g}=2x(x-1)e\mathrm{BDF}.$$

Second process, for form $a=e=x$. The equation of correspondence is here

$$\mathrm{g}-\chi-\chi'+\mathrm{F}(\mathrm{e}-\varepsilon-\varepsilon')=0;$$

but the points e being given as all the intersections of the curve $a(=e)$ by the line-system $c\mathrm{D}e$ which does not pass through a, we have $\mathrm{e}-\varepsilon-\varepsilon'=0$; so that $\mathrm{g}=\chi+\chi'$; and then

$$\chi=\mathrm{B}c\mathrm{D}x\mathrm{F}(x-1),\ \chi'=\mathrm{F}(x-1)\mathrm{D}c\mathrm{B}x,$$

giving the former result*.

* Of course, the result is obtained in the form belonging to the new form of specification. viz. here it is $=2x(x-1)c\mathrm{BDF}$; and so in other instances; but it is unnecessary to refer to this change.

Case 3. $D=F=x$. Reciprocation from 2; or else, *second process*,

$$\chi=BcXe(X-1)a,\ \chi'=Xe(X-1)cBa,$$
$$g=2X(X-1)Bace.$$

Third process: form $F=B=x$. We have here $g=\chi+\chi'-\text{Red.}$

$$\chi=XcDeXa,\ \chi'=XeDcXa(=\chi),$$
$$\chi+\chi'=2X^2Dace;$$

and the reductions are those of the first and second mode, as explained *ante*, Nos. 11, 12, viz. each of these is $=XDace$, and together they are $=2XDace$; whence the foregoing result.

Case 4. $a=D=x$.

$$\chi=BcXeFx,\ \chi'=FeXBx(=\chi),$$
$$g=2XxceBF.$$

Observe this is what the result for Case 1 becomes on writing therein $a=D=x$, viz. the opposite curves a, D may become one and the same curve without any alteration in the form of the result.

Case 5. $a=B=x$.

$$\chi=(X-2)cDeFx,\ \chi'=FeDcX(x-2),$$

where

$$(X-2)x+X(x-2)=2(Xx-X-x);$$

wherefore

$$g=2(Xx-X-x)ceDF.$$

Case 6. $a=c=e=x$: perhaps most easily by reciprocation of Case 7; or

Second process, functionally by taking the curve $a=c=e$ to be the aggregate curve $x+x'$. The triangle $aBcDeF$ is here in succession each of the eight triangles:

x B x D x F	x' B x' D x' F
x' „ x „ x „	x „ x' „ x' „
x „ x' „ x „	x' „ x „ x' „
x' „ x „ x „	x „ x' „ x' „

where the two top triangles give ϕx and $\phi x'$ respectively; the remaining triangles all belong to Case 2, and those of the first column give each $2(x^2-x)x'$BDF, and those of the second column each $2(x'^2-x')x$BDF. We have thus

$$\phi(x+x')-\phi x-\phi x'=\{6(x^2x'+xx'^2)-12xx'\}BDF.$$

Hence obtaining a particular solution and adding the constants, we have

$$\phi x=(2x^3-6x^2+\alpha x+\beta X+\gamma\xi)BDF;$$

it is easy to see that α, β, γ are independent of the curves B, D, F; and taking each of these to be a point, and the curve $a=c=e$ to be a conic, then it is known that $\phi x=2$, we have

$$2=16-24+2\alpha+2\beta+6\gamma,$$

that is

$$\alpha+\beta+3\gamma=5.$$

The case where the curve $a=c=e$ is a line gives $0=2-6+\alpha+3\gamma$, that is,

$$\alpha+3\gamma=4;$$

but it is not easy to find another condition; assuming however $\gamma=0$, we have $\alpha=4$, $\beta=1$, and thence

$$\phi x=(2x^3-6x^2+4x+\text{X})\text{BDF},$$

or say

$$\text{g}=\{2x(x-1)(x-2)+\text{X}\}\text{BDF}:$$

this is a good easy example of the functional process, the use of which begins to exhibit itself; and I have therefore given it, notwithstanding the difficulty as to the complete determination of the constants.

Third process. The equation of correspondence is

$$\text{g}-\chi-\chi'+\text{F}(\text{e}-\varepsilon-\varepsilon')=0,$$

but for the correspondence (a, e) we have

$$\text{e}-\varepsilon-\varepsilon'+\text{D}(\text{c}-\gamma-\gamma')=0,$$

and for the correspondence (a, c) we have

$$\text{c}-\gamma-\gamma'=\text{B}\Delta,$$

whence

$$\text{g}=\chi+\chi'+\text{BDF}.\Delta;$$

and then

$$\chi=\text{B}(x-1)\text{D}(x-1)\text{F}(x-1),\ \chi'=\text{F}(x-1)\text{D}(x-1)\text{B}(x-1)(=\chi):$$

that is

$$\chi+\chi'=\text{BDF}.2(x-1)^3.$$

Moreover

$$\Delta=\text{X}-2x+2+\varkappa$$

(if $\varkappa$ be the number of cusps of the curve $a=c=e$), and the resulting value is

$$\text{g}=\{2(x-1)^3+\text{X}-2x+2+\varkappa\}\text{BDF};$$

that is

$$=\{2x(x-1)(x-2)+\text{X}+\varkappa\}\text{BDF},$$

where, however, the term $\varkappa$BDF is to be rejected. I cannot quite explain this; I should rather have expected a rejection$=2\varkappa$BDF, introducing the term $-\varkappa$. For consider a tangent from the curve D from a cusp of the curve $a=c=e$: there are D such tangents; each gives in the neighbourhood of the cusp two points, say c, e; and from these we

draw B tangents cBa to the curve B, and F tangents eFa to the curve F; we have thus in respect of the given tangent of D, BF positions of a, or in all BDF positions of a which will ultimately coincide with the cusp; that is, BDF infinitesimal triangles of which the angles a, c, e coincide together at the cusp; and for all the cusps together $\varkappa$BDF such triangles: this would be what is wanted; the difficulty is that as (of the two intersections at the cusp) each in succession might be taken for c, and the other of them for e, it would seem that the foregoing number $\varkappa$BDF should be multiplied by 2.

Case 7. $B=D=F=x$. Here $g=\chi+\chi'-$Red. and

$$\chi=Xc(X-1)e(X-1)a,\ \chi'=Xe(X-1)c(X-1)a(=\chi);$$

that is,

$$\chi+\chi'=2X(X-1)^2ace.$$

The reductions of the two modes are as above, with only the variation that in the present case D is the same curve with the two curves $B=F$. That of the first mode is $=X(X-1)ace$, and that of the second mode is $(2\tau+3\iota)ace$, which is $=\{X(X-1)-x\}ace$; together they are $=\{2X(X-1)-x\}ace$, or subtracting, we have

$$g=\{2X(X-1)(X-2)+x\}ace.$$

Case 8. $a=c=B=x$.

$$\chi=(X-2)(x-3)DeFx,\ \chi'=FeDx(X-2)(x-3)(=\chi),$$
$$g=2x(x-3)(X-2)eDF.$$

Case 9. $D=F=e=x$. By reciprocation of 8.

$$\text{No.}=2X(X-3)(x-2)acB.$$

Case 10. $a=c=D=x$.

$$\chi=B(x-1)(X-2)eFx,\ \chi'=FeX(x-2)B(x-1),$$
$$g=2(x-1)(Xx-X-x)eBF.$$

Case 11. $D=F=a=x$. By reciprocation of 10.

$$\text{No.}=2(X-1)(Xx-X-x)ceB.$$

Second process: form $a=B=D=x$.

$$\chi=(X-2)c(X-1)eFx,\ \chi'=FeXc(X-1)(x-2),$$

giving the former result.

Case 12. $c=e=x$, $a=D=y$.

$$\chi=BxY(x-1)Fy,\ \chi'=FxY(x-1)By(=\chi),$$
$$g=2x(x-1)yYBF.$$

Case 13. $F=B=x$, $a=D=y$. By reciprocation of 12.

$$\text{No.}=2X(X-1)Yyce.$$

Case 14. $c=e=x,\ a=B=y$.

$$\chi=(Y-2)xD(x-1)Fy,\ \chi'=FxD(x-1)Y(y-2),$$
$$g=2x(x-1)(Yy-Y-y)DF.$$

Case 15. $F=B=x,\ D=e=y$. By reciprocation of 14.

$$\text{No. }=2X(X-1)(Yy-Y-y)ac.$$

Case 16. $c=e=x,\ D=F=y$.

$$\chi=BxY(x-1)(Y-1)a,\ \chi'=Yx(Y-1)(x-1)Ba(=\chi),$$
$$g=2x(x-1)Y(Y-1)aB.$$

Case 17. $c=e=x,\ B=F=y$.

$$\chi=D(x-1)Ya(Y-1)x,\ \chi'=Ya(Y-1)xD(x-1)(=\chi),$$
$$g=2x(x-1)Y(Y-1)aD.$$

But we have here aD as an axis of symmetry, so that each triangle is counted twice, or the number of distinct triangles is $=\frac{1}{2}g$.

Case 18. $a=D=x,\ c=B=y$.

$$\chi=Y(y-2)XeFx,\ \chi'=FeXy(Y-2)x(=\chi),$$
$$g=2xX(Yy-Y-y)eF.$$

Case 19. $c=F=x,\ e=B=y$.

$$\chi=YxDyXa,\ \chi'=XyDxYa(=\chi),$$
$$g=2xyXYaD.$$

Case 20. $c=D=x,\ e=F=y$.

$$\chi=Bx(X-2)y(Y-2)a,\ \chi'=Y(y-2)X(x-2)Ba,$$
$$g=\{xy(X-2)(Y-2)+XY(x-2)(y-2)\}aB$$
$$=2\{xyXY-xy(X+Y)-XY(x+y)+2xy+2XY\}aB.$$

Case 21. $c=B=x,\ e=F=y$.

$$\chi=X(x-2)Dy(Y-2)a,\ \chi'=Y(y-2)Dx(X-2)a,$$
$$g=\{X(Y-2)y(x-2)+Y(X-2)x(y-2)\}aD$$
$$=2\{xyXY-xy(X+Y)-XY(x+y)+2xY+2yX\}aD.$$

Case 22. $a=D=x,\ c=F=y,\ e=B=z$.

$$\chi=ZyXzYx,\ \chi'=YzXyZx(=\chi),$$
$$g=2xyzXYZ.$$

Case 23. $a=\mathrm{B}=x,\ c=\mathrm{D}=y,\ e=\mathrm{F}=z.$

$$\chi=(\mathrm{X}-2)y(\mathrm{Y}-2)z(\mathrm{Z}-2)x,\ \chi'=\mathrm{Z}(z-2)\mathrm{Y}(y-2)\mathrm{X}(x-2),$$
$$\begin{aligned}g&=xyz(\mathrm{X}-2)(\mathrm{Y}-2)(\mathrm{Z}-2)+\mathrm{XYZ}(x-2)(y-2)(z-2)\\&=2\{xyz\mathrm{XYZ}-xyz(\mathrm{YZ}+\mathrm{ZX}+\mathrm{XY})-\mathrm{XYZ}(yz+zx+xy)\\&\qquad+2xyz(\mathrm{X}+\mathrm{Y}+\mathrm{Z})+2\mathrm{XYZ}(x+y+z)-4xyz-4\mathrm{XYZ}\}.\end{aligned}$$

Case 24. $a=\mathrm{D}=x,\ c=\mathrm{B}=y,\ e=\mathrm{F}=z.$

$$\chi=\mathrm{Y}(y-2)\mathrm{X}z(\mathrm{Z}-2)x,\ \chi'=\mathrm{Z}(z-2)\mathrm{X}y(\mathrm{Y}-2)x,$$
$$\begin{aligned}g&=x\mathrm{X}\{\mathrm{Y}(\mathrm{Z}-2)z(y-2)+\mathrm{Z}(\mathrm{Y}-2)y(z-2)\}\\&=2x\mathrm{X}\{yz\mathrm{YZ}-yz(\mathrm{Y}+\mathrm{Z})-\mathrm{YZ}(y+z)+2y\mathrm{Z}+2z\mathrm{Y}\}\,*.\end{aligned}$$

Case 25. $a=c=x,\ \mathrm{D}=\mathrm{F}=y,\ e=\mathrm{B}=z.$

$$\chi=\mathrm{Z}(x-1)\mathrm{Y}z(\mathrm{Y}-1)x,\ \chi'=\mathrm{Y}z(\mathrm{Y}-1)x\mathrm{Z}(x-1)(=\chi),$$
$$g=2x(x-1)\mathrm{Y}(\mathrm{Y}-1)z\mathrm{Z}.$$

But we have here $e\mathrm{B}$ as an axis of symmetry, so that each triangle is counted twice, or the number of distinct triangles is $=\frac{1}{2}g$.

Case 26. $a=c=x,\ \mathrm{B}=\mathrm{D}=y,\ e=\mathrm{F}=z.$

$$\chi=\mathrm{Y}(x-1)(\mathrm{Y}-1)z(\mathrm{Z}-2)x,\ \chi'=\mathrm{Z}(z-2)\mathrm{Y}x(\mathrm{Y}-1)(x-1),$$
$$\begin{aligned}g&=x(x-1)\mathrm{Y}(\mathrm{Y}-1)\{z(\mathrm{Z}-2)+\mathrm{Z}(z-2)\}\\&=2x(x-1)\mathrm{Y}(\mathrm{Y}-1)(z\mathrm{Z}-z-\mathrm{Z}).\end{aligned}$$

Case 27. $a=c=e=x,\ \mathrm{B}=\mathrm{F}=y.$ By reciprocation of 28.

$$\text{No.}=\{2x(x-1)(x-2)+\mathrm{X}\}\mathrm{Y}(\mathrm{Y}-1)\mathrm{D},$$

where each triangle is counted twice, so that the number is really one half of this.

Case 28. $\mathrm{B}=\mathrm{D}=\mathrm{F}=x,\ c=e=y.$

Here $\quad g=\chi+\chi'-\text{Red.}$

$$\chi=\mathrm{X}y(\mathrm{X}-1)(y-1)(\mathrm{X}-1)a,\ \chi'=\mathrm{X}y(\mathrm{X}-1)(y-1)(\mathrm{X}-1)a(=\chi),$$
$$\chi+\chi'=ay(y-1)\,.\,2\mathrm{X}(\mathrm{X}-1)^2.$$

The reductions are those of the first and second mode as explained above, with the variation that the curves c and e are here identical, $c=e$, and that the curve D is identical with the curves $\mathrm{B}=\mathrm{F}$.

First-mode reduction is

$$a(\mathrm{C}+2\delta+3\varkappa)\mathrm{B}(\mathrm{B}-1)$$

(where δ, $\varkappa$ refer to the curve $c=e$), which is

$$=a\,c(c-1)\mathrm{B}(\mathrm{B}-1);$$

that is, the reduction is $=a\,y(y-1)\mathrm{X}(\mathrm{X}-1)$.

* Printed in the Table, erroneously, $2x\mathrm{X}\{\ldots+2yz+2\mathrm{YZ}\}$.

And the second-mode reduction is

$$a(2\tau+3\iota)c(c-1)$$

(where τ, ι refer to the curve $\mathrm{B}=\mathrm{D}=\mathrm{F}$), which is

$$=a\{\mathrm{B}(\mathrm{B}-1)-b\}c(c-1);$$

that is, the reduction is $=a\,y(y-1)\{\mathrm{X}(\mathrm{X}-1)-x\}$.

Hence the two together are $=a\,y(y-1)\{2\mathrm{X}(\mathrm{X}-1)-x\}$; and subtracting from $\chi+\chi'$ we have

$$\mathrm{g}=a\,y(y-1).\{2\mathrm{X}(\mathrm{X}-1)(\mathrm{X}-2)+x\};$$

but on account of the symmetry each triangle is reckoned twice, and the number of triangles is $=\tfrac{1}{2}\mathrm{g}$.

Case 29. $a=c=\mathrm{B}=x$, $\mathrm{D}=\mathrm{F}=y$.

$$\chi=(\mathrm{X}-2)(x-3)\mathrm{Y}e(\mathrm{Y}-1)x,\quad \chi'=\mathrm{Y}e(\mathrm{Y}-1)x(\mathrm{X}-2)(\mathrm{X}-3)(=\chi),$$
$$\mathrm{g}=2x(x-3)(\mathrm{X}-2)\mathrm{Y}(\mathrm{Y}-1)e.$$

Case 29. *Second process.* Taking the form

$$\mathrm{C}=\mathrm{D}=e=x,\quad \mathrm{B}=\mathrm{F}=y;$$

here

$$\text{No.}=\chi+\chi'-\text{Red.},$$

$$\chi=\mathrm{Y}x(\mathrm{X}-2)(x-3)\mathrm{Y}a,\quad =\chi',$$

and

$$\chi+\chi'=2\mathrm{Y}^2x(x-3)(\mathrm{X}-2)a.$$

There is a first-mode reduction,

$$a\mathrm{Y}\{2\tau+2\delta(\mathrm{X}-4)+3\varkappa(\mathrm{X}-3)\},$$

viz. this is

$$\begin{aligned}a\mathrm{Y}\{\quad &\mathrm{X}^2-\mathrm{X}+8x-3\xi\\ &+(\mathrm{X}-4)(x^2-x+8\mathrm{X}-3\xi)\\ &+(\mathrm{X}-3)(\qquad -9\mathrm{X}+3\xi)\},\end{aligned}$$

which is

$$=a\mathrm{Y}\{\mathrm{X}(x^2-x-6)-4x^2+12x\};$$

and a second-mode reduction,

$$a\mathrm{Y}\mathrm{X}(x-2)(x-3).$$

Hence the two together are

$$=a\mathrm{Y}\{\mathrm{X}(2x^2-6x)-4x^2+12x\}$$
$$=2\mathrm{Y}x(x-3)(\mathrm{X}-2)a,$$

whence the result is

$$=2(\mathrm{Y}^2-\mathrm{Y})x(x-3)(\mathrm{X}-2)a,$$

which agrees with that obtained above.

On account of the symmetry we must divide by 2.

Case 30. $e=\mathrm{D}=\mathrm{F}=x$, $a=c=y$. By reciprocation of 29.

$$\text{No.}=2\mathrm{X}(\mathrm{X}-3)(x-2)y(y-1)\mathrm{B}.$$

On account of the symmetry we must divide by 2.

Case 31. $c=e=\mathrm{D}=x$, $a=\mathrm{B}=y$.

$$\begin{aligned}&\chi=(\mathrm{Y}-2)x(\mathrm{X}-2)(x-3)\mathrm{F}y,\ \chi'=\mathrm{F}x(\mathrm{X}-2)(x-3)\mathrm{Y}(y-2),\\ &g=x(x-3)(\mathrm{X}-2)\mathrm{F}\{(\mathrm{Y}-2)y+\mathrm{Y}(y-2)\}\\ &\ \ =2x(x-3)(\mathrm{X}-2)(y\mathrm{Y}-y-\mathrm{Y})\mathrm{F}.\end{aligned}$$

Case 32. $\mathrm{F}=\mathrm{B}=a=x$, $\mathrm{D}=e=y$. By reciprocation of 31.

$$\text{No.}=2\mathrm{X}(\mathrm{X}-3)(x-2)(y\mathrm{Y}-y-\mathrm{Y})c.$$

Case 33. $\mathrm{B}=\mathrm{F}=y$, $a=e=\mathrm{D}=x$. By reciprocation of 34.

$$\text{No.}=2(x-1)(x\mathrm{X}-x-\mathrm{X})\mathrm{Y}(\mathrm{Y}-1)c.$$

Case 34. $c=e=y$, $\mathrm{B}=\mathrm{D}=a=x$. (In the Table, erroneously, $\mathrm{B}=\mathrm{D}=e=x$.)

$$\begin{aligned}&\chi=(\mathrm{X}-2)y(\mathrm{X}-1)(y-1)\mathrm{F}x,\ \chi'=\mathrm{F}y\mathrm{X}(y-1)(\mathrm{X}-1)(x-2),\\ &g=y(y-1)(\mathrm{X}-1)\{(\mathrm{X}-2)x+\mathrm{X}(\mathrm{X}-2)\}\mathrm{F}\\ &\ \ =2(\mathrm{X}-1)(x\mathrm{X}-x-\mathrm{X})y(y-1)\mathrm{F}.\end{aligned}$$

Case 35. $a=\mathrm{D}=y$, $c=e=\mathrm{B}=x$.

$$\begin{aligned}&\chi=\mathrm{X}(x-2)\mathrm{Y}(x-1)\mathrm{F}y,\ \chi'=\mathrm{F}x\mathrm{Y}(x-1)(\mathrm{X}-2)y,\\ &g=y\mathrm{Y}(x-1)\{\mathrm{X}(x-2)+(\mathrm{X}-2)x\}\mathrm{F}\\ &\ \ =2(x-1)(x\mathrm{X}-x-\mathrm{X})y\mathrm{Y}\mathrm{F}.\end{aligned}$$

Case 36. $a=\mathrm{D}=y$, $\mathrm{B}=\mathrm{F}=e=x$. By reciprocation of 35.

$$\text{No.}=2(\mathrm{X}-1)(\mathrm{X}x-x-\mathrm{X})y\mathrm{Y}c.$$

Case 37. $a=e=\mathrm{D}=x$, $c=\mathrm{B}=y$. By reciprocation of 38.

$$\text{No.}=2(x-1)\{xy\mathrm{XY}-xy(\mathrm{X}+\mathrm{Y})-\mathrm{XY}(x+y)+2xy+2\mathrm{XY}\}\mathrm{F}.$$

Case 38. $\mathrm{B}=\mathrm{D}=a=x$, $\mathrm{F}=e=y$.

$$\begin{aligned}&\chi=(\mathrm{X}-2)c(\mathrm{X}-1)y(\mathrm{Y}-2)x,\ \chi'=\mathrm{Y}(y-2)\mathrm{X}c(\mathrm{X}-1)(x-2),\\ &g=(\mathrm{X}-1)c\{xy(\mathrm{X}-2)(\mathrm{Y}-2)+\mathrm{XY}(x-2)(y-2)\}\\ &\ \ =2(\mathrm{X}-1)\{xy\mathrm{XY}-xy(\mathrm{X}+\mathrm{Y})-\mathrm{XY}(x+y)+2xy+2\mathrm{XY}\}c.\end{aligned}$$

Case 39. $a=c=e=\mathrm{B}=x$.

Functional process; the curve is assumed to be the aggregate of two curves, say $a=c=e=\mathrm{B}=x+x'$. Forming the enumeration

		Case.
$xXxDxF$	$x'X'x'Dx'F$	39
$x'Xx\ .\,x\,.$	&c.	10
$xX'x\ .\,x\,.$	.	6
$x'X'x\ .\,x\,.$	.	14
$xXx'\ .\,x\,.$	.	10
$x'Xx'\ .\,x\,.$	.	12
$xX'x'\ .\,x\,.$	.	14
$x'X'x'\ .\,x\,.$	.	8

(where the second column is derived from the first by a mere interchange of the accented and unaccented letters), I annex to each line the number of the case to which it belongs; thus $x'XxDxF$ is $B=c=e=x$, which is Case 10, and so in the other instances. Observing that cases 10 and 14 occur each twice, we have thus

$$\phi(x+x')-\phi x-\phi x'=\text{DF multiplied into}$$

$$\begin{array}{lll}
\quad 4(x-1)(Xx-X-x)x' & +\ \ldots & (10)\times 2\\
+\{2x(x-1)(x-2)+X\}X' & +\ \ldots & (6)\\
+4x(x-1)(X'x'-X'-x') & +\ \ldots & (14)\times 2\\
+2x(x-1)x'X' & +\ \ldots & (12)\\
+2x(x-3)(X-2)x' & +\ \ldots & (8)
\end{array}$$

where the $(\ldots)$'s refer to the like functions with the two sets of letters interchanged. Developing and collecting, this is

$$\phi(x+x')-\phi x-\phi x'=\text{DF multiplied into}$$

$$\begin{array}{l}
2XX'\\
+2X(3x^2x'+3xx'^2+x'^3-10xx'-5x'^2+6x')\\
+2X'(x^3+3x^2x'+3xx'^2-5x^2-10xx'+6x)\\
-12(x^2x'+xx'^2)+40xx',
\end{array}$$

and thence

$$\phi x=\text{DF multiplied into}$$

$$\begin{array}{ll}
X^2 & \\
+X(2x^3-10x^2+12x) & -LX\\
-4x^3+20x^2 & -lx-\lambda\xi,
\end{array}$$

where the constants L, l, λ have to be determined. Now for a cubic curve the number of triangles vanishes; that is, we have $\phi x=0$ in each of the three cases,

$$\begin{array}{lll}
x=3, & X=6, & \xi=18,\\
\quad ,, & X=4, & \xi=12,\\
\quad ,, & X=3, & \xi=10,
\end{array}$$

and we thus obtain the three equations

$$0=108-6\mathrm{L}-3l-18\lambda,$$
$$0=\ \ 88-4\mathrm{L}-3l-12\lambda,$$
$$0=\ \ 81-3\mathrm{L}-3l-10\lambda,$$

giving $\mathrm{L}=1$, $l=16$, $\lambda=3$. Whence, finally,

$$\phi x=\{\mathrm{X}^2+\mathrm{X}(2x^3-10x^2+12x-1)-4x^3+20x^2-16x-3\xi\}\,\mathrm{DF}.$$

Second process, by correspondence. We have

$$g-\chi-\chi'+\mathrm{F}(e-\varepsilon-\varepsilon')=0,$$
$$e-\varepsilon-\varepsilon'+\mathrm{D}(c-\gamma-\gamma')=0,$$

and thence

$$g-\chi-\chi'=\mathrm{DF}(c-\gamma-\gamma').$$

Moreover

$$\chi=(\mathrm{X}-2)(x-3)\mathrm{D}(x-1)\mathrm{F}(x-1),$$
$$\chi'=\mathrm{F}(x-1)\mathrm{D}(x-1)(\mathrm{X}-2)(x-1),\ =\chi,$$
$$\chi+\chi'=\mathrm{DF}(\mathrm{X}-2)2(x-3)(x-1)^2,$$

and

$$c-\gamma-\gamma'=2\tau+(\mathrm{X}-3)\varkappa-2(\mathrm{X}-2)(x-3),$$

as is easily obtained, but see also *post*, No. 29; hence

$$\begin{aligned} g=\mathrm{DF}\text{ multiplied into}&\\ &(\mathrm{X}-2).2(x-3)(x-1)^2\\ &+(\mathrm{X}-2).-2(x-3)\\ &+2\tau+(\mathrm{X}-3)\varkappa\,; \end{aligned}$$

but I reject the term $\mathrm{DF}.(\mathrm{X}-3)\varkappa$ as in fact giving a heterotypic solution; I do not go into the explanation of this. And then substituting for 2τ its value, we have

$$\begin{aligned} g=\mathrm{DF}\text{ multiplied into}&\\ &(\mathrm{X}-2).2x(x-1)(x-2)\\ &+\mathrm{X}^2-\mathrm{X}+8x-3\xi, \end{aligned}$$

where the second factor is

$$=\mathrm{X}^2+\mathrm{X}(2x^3-10x^2+12x-1)-4x^3+20x^2-16x-3\xi,$$

which is the foregoing result.

Case 40. $\mathrm{B}=\mathrm{D}=\mathrm{F}=e=x$. By reciprocation of 39,

$$\text{No.}=\{x^2+x(2\mathrm{X}^3-10\mathrm{X}^2+12\mathrm{X}-1)-4\mathrm{X}^3+20\mathrm{X}^2-16\mathrm{X}-3\xi\}ac.$$

Case 41. $c=e=\mathrm{D}=\mathrm{F}=x$.

$$\chi=\mathrm{B}x(\mathrm{X}-2)(x-3)(\mathrm{X}-3)a,$$
$$\chi'=\mathrm{X}(x-2)(\mathrm{X}-3)(x-3)\mathrm{B}a,$$
$$\begin{aligned} g &=(x-3)(\mathrm{X}-3)a\mathrm{B}\{x(\mathrm{X}-2)+\mathrm{X}(x-2)\} \\ &=2(x-3)(\mathrm{X}-3)(x\mathrm{X}-x-\mathrm{X})a\mathrm{B}. \end{aligned}$$

Case 42. $a=c=\mathrm{D}=\mathrm{F}=x$.

Functionally, viz. the curve is supposed to be the aggregate of two curves, say $a=c=\mathrm{D}=\mathrm{F}=x+x'$.

The enumeration is

		Case.
$x\ \mathrm{B}x\ \mathrm{X}\ e\mathrm{X}$	$x'\ \mathrm{B}\,x'\ \mathrm{X}'\ e\,\mathrm{X}'$,	(42)
$x'\ .\ x\ \mathrm{X}\ .\ \mathrm{X}$	&c.	(11)
$x\ .\ x'\,\mathrm{X}\ .\ \mathrm{X}$		(11)
$x'\ .\ x'\,\mathrm{X}\ .\ \mathrm{X}$		(17)
$x\ .\ x\ \mathrm{X}'.\ \mathrm{X}$		(10)
$x'\ .\ x\ \mathrm{X}'.\ \mathrm{X}$		(19)
$x\ .\ x'\,\mathrm{X}'.\ \mathrm{X}$		(21)
$x'\ .\ x'\,\mathrm{X}'.\ \mathrm{X}$		(10)

whence

$$\phi(x+x')-\phi x-\phi x'=e\mathrm{B}\text{ multiplied into}$$
$$\begin{array}{lll} 4(\mathrm{X}-1)(\mathrm{X}x-\mathrm{X}-x)x' & +\,.. & (11)\times 2 \\ +2x(x-1)\mathrm{X}'(\mathrm{X}'-1) & +\,.. & (17) \\ +4(x-1)(\mathrm{X}x-x-x)\mathrm{X}' & +\,.. & (10)\times 2 \\ +2x\mathrm{X}x'\mathrm{X}' & +\,.. & (19) \\ +2xx'\mathrm{X}\mathrm{X}'-2(x+x')\mathrm{X}\mathrm{X}'-2(\mathrm{X}+\mathrm{X}')xx'+4(\mathrm{X}x'+\mathrm{X}'x) & +\,.. & (21) \end{array}$$

where the (..)'s refer to the like functions with the two sets of letters interchanged. Developing and collecting, we have

$$\phi(x+x')-\phi x-\phi x'=e\mathrm{B}\text{ multiplied into}$$
$$\begin{array}{l} \mathrm{X}^2\ \ (4xx'+2x'^2-6x') \\ +\mathrm{X}\mathrm{X}'(4x^2+8xx'+4x'^2-12x-12x'+8) \\ +\mathrm{X}'^2\ (2x^2+4xx'-6x) \\ +\mathrm{X}\ \ (-12xx'-6x'^2\ +18x') \\ +\mathrm{X}'\ (-\ 6x^2\ -12xx'+18x) \\ +\qquad 8xx', \end{array}$$

and consequently

$$
\begin{aligned}
\phi x = e\mathrm{B} &\text{ multiplied into} \\
&\mathrm{X}^2(2x^2-6x+4) \\
&+\mathrm{X}(-6x^2+18x+\mathrm{L}) \\
&+ \quad 4x^2+lx \quad +\lambda\xi,
\end{aligned}
$$

where the constants L, l, λ have to be determined. The number of triangles vanishes when the curve is a line or a conic, that is $\phi x=0$ for $x=1, \mathrm{X}=0, \xi=0$, and for $x=\mathrm{X}=2$, $\xi=6$; we thus have

$$
\begin{aligned}
0 &= 4+l, \\
0 &= 40+2\mathrm{L}+2l+6\lambda.
\end{aligned}
$$

Moreover, the data being sibireciprocal, the result must be so likewise; we must therefore have $\mathrm{L}=l$. We thus obtain $\mathrm{L}=l=\lambda=-4$; so that finally

$$\phi x=\{\mathrm{X}^2(2x^2-6x+4)+\mathrm{X}(-6x^2+18x-4)+4x^2-4x-4\xi\}e\mathrm{B}.$$

Second process, by correspondence: form $a=c=\mathrm{D}=\mathrm{F}=x$. We have

$$\mathrm{c}-\chi-\chi'+2(\mathrm{f}-\phi-\phi')=0;$$

also from the special consideration that the points D, F are given as the intersections of the curve x, by the first polar of the point e, which first polar does not pass through a, we have

$$(\mathrm{f}-\phi-\phi')+e(\mathrm{d}-\delta-\delta')=0,$$

and by the consideration that c, D are given as intersections, c a double intersection, of the curve with the first polar of the point c, which first polar does not pass through a,

$$\mathrm{d}-\delta-\delta'+2(\mathrm{c}-\gamma-\gamma')=0,$$

whence

$$\mathrm{g}-\chi-\chi'=-4e(c-\gamma-\gamma')$$

and

$$\mathrm{c}-\gamma-\gamma'=\mathrm{B}\Delta,$$

so that this is

$$
\begin{aligned}
\mathrm{g}-\chi-\chi' &= -4\mathrm{B}e\Delta \\
&= -4\mathrm{B}e(-2\mathrm{X}-2x+2+\xi).
\end{aligned}
$$

Also

$$
\begin{aligned}
\chi &= \mathrm{B}(x-1)(\mathrm{X}-2)e(\mathrm{X}-1)(x-2) \\
\chi' &= (\mathrm{X}-2)e(\mathrm{X}-1)(x-2)\mathrm{B}(x-1), =\chi,
\end{aligned}
$$

so that

$$
\begin{aligned}
\mathrm{g}=\mathrm{B}e &\text{ multiplied into} \\
&2(\mathrm{X}-1)(\mathrm{X}-2)(x-1)(x-2)-4(-2\mathrm{X}-2x+2+\xi),
\end{aligned}
$$

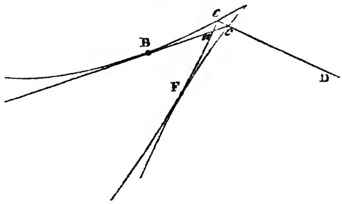
viz. this is

$$\mathrm{B}e\{\mathrm{X}^2(2x^2-6x+4)+\mathrm{X}(-6x^2+18x-4)+4x^2-4x-4\xi\}.$$

Third process: form $c=e=\mathrm{F}=\mathrm{B}=x$.

$$\mathrm{g}=\chi+\chi'-\text{Red.},$$
$$\chi=\mathrm{X}(x-2)\mathrm{D}(x-1)(\mathrm{X}-2)a,$$
$$\chi'=\mathrm{X}(x-2)\mathrm{D}(x-1)(\mathrm{X}-2)a,\ =\chi,$$
$$\chi+\chi'=a\mathrm{D}\,.\,2\mathrm{X}(\mathrm{X}-2)(x-1)(x-2).$$

Fig. 4.



The first-mode reduction is here

$$a\mathrm{D}[(\mathrm{X}-2)\mathrm{X}+(\mathrm{X}-4)2\delta+(\mathrm{X}-3)3\kappa+\kappa];$$

where the last term $a\mathrm{D}\kappa$ arises from the tangents $c\mathrm{B}a$ and $e\mathrm{F}a$, each coinciding with a cuspidal tangent, as shown in the figure.

The second-mode reduction is

$$=a\mathrm{D}\,.\,\mathrm{X}(x-2)(x-3),$$

so that the two reductions together are

$$=a\mathrm{D}\{(\mathrm{X}-2)\mathrm{X}+(\mathrm{X}-4)2\delta+(\mathrm{X}-3)3\kappa+\kappa+\mathrm{X}(x-2)(x-3)\},$$

viz. this is

$$=a\mathrm{D}\{(\mathrm{X}-2)\mathrm{X}+(\mathrm{X}-4)(2\delta+3\kappa)+4\kappa+\mathrm{X}(x-2)(x-3)\};$$

or substituting for $2\delta+3\kappa$ and κ the values $x^2-x-\mathrm{X}$ and $-3\mathrm{X}+\xi$ respectively, and reducing, it is

$$a\mathrm{D}\{\mathrm{X}(2x^2-6x-4)-4x^2+4x+4\xi\}.$$

Hence subtracting from $\chi+\chi'$, written in the form

$$a\mathrm{D}\{\mathrm{X}^2(2x^2-6x+4)+\mathrm{X}(-4x^2+12x-8)\},$$

the result is

$$=a\mathrm{D}\{\mathrm{X}^2(2x^2-6x+4)+\mathrm{X}(-6x^2+18x-4)+4x^2-4x-4\xi\}.$$

On account of the symmetry we must divide by 2.

Case 43. $a=c=e=x$, $\mathrm{B}=\mathrm{D}=\mathrm{F}=y$.

Suppose for a moment that the angle a is a free point; the locus of a is a curve the order of which is obtained from Case 28, by writing $c=e=x$, $\mathrm{B}=\mathrm{D}=\mathrm{F}=y$; the locus in question meets a curve order a in $\{2\mathrm{Y}(\mathrm{Y}-1)(\mathrm{Y}-2)+y\}x(x-1)a$ points; wherefore the order of the locus is

$$=\{2\mathrm{Y}(\mathrm{Y}-1)(\mathrm{Y}-2)+y\}x(x-1),$$

and this locus meets the curve $a=c=e=x$ in a number of points

$$=\{2\mathrm{Y}(\mathrm{Y}-1)(\mathrm{Y}-2)+y\}x^2(x-1),$$

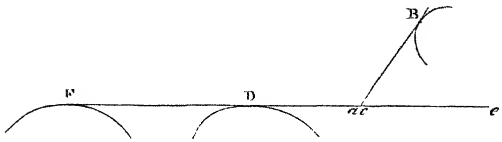
viz. this is the number of positions of the angle a; but several of these belong to special forms of the triangle $a\mathrm{B}c\mathrm{D}e\mathrm{F}$, giving heterotypic solutions, which are to be rejected; the required number is thus

$$\{2\mathrm{Y}(\mathrm{Y}-1)(\mathrm{Y}-2)+y\}x^2(x-1)-\text{Red.}$$

The reduction is due first and secondly to triangles wherein the angle a coincides with an angle c or e, and thirdly to triangles wherein the angles a, c, e all coincide.

1°. Take for the side cDe a double tangent of the curve $B=D=F$, this meets the

Fig. 5.



curve $a=c=e$ in x points, and selecting any one of them for e and any other for c, we have from the last-mentioned point $Y-2$ tangents to the curve $B=D=F$; and in respect of each of these a position of a coincident with c. The reduction on this account is $2\tau x(x-1)(Y-2)$; but since we may in the figure interchange c and e, B and F, we have the same number belonging to the coincidence of the angles a, e, or together the reduction is $=4\tau x(x-1)(Y-2)$.

But instead of a double tangent we may have cDe a stationary tangent; we have thus reductions $3\iota\, x(x-1)(Y-2)$ and $3\iota\, x(x-1)(Y-2)$, together $6\iota\, x(x-1)(Y-2)$; and for the double and stationary tangents together we have

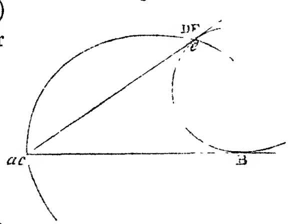
$$(4\tau+6\iota)x(x-1)(Y-2)$$

$$=2\{Y(Y-1)-y\}x(x-1)(Y-2),$$

that is,

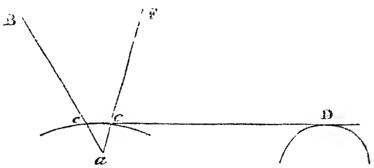
$$=2x(x-1)Y(Y-1)(Y-2)-2x(x-1)y(Y-2).$$

Fig. 6.



2°. The side cDe may be taken to be a tangent to the curve $B=D=F$ at any one of its intersections with the curve $a=c=e$. Taking then the point e at the intersection in question, and the point c at any other of the intersections of the tangent with the curve $a=c=e$, and from c drawing any other tangent to the curve $B=D=F$, there is in respect of each of these tangents a position of a at c; and the reduction on this account is $=xy(x-1)(Y-1)$. But interchanging in the figure the letters c, e, B, F, there is an equal reduction belonging to the coincidence of a, e; and the whole reduction in this manner is $=2x(x-1)y(Y-1)$.

Fig. 7.



3°. If the side cDe intersects the curve $a=c=e$ in two coincident points, then taking these in either order for the points c, e, and from the two points respectively drawing two other tangents to the curve $D=B=F$, we have a triangle wherein the angles a, c, e all coincide. The side cDe may be a proper tangent to the curve $a=c=e$, or it may

pass through a node or a cusp of this curve, viz. it is either a common tangent of the curves B=D=F and $a=c=e$ (as in the figure, except that for greater distinctness the points c and e are there drawn nearly instead of actually coincident), or it may be a tangent to the curve B=D=F from a node or a cusp of the curve $a=c=e$; we have thus the numbers

Common tangent	$XY(Y-1)(Y-2)$,
Tangent from node	$2\delta Y(Y-1)(Y-2)$,
Tangent from cusp	$2\varkappa Y(Y-1)(Y-2)$;

but (as we are counting intersections with the curve $a=c=e$) the second of these, as being at a node of this curve, is to be taken 2 times; and the third, as being at a cusp, 3 times; and the three together are thus

$$\begin{aligned}&(X+4\delta+6\varkappa\quad)Y(Y-1)(Y-2),\\ &=\{2x(x-1)-X\}Y(Y-1)(Y-2).\end{aligned}$$

The reductions 1°, 2°, 3° altogether are

$$\begin{aligned}&2x(x-1)Y(Y-1)(Y-2)\\ &-2x(x-1)y(Y-2)\\ &+2x(x-1)y(Y-1)\\ &+2x(x-1)Y(Y-1)(Y-2)\\ &-XY(Y-1)(Y-2),\end{aligned}$$

which is

$$\begin{aligned}=&4x(x-1)Y(Y-1)(Y-2)\\ &+2x(x-1)y\\ &-XY(Y-1)(Y-2);\end{aligned}$$

and subtracting from the before-mentioned number

$$\begin{aligned}&2x^2(x-1)Y(Y-1)(Y-2)\\ &+x^2(x-1)y,\end{aligned}$$

the required number of positions of the angle a is

$$\begin{aligned}=&2x(x-1)(x-2)Y(Y-1)(Y-2)\\ &+yx(x-1)(x-2)y+XY(Y-1)(Y-2).\end{aligned}$$

The number of triangles is on account of the symmetry equal to one-sixth of this number.

Case 44. $e=\mathrm{D}=\mathrm{F}=x,\ a=c=\mathrm{B}=y$.

$$\chi=(\mathrm{Y}-2)(y-3)\mathrm{X}(x-2)(\mathrm{X}-3)y,$$
$$\chi'=\mathrm{X}(x-2)(\mathrm{X}-3)y(\mathrm{Y}-2)(y-3)(=\chi),$$
$$g=2(x-2)\mathrm{X}(\mathrm{X}-3)(\mathrm{Y}-2)y(y-3):$$

there is a division by 2 on account of the symmetry.

Case 45. $a=\mathrm{D}=\mathrm{B}=x,\ c=e=\mathrm{F}=y$.

$$\chi=(\mathrm{X}-2)y(\mathrm{X}-1)(y-1)(\mathrm{Y}-2)x,$$
$$\chi'=\mathrm{Y}(y-2)\mathrm{X}(y-1)(\mathrm{X}-1)(x-2),$$
$$g=(\mathrm{X}-1)(y-1)\{xy(\mathrm{X}-2)(\mathrm{Y}-2)+\mathrm{XY}(x-2)(y-2)\}$$
$$=2(\mathrm{X}-1)(y-1)\{\mathrm{XY}xy-\mathrm{XY}(x+y)-xy(\mathrm{X}+\mathrm{Y})+2xy+2\mathrm{XY}\}.$$

Case 46. $a=c=y,\ \mathrm{B}=\mathrm{D}=\mathrm{F}=e=x$. By reciprocation of 47.

$$\text{No.}=y(y-1)\{x^2+x(2\mathrm{X}^3-10\mathrm{X}^2+12\mathrm{X}-1)-4\mathrm{X}^3+20\mathrm{X}^2-16\mathrm{X}-3\xi\}:$$

there is a division by 2 on account of the symmetry.

Case 47. $\mathrm{D}=\mathrm{F}=y,\ a=c=e=\mathrm{B}=x$.

The functional process is exactly the same as for No. 39 ($a=c=e=\mathrm{B}=x$), with only $\mathrm{Y}(\mathrm{Y}-1)$ written instead of DF; hence

$$\text{No.}=\mathrm{Y}(\mathrm{Y}-1)\{\mathrm{X}^2+\mathrm{X}(2x^3-10x^2+12x-1)-4x^3+20x^2-16x-3\xi\}:$$

there is a division by 2 on account of the symmetry.

Case 48. $a=c=\mathrm{D}=\mathrm{F}=x,\ e=\mathrm{B}=y$.

The functional process, writing $a=c=\mathrm{D}=\mathrm{F}=x+x'$, would be precisely the same as for Case 42, with only the factor $y\mathrm{Y}$ written instead of $e\mathrm{B}$; and we have thus the like result, viz.

$$\text{No.}=\{\mathrm{X}^2(2x^2-6x+4)+\mathrm{X}(-6x^2+18x-4)+4x^2-4x-4\xi\}y\mathrm{Y},$$

which on account of the symmetry must be divided by 2.

Case 49. $a=\mathrm{B}=y,\ c=e=\mathrm{D}=\mathrm{F}=x$.

$$\chi=(\mathrm{Y}-2)x(\mathrm{X}-2)(x-3)(\mathrm{X}-3)y,$$
$$\chi'=\mathrm{X}(x-2)(\mathrm{X}-3)(x-3)\mathrm{Y}(y-2),$$
$$g=(x-3)(\mathrm{X}-3)\{xy(\mathrm{X}-2)(\mathrm{Y}-2)+\mathrm{XY}(x-2)(y-2)\}$$
$$=2(x-3)(\mathrm{X}-3)\{xy\mathrm{XY}-(x+y)\mathrm{XY}-(\mathrm{X}+\mathrm{Y})xy+2xy+2\mathrm{XY}\}.$$

Case 50. $c=e=\mathrm{B}=\mathrm{D}=\mathrm{F}=x$.

Functional process; by taking the curve $c=e=\mathrm{B}=\mathrm{D}=\mathrm{F}$ as the aggregate of two curves, say $=x+x'$. The cases are

		Case.
$a\mathrm{X}\,x\,\mathrm{X}\,x\,\mathrm{X}$	$a\mathrm{X}'x'\mathrm{X}'x'\mathrm{X}'$	50
$.\,\mathrm{X}'x\,\mathrm{X}\,x\,.$	&c.	41
$.\,\mathrm{X}\,x'\mathrm{X}\,x\,.$	.	40
$.\,\mathrm{X}'x'\mathrm{X}\,x\,.$	.	32
$.\,\mathrm{X}\,x\,\mathrm{X}'x\,.$	.	42
$.\,\mathrm{X}'x\,\mathrm{X}'x\,.$	.	33
$.\,\mathrm{X}\,x'\mathrm{X}'x\,.$	.	38
$.\,\mathrm{X}'x'\mathrm{X}'x\,.$	.	32
$.\,\mathrm{X}\,x\,\mathrm{X}\,x'\,.$	.	40
$.\,\mathrm{X}'x\,\mathrm{X}\,x'\,.$	.	36
$.\,\mathrm{X}\,x'\mathrm{X}\,x'\,.$	.	28
$.\,\mathrm{X}'x'\mathrm{X}\,x'\,.$	.	33
$.\,\mathrm{X}\,x\,\mathrm{X}'x'\,.$	.	38
$.\,\mathrm{X}'x\,\mathrm{X}'x'\,.$	.	36
$.\,\mathrm{X}\,x'\mathrm{X}'x'\,.$	.	29
$.\,\mathrm{X}'x'\mathrm{X}'x'\mathrm{X}$	.	41

and we thus have

$$\phi(x+x')-\phi x-\phi x'=a \text{ multiplied into}$$

$$\begin{aligned}
= \;& 4(x-3)(\mathrm{X}-3)(x\mathrm{X}-x-\mathrm{X})\mathrm{X}'+\,.\,. && 2(41)\\
&+2x'[x^2+x(2\mathrm{X}^3-10\mathrm{X}^2+12\mathrm{X}-1)-4\mathrm{X}^3+20\mathrm{X}^2-16\mathrm{X}-3\xi]+\,.\,. && 2(40)\\
&+4\mathrm{X}(\mathrm{X}-3)(x-2)(x'\mathrm{X}'-\mathrm{X}'-x')+\,.\,. && 2(32)\\
&+[\mathrm{X}^2(2x^2-6x+4)+\mathrm{X}(-6x^2+18x-4)+4x^2-4x-4\xi]\mathrm{X}'+\,.\,. && (42)\\
&+4(x-1)(x\mathrm{X}-x-\mathrm{X})(\mathrm{X}'^2-\mathrm{X}')+\,.\,. && 2(33)\\
&+4(\mathrm{X}-1)[\mathrm{X}\mathrm{X}'xx'-\mathrm{X}\mathrm{X}'(x+x')-xx'(\mathrm{X}+\mathrm{X}')+2\mathrm{X}\mathrm{X}'+2xx']+\,.\,. && 2(38)\\
&+4x\mathrm{X}(\mathrm{X}'-1)(\mathrm{X}'x'-\mathrm{X}'-x')+\,.\,. && 2(36)\\
&+(x^2-x)(2\mathrm{X}'^3-6\mathrm{X}'^2+4\mathrm{X}'+x')+\,.\,. && (28)\\
&+2x(x-3)(\mathrm{X}-2)(\mathrm{X}'^2-\mathrm{X}')+\,.\,. && (29)
\end{aligned}$$

where as before the (. .)'s refer to the like functions with the two sets of letters interchanged. Developing and collecting, this is

$$\phi(x+x')-\phi x-\phi x'=a \text{ multiplied into}$$

$$\begin{aligned}
&3x^2x'+3xx'^2\\
&+x^2\,.\,6\mathrm{X}^2\mathrm{X}'+6\mathrm{X}\mathrm{X}'^2+2\mathrm{X}'^3\\
&\qquad-28\mathrm{X}\mathrm{X}'-14\mathrm{X}'^2\\
&\qquad+28\mathrm{X}'
\end{aligned}$$

$$
\begin{aligned}
&+xx' \,.\, 4X^3+12X^2X'+12XX'^2+4X'^3 \\
&\qquad -28X^2-56XX'-28X'^2 \\
&\qquad +56X+56X' \\
&\qquad -22 \\
&+x'^2 \,.\, 2X^3+6X^2X'+6XX'^2 \\
&\qquad -14X^2-28XX' \\
&\qquad +28X \\
&+x \,.\, -30X^2X'-30XX'^2-10X'^3 \\
&\qquad +140XX'+70X'^2 \\
&\qquad -116X'-6\xi' \\
&+x' \,.\, -10X^3-30X^2X'-30XX'^2 \\
&\qquad +70X^2+140XX' \\
&\qquad -116X-6\xi \\
&\qquad +36X^2X'+36XX'^2 \\
&\qquad -152XX' \\
&\qquad -4(X\xi'+X'\xi)\,;
\end{aligned}
$$

whence

$\phi x = a$ multiplied into

$$
\begin{aligned}
&x^3(&& && && +\ 1) \\
&+x^2(&& 2X^3-14X^2+ && 28X && -11) \\
&+x\,(-10X^3+70X^2 && -116X && && +l) \\
&+ && 12X^3-76X^2 && && +LX \\
&+\xi\,(-6x-4X && && && +\lambda),
\end{aligned}
$$

where the constants l, L, λ have to be determined. We should have $\phi x=0$ for a cubic curve; viz. $x=3$: $X=6$, $\xi=18$; $X=4$, $\xi=12$; or $X=3$, $\xi=10$. Writing first $x=3$, the equation is

$$8X^2-96X-72-\xi(18+4X)+3l+XL+\xi\lambda=0,$$

giving in the three cases respectively

$$
\begin{aligned}
3l+6L+18\lambda &= 1116, \\
3l+4L+12\lambda &= 736, \\
3l+3L+10\lambda &= 588\,;
\end{aligned}
$$

and we have then $l=-8$, $L=64$, $\lambda=42$, so that the required number is

$$
\begin{aligned}
= \ & x^3(&& && && + 1) \\
& +x^2(&& 2X^3-14X^2+ && 28X-11) \\
& +x(-10X^3+70X^2-116X- && 8) \\
& + \quad 12X^3-76X^2+ && 64X \\
& +\xi(-6x-4X+42 &&).
\end{aligned}
$$

As a verification, observe that for a conic, $x=X=2$, $\xi=6$, this is $=0$.

Second process, by correspondence: form $c=e=B=D=F=x$.

We have

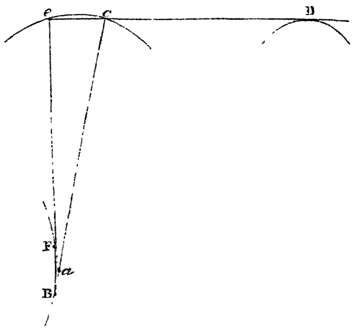
$$
\begin{aligned}
& g=\chi+\chi'-\text{Red.}, \\
& \chi=X(x-2)(X-3)(x-3)(X-3)a, \\
& \chi'=X(x-2)(X-3)(x-3)(X-3)a, =\chi. \\
& \chi+\chi'=a \text{ into} \\
& 2(x-2)(x-3)X(X-3)^2.
\end{aligned}
$$

There is a first-mode reduction, which is

$$=a\{2\delta(X-4)(X-5)+3\varkappa(X-3)(X-4)+\varkappa(X-3)+2\tau(X-3)\},$$

where the term $a\,.\,2\tau(X-3)$ arises, as shown in the figure, and a second-mode reduction, which is

Fig. 8.



$$=a\{2\tau(x-4)(x-5)+3\iota(x-3)(x-4)\};$$

and the two together are $=a$ into

$$
\begin{aligned}
& (X-4)(X-5)(x^2-x+8X \quad -3\xi) \\
& +(X-3)(X-4)(\qquad 9X \quad +3\xi) \\
& + \qquad (X-3)\begin{pmatrix} -3X \quad + \xi \\ +X^2- X+8x-3\xi \end{pmatrix} \\
& +(x-4)(x-5)(X^2-X+8x \quad -3\xi) \\
& +(x-3)(x-4)(\qquad -9x+3\xi);
\end{aligned}
$$

that is, $=a$ into

$$
\begin{aligned}
& -x^3 \\
& +x^2.2X^2-10X+11 \\
& +x.-10X^2+26X+8 \\
& \quad +4X^2+44X \\
& +\xi(6x+4X-42);
\end{aligned}
$$

and subtracting this from the foregoing value of $\chi+\chi'$, which is $=a$ into

$$
\begin{aligned}
& x^2(\quad 2X^3-12X^2+ 18X) \\
& +x(-10X^3+60X^2- 90X) \\
& + \qquad 12X^3-72X^2+108X,
\end{aligned}
$$

the result is as before.

There is a division by 2 on account of the symmetry.

Case 51. $a=c=e=\mathrm{B}=\mathrm{D}=x$. By reciprocation of 50,

$$
\begin{aligned}
\text{No. is} = \ & \mathrm{X}^3(\qquad\qquad\qquad\qquad\qquad +\ 1)\\
&+\mathrm{X}^2(\quad 2x^3-14x^2+\ 28x-11)\\
&+\mathrm{X}\,(-10x^3+70x^2-116x-\ 8)\\
&+\qquad 12x^3-76x^2+\ 64x\\
&+\xi(-6\mathrm{X}-4x+42).
\end{aligned}
$$

There is a division by 2 on account of the symmetry.

Case 52. $a=c=e=\mathrm{B}=\mathrm{D}=\mathrm{F}=x$.

Functional process, by taking the curve to be the aggregate of two curves, say $=x+x'$. The enumeration of the cases is conveniently made in a somewhat different manner from that heretofore employed, viz. we may write

x or x'	x' or x	Case	times
$\Vert$	$\Vert$		
all	none	(52)	1
a	residue	(50)	3
B	,,	(51)	3
a, c	,,	(46)	3
B, D	,,	(47)	3
a, D	,,	(48)	3
a, B	,,	(49)	6
a, c, e	B, D, F	(43)	1
a, B, F	c, e, D	(44)	3
a, B, D	c, e, F	(45)	6
			32;

and the functional equation then is

$$
\begin{aligned}
&\phi(x+x')-\phi x-\phi x'\\
&=3x'\left\{\begin{array}{l}
x^3(\qquad\qquad\qquad\qquad\qquad +\ 1)\\
x^2(\quad 2\mathrm{X}^3-14\mathrm{X}^2+\ 28\mathrm{X}-11)\\
x\,(-10\mathrm{X}^3+70\mathrm{X}^2-116\mathrm{X}-\ 8)\\
\quad +12\mathrm{X}^3-76\mathrm{X}^2+\ 64\mathrm{X}\\
\quad +\xi(-6x-4\mathrm{X}\qquad\qquad +42)
\end{array}\right\}+\ldots \qquad (50)\times 3
\end{aligned}
$$

$$+3X' \left\{\begin{array}{l} X^3(\qquad\qquad\qquad\quad +\ 1) \\ X^2(\quad 2x^3-14x^2+\ 28x-11) \\ X\,(-10x^3+70x^2-116x-\ 8) \\ \quad +12x^3-76x^2+\ 64x \\ \quad +\xi(-6X-4x \qquad +42) \end{array}\right\} \qquad\qquad (51)\times 3$$

$$+3(x'^2-x') \left\{\begin{array}{l} \quad x^2 \\ +x\,(2X^3-10X^2+12X-1) \\ - \quad 4X^3+20X^2-16X-3\xi \end{array}\right\} \quad +\ldots \qquad (46)\times 3$$

$$+3(X'^2-X') \left\{\begin{array}{l} \quad X^2 \\ +X(2x^3-10x^2+12x-1) \\ - \quad 4x^3+20x^2-16x-3\xi \end{array}\right\} \quad +\ldots \qquad (47)\times 3$$

$$+3x'X' \left\{\begin{array}{l} \quad X^2(\ \ 2x^2-\ \ 6x+4) \\ +X\,(-6x^2+18x-4) \\ + \qquad 4x^2-\ \ 4x-4\xi \end{array}\right\} \quad +\ldots \qquad (48)\times 3$$

$$+12(x'-3)(X'-3)\{xx'XX'-xx'(X+X')-XX'(x+x')+2xx'+2XX'\} \quad +\ldots \quad (49)\times 6$$

$$+\{2x'(x'-1)(x'-2)X(X-1)(X-2)+xx'(x'-1)(x'-2)+X'X(X-1)(X-2)\} \quad +\ldots \quad (43$$

$$+6(x'-2)X'(X'-3)(X-2)(x-3) \qquad\qquad +\ldots \quad (44)\times 3$$

$$+12(X'-1)(x-1)\{xx'XX'-xx'(X+X')-XX'(x+x')+2xx')+2XX'\} \quad +\ldots \quad (45)\times 6$$

where as before the (. .)'s refer to the like functions with the two sets of letters interchanged. Developing and collecting, this is found to be

$$=4X^3X'+6X^2X'^2+4XX'^3$$

$$+X^3 \left\{\begin{array}{l} \quad 6x^2x'\ +6xx'^2+2x'^3 \\ -36xx'-18x'^2 \\ +52x' \end{array}\right\}$$

$$+(X^2X'+XX'^2) \left\{\begin{array}{l} \qquad 6x^3+\ 18x^2x'+18xx'^2+6x'^3 \\ -\ \ 54x^2-108\,xx'-\ 54x'^2 \\ +156x\ +156x' \\ -138 \end{array}\right\}$$

$$+X'^3 \left\{\begin{array}{l} \quad 2x^3+6x^2x'+6xx'^2 \\ -18x^2-36xx' \\ +52x \end{array}\right\}$$

$$+\&c.\ \&c.$$

I abstain from writing down the remaining terms, as they can at once be obtained backwards from the value of ϕx; they were in fact found directly, and the integration of the functional equation then gives

$$
\begin{aligned}
\phi x = & \ X^4(\qquad\qquad\qquad\qquad + 1) \\
& + X^3(\ 2x^3 - 18x^2 + 52x - 46) \\
& + X^2(-18x^3 + 162x^2 - 420x + 221) \\
& + X\ (\ 52x^3 - 420x^2 + 704x + l\ ) \\
& + \ x^4 - 46x^3 + 221x^2 + lx \\
& + \xi \left\{ \begin{array}{l} X^2(\qquad\qquad - 9) \\ + X\ (\ - 12x + 135) \\ -9x^2 + 135x + \lambda \end{array} \right\}
\end{aligned}
$$

where the constants l, λ have to be determined; I have in the first instance written $l(X+x)+\lambda\xi$, instead of $LX+lx+\lambda\xi$, thus introducing two constants only, since it is clear from the symmetry in regard to x, X that we must have $l=L$. We must have $\phi x=0$, when the curve is a conic or cubic. Writing $x=2$, we have

$$\phi x = X^4 + 2X^3 - 115X^2 + 144X + 532 + \xi(-9X^2 + 111X + 234) + l(2+X) + \xi\lambda,$$

and then for the conic, $X=2$, $\xi=6$.

Writing $x=3$, we have

$$\phi x = X^4 + 2X^3 - 67X^2 - 264X + 828 + \xi(-9X^2 + 99X + 324) + l(3+X) + \lambda\xi,$$

and then for the three cases of the cubic $X=6$, $\xi=18$; $X=4$, $\xi=12$; and $X=3$, $\xi=10$. We have thus the four equations

$$
\begin{aligned}
2912 + 4l + 6\lambda &= 0, \\
9252 + 9l + 18\lambda &= 0, \\
5796 + 7l + 12\lambda &= 0, \\
4968 + 6l + 10\lambda &= 0,
\end{aligned}
$$

all satisfied by $l=+172$, $\lambda=-600$. Whence, finally,

$$
\begin{aligned}
\phi x = & \ X^4(\qquad\qquad\qquad\qquad + 1) \\
& + X^3(\ 2x^3 - 18x^2 + 52x - 46) \\
& + X^2(-18x^3 + 162x^2 - 420x + 221) \\
& + X\ (\ 52x^3 - 420x^2 + 704x + 172) \\
& + \ x^4 - 46x^3 + 221x^2 + 172x \\
& + \xi \left\{ \begin{array}{l} X^2(\qquad\qquad - 9) \\ + X\ (\ - 12x + 135) \\ -9x^2 + 135x - 600 \end{array} \right\}
\end{aligned}
$$

but on account of the symmetry the number of triangles is $=$one sixth of this expression.

The Case 52, *as belonging to a different series of Problems.* Art. Nos. 22 to 36.

22. In the foregoing Case 52, where all the curves are one and the same curve, we have the unclosed trilateral $a\mathrm{B}c\mathrm{D}e\mathrm{F}g$, and we seek for the number of the united points (a, g). But we may consider this as belonging to a series of questions, viz. we may seek for the number of the united points (a, B), (a, c), (a, D), (a, e), (a, F), (the last four of these giving by reciprocity the numbers of the united points (B, D), (B, e), (B, F), (B, g)), and finally the number of the united points (a, g). It is very instructive to consider this series of questions, and the more so that in those which precede (a, F) there are only special solutions having reference to the singular points and tangents of the curve, and that the solutions thus explain themselves.

23. Thus the first case is that of the united points (a, B), viz. we have here a point a on the curve, and from it we draw to the curve a tangent $a\mathrm{B}$ touching it at B; the points a and B are to coincide together. Observe that from a point in general a of the curve we have $\mathrm{X}-2$ tangents (X the class as heretofore), viz. we disregard altogether the tangent *at* the point, counting as 2 of the X tangents from a point not on the curve, and attend exclusively to the $\mathrm{X}-2$ tangents *from* the point. Now if the point a is an inflection, or if it is a cusp, there are only $\mathrm{X}-3$ tangents, or, to speak more accurately, one of the $\mathrm{X}-2$ tangents has come to coincide with the tangent *at* the point; such tangent is a tangent of three-pointic intersection, viz. we have the point a and the point B (counting, as a point of contact, twice) all three coinciding; that is, we have a position of the united point (a, B); and the number of these united points is $=\iota+\varkappa$.

24. It is important to notice that neither a point of contact of a double tangent, nor a double point, is a united point. In the case of the point of contact of a double tangent, one of the tangents from the point coincides with the double tangent; but the point B is here the other point of contact of this tangent, so that the points a, B are not coincident. In the case of a double point, regarding the assumed position of a at the double point as belonging to one of the two branches, then of the $\mathrm{X}-2$ tangents there are two, each coinciding with the tangent to the other branch; hence, attending to either of these, the point B belongs to the other branch, and thus, though a and B are each of them at the double point, the two do not constitute a united point. (In illustration remark that for a unicursal curve, the position of a answers to a value $=\lambda$, and that of B to a value $=\mu$ of the parameter θ, viz. λ, μ are the two values of θ at the double point; contrariwise in the foregoing case of a cusp, where there is a single value $\lambda=\mu$. Hence the whole number of the united points (a, B) is $=\iota+\varkappa$, and this is in fact the value given (as will presently appear) by the theory of correspondence.

I recall that I use Δ, $=2\mathrm{D}$, to denote twice the deficiency of the curve, viz. that we have $\Delta=\mathrm{X}-2x+2+\varkappa$, $=-2x-2\mathrm{X}+2+\xi$.

25. The several cases are

United points.

$(a,\ \mathrm{B})\qquad \mathrm{b}-\beta-\beta'=2\Delta,$

$(a,\ c\ )\qquad \mathrm{c}-\gamma-\gamma'+2(\mathrm{b}-\beta-\beta')=(\mathrm{X}-2)\Delta,$

(B, D)	$c_0-\gamma_0-\gamma'_0$ by reciprocity,
$(a$, D)	$d-\delta-\delta'+2(c_0-\gamma_0-\gamma'_0)+(X-3)(b-\beta-\beta')=0,$
(a, e)	$e-\varepsilon-\varepsilon'+2(d-\delta-\delta')+(X-3)(c-\gamma-\gamma')=0,$
(B, F)	$e_0-\varepsilon_0-\varepsilon'_0$ by reciprocity,
$(a$, F)	$f-\phi-\phi'+2(e_0-\varepsilon_0-\varepsilon'_0)+(X-3)(d-\delta-\delta')=0,$
(a, g)	$g-\chi-\chi'+2(f-\phi-\phi')+(X-3)(e-\varepsilon-\varepsilon')=0,$
(B, H)	$g_0-\chi_0-\chi'_0$ by reciprocity,

and so on.

26. The mode of obtaining these equations appears *ante*, Nos. 5 and 6, but for greater clearness I will explain it in regard to a pair of the equations, say those for (a, e), $(a$, D). Regarding a as given, we draw from a the tangents aBc, touching at B and besides intersecting at c (viz. the number of tangents is $=X-2$, and the number of the points c is $=(X-2)(x-3)$); from each of the positions of c we draw to the curve the $(X-3)$ tangents cDe touching at D and intersecting at e; the whole number of these tangents is $=(X-2)(x-3)(X-3)$; and this is also the number of the points D, but the number of the points e is $=(X-2)(x-3)(X-3)(x-3)$. Now this system of the $(X-2)(x-3)(X-3)$ tangents is the curve Θ of the general theory (*ante*, Nos. 3, 4), viz. the curve Θ (which does not pass through a) intersects the given curve in the three classes of points c, D, e, the number of intersections at a point e being $=1$, at a point D being $=2$, and at a point c being $=X-3$. And we have thus the equation

$$e-\varepsilon-\varepsilon'+2(d-\delta-\delta')+(X-3)(c-\gamma-\gamma')=0,$$

where e, d, c are the numbers of united points and $(\varepsilon, \varepsilon')$, (δ, δ'), (γ, γ') the correspondences in the three cases respectively.

27. Observe that we cannot, starting from a, obtain in this manner the equation for the number of the united points $(a$, D); for we introduce per force the points e, and thus obtain the foregoing equation for (a, e). But starting from D, the tangent at this point besides intersects the curve in $(x-2)$ points, each of which is a position of c; and from each of these drawing a tangent cBa to the curve, we have the curve Θ consisting of these $(x-2)(X-3)$ tangents, not passing through D, but intersecting the given curve in the three classes of points c, B, a, viz. the number of intersections at each point c is $=X-3$, at each point B it is $=2$, and each point a it is $=1$; and we have thus the equation

$$(d-\delta-\delta')+2(c_0-\gamma_0-\gamma'_0)+(X-3)(b-\beta-\beta')=0,$$

where the numbers (d, δ, δ'), $(c_0, \gamma_0, \gamma'_0)$, (b, β, β') refer to the correspondences (D, a), (D, B), and (D, c) (or what is the same thing $(a$, B)) respectively.

28. Correspondence $(a$, B).

We have

$$\beta=X-2,\quad \beta'=x-2,$$

and thence

$$\begin{aligned} b&=x+X-4+2\Delta \\ &=-3x-3X+2\xi, \end{aligned}$$

which is the solution: the value obtained above was $b=\iota+\varkappa$, and we in fact have identically

$$\iota+\varkappa=-3x-3X+2\xi.$$

It was in this manner that I originally applied the principle of correspondence to investigating the number of inflections of a curve, regarding, however, the term $\varkappa$ as a special solution; it is better to put the cusp and inflection on the same footing as above.

29. Correspondence (a, c).

Since $b-\beta-\beta'=2\Delta$, we have here

$$c-\gamma-\gamma'=(X-6)\Delta,$$

and

$$\gamma=\gamma'=(X-2)(x-3),$$

whence

$$\begin{aligned} c&=2(X-2)(x-3)+(X-6)(-2x-2X+2+\xi)\\ &=-2X^2+8X+8x+(X-6)\xi; \end{aligned}$$

this is in fact$=2\tau+(X-3)\varkappa$, viz. we have

$$2\tau=X^2-X+8x-3\xi$$

$$(X-3)\varkappa=(X-3)(-3X+\xi)=-3X^2+9X+(X-3)\xi,$$

and therefore

$$2\tau+(X-3)\varkappa=\text{as above},$$

viz. the united points (a, c) are the 2τ points of contact of the double tangents, and the $\varkappa$ cusps each $(X-3)$ times in respect of the $(X-3)$ tangents from it to the curve. This is the way in which I originally applied the principle to finding the number of double tangents of a curve.

30. Correspondence (B, D). By reciprocation.

$$\begin{aligned} &c_0-\gamma_0-\gamma'_0=(x-6)\Delta.\\ &c_0=-2x^2+8x+8X+(x-6)\xi\\ &\quad=\ 2\delta\ +(x-3)\iota. \end{aligned}$$

31. It may be remarked, as regards the cases which follow, that although the result in terms of $(\delta, \varkappa, \iota, \tau)$ when once known can be explained and verified easily enough, there is great risk of oversight if we endeavour to find it in the first instance; while on the other hand the transformation from the form in terms of (x, X, ξ), as given by the principle of correspondence, to the required form in terms of $(\delta, \varkappa, \iota, \tau)$ is by no means easy. I in fact first obtained the expression in (x, X, ξ), and then, knowing in some measure the form of the other expression, was able to find it by the actual transformation of the expression in (x, X, ξ).

32. Correspondence (a, D).

From the values of $c_0-\gamma_0-\gamma'_0$ and $b-\beta-\beta'$ we have

$$d-\delta-\delta'=-(2X+2x-18)\Delta,$$

and then

$$\delta=(X-2)(x-3)(X-3),\ \delta'=(x-2)(X-3)(x-3),$$

whence

$$\begin{aligned}d=&(x-3)(X-3)(X+x-4)\\&+(-2X-2x+18)(-2X-2x+2+\xi),\end{aligned}$$

which is

$$\begin{aligned}=\ &X^2(\quad x+1)\\&+X(x^2-2x-19)\\&+\quad x^2-19x\\&+\xi(-2X-2x+18).\end{aligned}$$

And we then, by means of the equations

$$\begin{aligned}(x-4)2\tau&=(x-4)(X^2-X+8x-3\xi),\\(X-4)2\delta&=(X-4)(x^2-x+8X-3\xi),\\(x-3)\iota&=(x-3)(-3x+\xi),\\(X-3)\kappa&=(X-3)(-3X+\xi),\end{aligned}$$

verify that

$$d=(x-4)2\tau+(X-4)2\delta+(x-3)\iota+(X-3)\kappa.$$

33. Correspondence (a, e).

From the values of $d-\delta-\delta'$, $c-\gamma-\gamma'$ we have

$$e-\varepsilon-\varepsilon'=(-X^2+13X+4x-54)\Delta,$$

and then

$$\varepsilon=\varepsilon'=(X-2)(x-3)(X-3)(x-3);$$

that is

$$\begin{aligned}e=&2(x-3)^2(X-2)(X-3)\\&+(-X^2+13X+4x-54)(-2X-2x+2+\xi),\end{aligned}$$

which is

$$\begin{aligned}=\ &X^3(\qquad 2)\\&+X^2(2x^2-10x-10)\\&+X(-10x^2+26x+44)\\&+\quad 4x^2+44x\\&+\xi(-X^2+13X+4x-54),\end{aligned}$$

and then

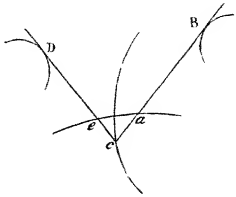
$$\begin{aligned}(x-4)(x-5)2\tau&=(x-4)(x-5)(X^2-X+8x-3\xi),\\\{(X-4)(X-5)+x-3\}2\delta&=\{(X-4)(X-5)+x-3\}(x^2-x+8X-3\xi),\\\{3(x-3)(x-4)+x-3\}\iota&=(x-3)(3x-11)(-3x+\xi)\\2(X-3)(X-4)\kappa&=2(X-3)(X-4)(-3X+\xi);\end{aligned}$$

and summing these values and comparing,

$$\begin{aligned}c=&(x-4)(x-5)2\tau+2(X-3)(X-4)\kappa\\&+[(X-4)(X-5)+x-3]2\delta+[3(x-3)(x-4)+x-3]\iota.\end{aligned}$$

The united points (a, e) are in fact, 1°, each of the $x-4$ intersections of a double tangent with the curve, in respect of the two contacts and of the remaining $x-5$ intersections; 2°, each double point in respect of the two branches and of the pairs of tangents from it to the curve; 3°, each of the $x-3$ intersections of each of the tangents at a double point with the curve; 4°, each of the $x-3$ intersections of a tangent at an inflection (stationary tangent) with the curve, in respect of the $(x-4)$ remaining intersections; 5°, each inflection in respect of the $x-3$ intersections of the tangent with the curve; and 6°, each cusp in respect of the pairs of tangents from it to the curve. Thus (2°), the double point in respect of the branch which contains c, and of the two tangents from it to the curve, is a position of the united point (a, e), as appearing in the figure.

Fig. 9.



34. Correspondence (B, F). By reciprocation of (a, e).

$$e_0-\varepsilon_0-\varepsilon'_0=(-x^2+13x+4X-54)\Delta,$$

$$e_0=(X-4)(X-5)2\delta+2(x-3)(x-4)\iota$$
$$\quad+[(x-4)(x-5)+X-3]2\tau+[3(X-3)(X-4)+(X-3)]\kappa.$$

35. Correspondence (a, F). By means of the values of $e_0-\varepsilon-\varepsilon'$ and $d-\delta-\delta'$, we have

$$f-\phi-\phi'=(2X^2+2Xx+2x^2-32X-32x+162)\Delta,$$

and then

$$\phi=(X-2)(x-3)(X-3)(x-3)(X-3),$$
$$\phi'=(x-2)(X-3)(x-3)(X-3)(x-3),$$

whence

$$f=(X+x-4)(x-3)^2(X-3)^2$$
$$\quad+(2X^2+2Xx+2x^2-32X-32x+162)(-2X-2x+2+\xi),$$

which is

$$\begin{aligned}
= \; & X^3(x^2-6x+5)\\
&+X^2(x^3-16x^2+61x-22)\\
&+X(-6x^3+61x^2-120x-91)\\
&+5x^3-22x^2-91x\\
&+\xi\left\{\begin{aligned}&X^2(2)\\&+X(2x-32)\\&+2x^2-32x+132).\end{aligned}\right\}
\end{aligned}$$

This result includes proper solutions of the problem of finding the number of the triangles $aBcDeF$, which are such that the side ea touches the curve at a; and also heterotypic solutions having reference to the singular points of the curve; but I have not determined the number of solutions of each kind.

36. Correspondence (a, g): from the values of $\mathrm{f}-\varphi-\varphi'$ and $\mathrm{e}-\varepsilon-\varepsilon'$, we have

$$\mathrm{g}-\chi-\chi'=(\mathrm{X}^3-20\mathrm{X}^2-8\mathrm{X}x-4x^2+125\mathrm{X}+44x-486)\Delta,$$

and then

$$\chi=\chi'=(\mathrm{X}-2)(x-3)(\mathrm{X}-3)(x-3)(\mathrm{X}-3)(x-3),$$

wherefore

$$\begin{aligned}\mathrm{g}=&\,2(\mathrm{X}-2)(\mathrm{X}-3)^2(x-3)^3\\ &+(\mathrm{X}^3-20\mathrm{X}^2-8\mathrm{X}x+125\mathrm{X}+44x-486)(-2\mathrm{X}-2x+2+\xi),\end{aligned}$$

viz. this is

$$\begin{aligned}\mathrm{g}=&\quad \mathrm{X}^4(\qquad\qquad\qquad\qquad -\ 2)\\ &+\mathrm{X}^3(\quad 2x^3-\ 18x^2+\ 52x-\ 12)\\ &+\mathrm{X}^2(\ -16x^3+144x^2-376x+142)\\ &+\mathrm{X}\,(\quad 42x^3-362x^2+780x+\ 88)\\ &\qquad\ -36x^3+236x^2+\ 88x\\ &+\xi\left\{\begin{aligned}&\ \mathrm{X}^3(\qquad\qquad 1)\\ &+\mathrm{X}^2(\qquad -\ 20)\\ &+\mathrm{X}\,(\ -\ 8x+125)\\ &\qquad +44x-486.\end{aligned}\right\}\end{aligned}$$

Comparing with the expression of φx, Case 52, we have

$$\begin{aligned}\mathrm{g}-\varphi x=&\quad \mathrm{X}^4(\qquad\qquad\qquad\qquad -\ 3)\\ &+\mathrm{X}^3(\qquad\qquad\qquad\qquad +34)\\ &+\mathrm{X}^2(\qquad 2x^3-18x^2+44x-79)\\ &+\mathrm{X}\,(\ -10x^3+58x^2+76x-84)\\ &\qquad -x^4+10x^3+15x^2-84x\\ &+\xi\left\{\begin{aligned}&\ \mathrm{X}^3(\qquad\qquad 1)\\ &+\mathrm{X}^2(\qquad -\ 11)\\ &+\mathrm{X}\,(\qquad 4x-\ 10)\\ &\qquad +9x^2-91x+114,\end{aligned}\right\}\end{aligned}$$

which difference must be the number of heterotypic solutions having relation to the singularities of the curve; but I have not further considered this.

XVI. *On the Unequal Distribution of Weight and Support in Ships, and its Effects in Still Water, in Waves, and in Exceptional Positions on Shore.* *By* E. J. REED, *C.B., Vice-President of the Institution of Naval Architects. Communicated by Professor* G. G. STOKES, *Sec. R.S.*

Received December 31, 1870,—Read February 9, 1871.

THE object of the present paper is to bring within the grasp of calculation a much neglected division of ship building scienceand art. Many writers of great ability (French, Spanish, Dutch, Swedish, and English) have studied and explained the forces brought into action upon a ship by her own weight and stability, and by the action of the wind upon her sails and of the waves upon her hull; and the result of their investigations has been to encourage the construction of ships of such forms and such dispositions of weights as conduce to moderate and easy motions in the waves of the sea. The relative positions of the centre of gravity and the metacentre, the excursions of the centres of gravity and buoyancy, the inclinations of the axis of rotation, and many other like questions have been very fully and thoroughly discussed, especially by modern English naval architects; in some cases, I venture to say, with even more elaboration and minuteness of inquiry than their intrinsic importance demands. But while the means of securing ease and moderation of movement of the ship at sea have been thus elaborately studied, in order, mainly, as we have been told, to save the fabric of the ship and its fastenings from excessive strains, comparatively few writers upon naval architecture have pursued the subject to its legitimate and necessary development, by seeking to investigate the actual longitudinal bending- and shearing-strains to which the fabric is in fact exposed in ships of various forms under the various circumstances to which every ship is liable.

But more than this: not only has the question of internal strain and strength in the ship been left undeveloped, but a serious fallacy has underlain many of the writings even of men of the greatest eminence upon this subject, viz. the fallacy of considering *ease of motion* identical with *moderation of strain.* No doubt ease of motion is very desirable in all ships, and violence of motion tends to distress any given fabric; but at the same time it is quite practicable, as will clearly appear hereafter, to so design and build two ships, that in a sea-way the easier of the two shall be the more distressed even with precisely the same structural arrangements, and therefore it is obviously very desirable to examine the actual strains, both static and dynamic, with which we have to deal. The weakness exhibited by many ships, notwithstanding the greatest care on the part of the designers, has long pointed to the necessity of further investigation in this direction; but two modern events—the introduction of armoured ships, and the use of iron and

steel in shipbuilding—have added much to the urgency of the inquiry. A long armoured ship, say, like the 'Minotaur' or 'Agincourt' (400 feet in length, and with fine tapering extremities burdened with towering masses of armour), when pitching in Atlantic waves, undergoes a succession of stresses of great magnitude, undoubtedly requiring to be brought as much as possible within the grasp of calculation, the more so as these stresses undergo continual changes, sweeping through the fabric, so to speak, with prodigious velocity. The employment of iron and steel, and the improvements which the manufacture of both is undergoing, fortunately facilitate the concentration of the strength of the ship in those parts which are subject to the greatest stresses; and to further this object, a closer knowledge of these stresses than has hitherto been possessed is much needed.

BOUGUER, the author of the famous 'Traité du Navire' (1746), was one of the earliest authors who gave consideration to the strains brought upon ships by the unequal distribution of weight and buoyancy. The approximate and imperfect character of his theoretical investigations will readily be seen from the fact that he assumes the immersed portion of the ship to be formed of the halves of two equal right cones, set base to base, with their common axis in the water-line. He then constructs a curve the ordinates of which represent the areas of the corresponding sections of the cones, and this he calls the curve of buoyancy for the body. It necessarily comes out a parabola for each cone. He next assumes that the weight of the ship and her lading is uniformly distributed throughout the length, so that it can be represented by a rectangle which stands on the same base as the parabolic curve, and includes an equal area, the latter condition being fixed by the necessity for the displacement equalling the total weight. His investigations have, however, the merit of exhibiting an early example of that graphical or geometrical method of illustrating the distribution of weight and buoyancy which has since been employed by many authors, which Professor RANKINE in particular has recently used with great advantage, and which I have adopted throughout the following investigations.

About ten years after the publication of BOUGUER's work, DANIEL BERNOULLI wrote his celebrated memoir "Principes Hydrostatiques et Méchaniques," which obtained the prize offered by the Academy of Sciences in 1757 for the best essay on the means of preventing pitching and rolling. Being limited to the discussion of the one branch of the subject proposed by the Academy, BERNOULLI did not discuss, as the title of his work might lead us to expect, the strength and strains of ships; but two years afterwards the Academy, in order to supply this want, supplemented their previous question by another having reference to the strains caused by pitching and rolling, and this elicited EULER's well-known memoir*. Although even this treatise says little of statical strains, and contains but a cursory notice of the longitudinal bending-strains of a ship afloat in still water, it is remarkable on the ground that it presents us with the first instance in which we find account taken of the bending effect of the longitudinal pressure of the

* "Examen des efforts qui ont à soutenir toutes les parties d'un vaisseau dans le roulis et dans le tangage; ou recherches sur la diminution de ces mouvements." Par M. L. EULER, Directeur de l'Académie Royale des Sciences et Belles-Lettres de Prusse.

water on the immersed part of a ship. EULER seems to consider that it will tend to reduce hogging; the reverse is really true, as we shall see hereafter. In his later work (the 'Théorie complette') EULER omits all consideration of the effect of this pressure. No quantitative or practical results are arrived at in this division of EULER's work.

In DON JUAN's celebrated 'Examen Maritime' (1771), the graphical method of investigating the forces which result from the varying distribution of weight and buoyancy is again adopted, and the author attempts to make an approximation to the magnitude of the resulting bending-moments. He assumes that the immersed part of a ship may be represented by two surfaces of revolution, the fore body being generated "by the revolution of a semiellipse, and the after body by the revolution of a parabola," about a common axis coinciding with the middle line of the section made by the plane of floatation. No practical importance attaches to his results, since they rest upon so many assumptions; but the work is interesting on account of the grasp of the general conditions of the problem which DON JUAN displays.

ROMME, in 'L'art de la Marine' (1787), gives some attention to this subject, in the main following BOUGUER; but the chief interest of his observation lies in the information which he gives respecting the extent to which the ships of his day yielded to the bending-strains, and the practical methods resorted to for mitigating the evil.

This rapid glance at what the earlier writers had to say upon the subject brings us down to the present century, when our own countrymen took it up. Sir ROBERT SEPPINGS was the first who applied himself to it, but he did little or nothing to advance the science of the question. Dr. YOUNG was, however, fortunately called upon to report on SEPPINGS's practical improvements, and brought great ability to bear upon the whole subject of the strains of ships, giving the valuable results of his investigations, first in a Report to the Admiralty (1811), and afterwards in a paper submitted to the Royal Society, and published in its Transactions (1814). A perusal of Dr. YOUNG's paper clearly shows how very necessary it is, in discussing this subject, to place one's self in possession of such detailed calculations as I have now had carried out for several typical ships, the results of which I shall presently place on record. The "causes of arching" are first considered, the unequal distribution of weight and buoyancy being assigned as the chief cause, and hogging being named as the ordinary effect, although Dr. YOUNG thinks that there are many cases "in which a strain of a very different kind is produced." Alterations in stowage are also mentioned as causes of variation in the strains. As a type of ordinary ships, an English 74-gun ship is taken, for which the details of weight and buoyancy had been calculated for various parts of the length. Dr. YOUNG makes various arbitrary assumptions with respect to the positions of the centres of action of the resultant vertical forces of which he thus knows the amounts, and by this means obtains a balance of the moments of the upward and downward forces about either end of the ship, which is obviously necessary in order to satisfy the hydrostatical conditions of equilibrium for the ship. The result at which he arrives is graphically represented. Dr. YOUNG indicates a correct method of calculating the bending-moments at various parts of the ship, and makes an

approximation to the maximum bending-moment, taking account in each case of the action of the forces on one side only of the station considered. He also attempts, but with less success, to find expressions for the deflections corresponding to the bending-moments, in order to estimate the amount of hogging. Next he shows that the longitudinal fluid pressure has a considerable bending-effect on a ship afloat, following, no doubt unconsciously, in the path of EULER, but not falling into the error which EULER had made in supposing that this pressure tended to *reduce* the hogging caused by the vertical forces. The great merit of Dr. YOUNG's paper consists in the fact that, throughout the investigations of the strains of ships, quantitative results based upon an actual ship are given. No preceding writer, to whom I have referred, followed this course, although BOUGUER and DON JUAN took hypothetical cases to illustrate their methods. Dr. YOUNG also takes a much more complete review of the principal causes of straining than any of his predecessors. In dealing with the question of the strength of ships, however, he is not so successful.

About two years after the publication of Dr. YOUNG's paper in the Philosophical Transactions, appeared another written by the eminent French geometrician DUPIN. It was professedly "A Theoretical Examination of the Structure of English Vessels," meaning thereby vessels built on the diagonal system. Although DUPIN followed Dr. YOUNG and SEPPINGS in the discussion of this subject, he contributed several additional features possessing both interest and importance.

Without further preface I shall pass, first, to the consideration of the actual distribution of weight and buoyancy in various classes of ships when floating in still water. The testimony of BOUGUER, EULER, and DON JUAN puts it almost beyond doubt that in the older types of wood sailing-ships there was generally a great excess of buoyancy in the middle, and deficiencies of buoyancy at the ends only. In later sailing-ships, such as that referred to by Dr. YOUNG, there were portions of the amidship length (in wake of water, ballast, and other concentrated weights) of which the weight exceeded the buoyancy; and this excess, as well as that due to the heavy extremities, was counterbalanced by the surplus buoyancy of the portions of the ship intermediate between the middle and the extremities. With the introduction of steam as a propelling agent, and of very largely increased lengths and proportions for ships, a vastly different state of things has been brought about in the distribution of weight and buoyancy. At the ends of ships there still remains an excess of weight, exaggerated in many cases by the adoption of very fine under-water lines in combination with heavy bows and sterns above water; but the distribution of weights in the fuller parts of the ship becomes much changed. How great the change has been we may infer from the fact that at present merchant steam-ships are in actual employment of which the length is 400 feet, and the proportion of length to breadth exceeds 10 to 1, both length and proportion having been more than doubled since the introduction of iron into ship-construction and steam into ship-propulsion. Ships of even greater length, both actually and proportionately, to breadth are being constructed for trading between Europe and the East through the

FIG. I.

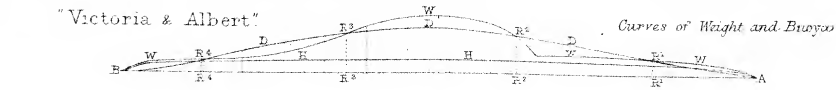


FIG. II.

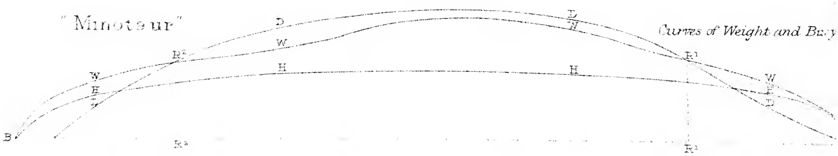


FIG. III.

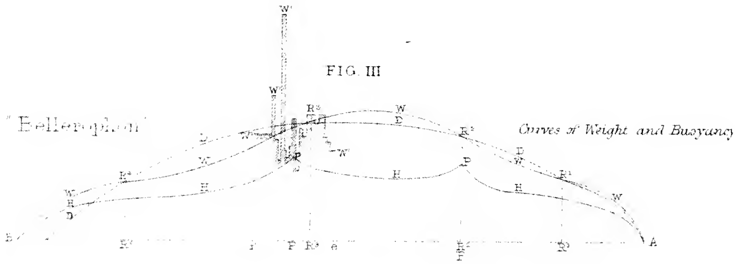


FIG. IV.

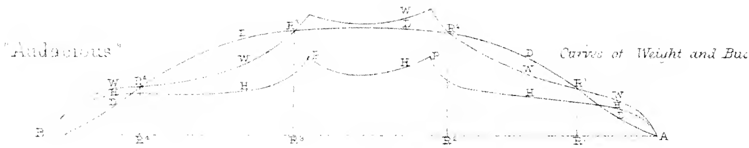


FIG. V.

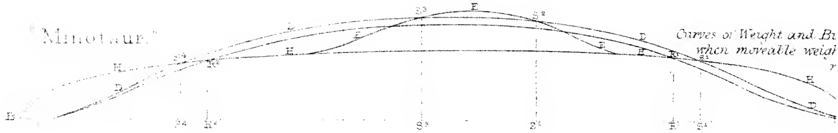


FIG. VI.

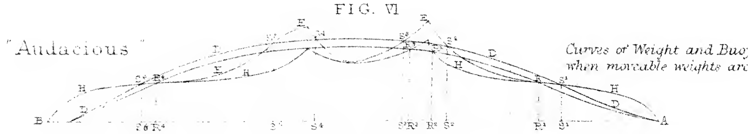
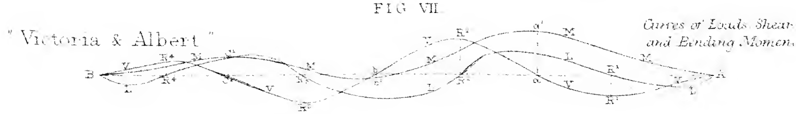


FIG. VII.



Scale used in diag^{m} VII.

For areas of curves L.L.L. 3 square inches = 4000 tons.

For ordinates of curves V.V.V. 1 inch = 400 tons.

For ordinates of curves M.M.M. 1 inch = 16000 foot tons.

Suez Canal. We usually find the weights of engines, boilers, and coals concentrated at some part of a ship. In a paddle-steamer they are found near the middle of the length, in full-powered screw-steamers rather abaft the middle, and in auxiliary screw-steamers very far aft. Wherever they come their weight obviously increases the downward pressure at that part very considerably; in some cases they cause, while in others they exaggerate, an excess of weight over buoyancy, and in others they bring up the weight very nearly to an equality with the buoyancy. No general law can now be laid down for the strains of all ships, and no general statement can be made to include all the conditions in which any particular ship may be placed by means of variations in her stowage or in the weights she has on board. Having given the details of the weights and buoyancies of various parts, however, the calculation of the resulting still-water strains is practicable, but involves considerable labour. I have taken the cases of three or four typical ships, and have had the DISTRIBUTION OF THE WEIGHT AND BUOYANCY very carefully calculated and graphically recorded. Each example is a ship of modern type, and the results are wholly unlike any which have before been published. In fact, owing to the great labour involved, or to some other cause, only the most meagre and unsatisfactory attempts to discover and exhibit the actual strains of ships have previously been made and recorded.

The first case represents the conditions of long fine paddle-steamers, of high speed, employed as yachts, or blockade-runners, or on other services where great cargo-carrying power is of comparatively minor moment. The case I have selected is that of the Royal Yacht 'Victoria and Albert,' and the diagram in Plate XVI. fig. 1 has been prepared in order to indicate the distribution of weight and buoyancy. In making the calculations required for this purpose, the total length (300 feet) has been divided into 20-feet spaces, and transverse planes of division have been supposed to be drawn in order to form the foremost and aftermost boundaries of the spaces. For each division of the ship the buoyancy, the weight of the hull, and the weight of the equipment have been determined; and the sum of the two latter qualities of course gives the total weight of ship and lading for any particular 20-feet space. A base-line A B (fig. 1) has been taken to represent the ship's length, and a series of equidistant ordinates has been erected, each ordinate representing, in position, the centre plane of a 20-feet space. The positions of the imaginary planes of division in the ship are indicated in the figure at the middle points of the parts of A B lying between the feet of the ordinates; and the distance between consecutive ordinates is, I need hardly say, 20 feet on the scale by which A B is set off. Upon these ordinates there have been set off on a certain scale of tons per inch*,—(1) a length representing the buoyancy of the division of the ship with which the ordinate corresponds, divided by the length of the division; the ordinate will therefore represent the average buoyancy of the division per unit of length: (2) a length representing in a similar way, and on a similar scale, the average weight of hull per unit of length for that division: (3) a length similarly representing the weight of hull and equipment for that division. Through the

* The various scales employed in constructing the diagrams are specified on the respective Plates.

three sets of points thus obtained three curves have been drawn. The curve D D represents the displacement or buoyancy, the curve H H represents the weight of hull, and the curve W W represents the total weight of hull and equipment. From this explanation it will be obvious that, by choosing a proper scale, the areas lying above the line A B, and enclosed by the various curves as well as by any two ordinates, may be taken as representatives of the buoyancy, total weight, and weight of hull, respectively, for the corresponding part of the ship. Hereafter it will appear preferable to adopt the latter mode of representation, and in the various diagrams of a character similar to fig. 1 this plan is followed.

These curves are not minutely accurate representations of the distribution of weight and buoyancy; but for our present purpose they are sufficiently close approximations to such representations. Our chief interest centres in the comparison of the curve of buoyancy with the curve of total weight of hull and equipment; but the curve H H of weight of hull has an interest attaching to it also, as it enables us to determine the straining-effect of the equipment, and to illustrate the importance of careful stowage of the weights carried. For the present I shall only make an examination of the distribution of the weight and buoyancy, and for this purpose shall compare the curves W W and D D. These curves, it will be noticed, cross each other at four points marked R^1, R^2, R^3, R^4 in fig. 1; at these stations the weight equals the buoyancy, and the ship is there "water-borne." Before the foremost water-borne section $R^1 R^1$, which is 50 feet from the bow, the weight exceeds the buoyancy by 85 tons; between this section and the water-borne section $R^2 R^2$ next abaft it, a length of about 68 feet, the buoyancy exceeds the weight by 225 tons; between the two water-borne sections $R^2 R^2$ and $R^3 R^3$, a length of 82 feet of the midship length (in which come the engines, boilers, and coals), the weight exceeds the buoyancy by 210 tons; and from $R^3 R^3$ to $R^4 R^4$, a length of 70 feet, the buoyancy exceeds the weight by 130 tons; while abaft $R^4 R^4$, which is 30 feet from the stern, the weight exceeds the buoyancy by 60 tons. These excesses and defects of buoyancy are graphically represented by the areas of the spaces enclosed by the two curves D D D and W W W between their various points of intersection. The hydrostatical conditions of equilibrium are, of course, satisfied by the distribution of the weight and buoyancy.

These figures will show the vastly different condition of many modern steam-ships as compared with the older types of sailing-ships, which had an excess of weight only at the extremities.

Some modern ships, however, have a distribution of weight and buoyancy similar in kind, although extremely different in degree, to that of their predecessors; and as an example of these I have taken the iron-clad frigate 'Minotaur.' This ship is armoured throughout the length, or, to use a more common phrase, is "completely protected," and may be considered a fair representative of extremely long fine ships so protected, with V-shaped vertical transverse sections at the bow. Her length is 400 feet; the heavy weights of engines, boilers, water, powder, and provisions are distributed over a consi-

derable portion of the length; the guns are also distributed along the broadside; and the weight of hull is nearly uniform, except at the extremities. We should naturally expect, therefore, that the weight would considerably exceed the buoyancy at the bow and stern, and that the buoyancy would exceed the weight throughout the amidship section. The curves in Plate XVI. fig. 2 show that this is actually the case. They are constructed and marked similarly to those of the 'Victoria and Albert.' In this case there are only two water-borne sections $R^1 R^1$, $R^2 R^2$. The first is about 80 feet from the stem, and before it the weight exceeds the buoyancy by about 420 tons; the second is 70 feet from the stern, and on this length there is an excess of weight of about 450 tons; between $R^1 R^1$ and $R^2 R^2$, a length of 250 feet, the buoyancy exceeds the weight by the sum of these excesses—870 tons. It will be observed that at the stern the curve of buoyancy D D in fig. 2 is ended at some distance before the curve of total weight W W; the same thing, although in a less prominent degree, is observable in fig. 1. The overhang of the stern above water is the cause of this method of ending the curves; and in the 'Minotaur' the distance between the points where they terminate is greater than in the 'Victoria and Albert,' because she is a larger ship and has a screw-propeller.

My third illustration is taken from the 'Bellerophon,' for which ship the curves of total weight, weight of hull, and buoyancy shown in fig. 3 have been constructed in the manner previously described for the 'Victoria and Albert.' This case may be taken as a representative of the distribution of weight and buoyancy in iron-clads of moderate length and proportions, with central batteries and armour-belts, and with a fall-back stem. The advantages of this bow in giving increased buoyancy are well illustrated by the fact that the foremost water-borne section $R^1 R^1$ in Plate XVI. fig. 3 is only 40 feet from the bow, and that the excess of weight over buoyancy on this length is only 45 tons. In the 'Minotaur,' as we have seen, the foremost water-borne section is 80 feet from the bow, and the excess of weight on this length is 420 tons. Part of this difference is undoubtedly due to the different arrangement of the armour of the two ships; but as the 'Bellerophon' has an armoured bow-battery about 20 feet long, in wake of which the armour-plating reaches up to the height of the upper deck, and as the 'Bellerophon's' draught of water is much less, forward, than that of the 'Minotaur,' it will be obvious that the very different form of the bows greatly influences the case. The aftermost water-borne section $R^4 R^4$ in the 'Bellerophon' is about 50 feet before the stern, the excess of weight on this length being about 220 tons; it will be remembered that for 70 feet of the after part of the 'Minotaur' the weight exceeds the buoyancy by 450 tons, this large excess being due, in great part, to the armour-plating on the stern*. In the 'Bellerophon' (as in the 'Victoria and Albert') there are four water-borne sections, the excess of weight over buoyancy on 70 feet of the length amidships between $R^2 R^2$ and $R^3 R^3$ amounting to 250 tons. This excess is caused by the concentration of the weights of the armour and armament of the central battery in a comparatively short length amidships;

* The 'Minotaur' is a much larger ship than the 'Bellerophon;' but the difference of size is almost entirely a difference of length, the breadths and heights of the two ships differing comparatively little.

while the heavy weights of machinery, coals, powder, water, &c. are also more concentrated than they would be in a long ship like the 'Minotaur.' On the length of 50 feet between the water-borne sections $R^1 R^1$ and $R^2 R^2$ the buoyancy exceeds the weight by about 105 tons; and between $R^3 R^3$ and $R^4 R^4$ there is an excess of buoyancy of 410 tons on a length of 90 feet. There is, it will be observed, a certain amount of resemblance between the distribution of the weight and buoyancy in the 'Victoria and Albert' and the 'Bellerophon,' and in both these vessels the conditions are very different from those of the 'Minotaur.'

A few words are needed to explain the peculiar form of the curve H H in fig. 3 of the 'Bellerophon's' weight of hull. In both the preceding cases this curve is very flat, and throughout the length is concave to the base-line A B; while in the 'Bellerophon' it is comparatively irregular, having cusps at the ordinates P P, P P. This difference is caused by the disposition of the armour in the 'Bellerophon.' Her central battery is shut in by transverse armoured bulk-heads, the weights of which are included in those of the divisions of the ship represensed by the ordinates P P, P P; hence it follows that there should be cusps in the curve H H at those ordinates. Between those ordinates the curve is nearly straight, as the armour on the ship's sides is of about uniform weight for the length of the battery, and the weight of hull per foot of length is also nearly uniform. Before and abaft the central battery the armour-belt only reaches to the main deck (except in wake of the armoured bow-battery), and its weight per foot of length of the ship is considerably less than it is in wake of the central battery; this reduction, in combination with the gradual *fineing* of the vessel, leads to the form of curve shown in the diagram.

It will be remembered that I have already intimated that the graphical method of representation adopted in this paper is not minutely accurate; and the case just considered being an extreme one, affords an excellent opportunity for illustrating this fact, and showing the extent of the error thus introduced. From the description of the method given above, it will be obvious that we have assumed spaces bounded by curves, which are practically continuous, to be graphic representations of the distribution of the weight and buoyancy. The assumption is fairly accurate for most of the divisions, especially in the case of the curve of buoyancy; but it is not nearly exact for the curve of weight in some parts of a ship, particularly where there are any great weights concentrated in a very small length. The transverse armoured bulk-heads enclosing the central battery of the 'Bellerophon' are each really concentrated in a space, measured fore and aft, of less than 2 feet: and it is obvious that, by spreading the weight over the 20-feet division in which such a bulk-head comes, we materially modify the distribution, and obtain an inaccurate curve of weight, even when we make some allowance for the bulk-heads, as is done in the cusps P, P in fig. 3. If we place the planes of division 18 inches apart instead of 20 feet, and calculate carefully the weights of the 18-inch lengths, we shall, of course, obtain a curve of weight much more accurately representing the actual distribution in the ship, and shall then be able to judge of the amount of error

in the curve W W obtained previously. This I have done in the present case for a length of about 40 feet in the neighbourhood of one of the armoured bulk-heads, and it has been found that the curve of weights between the stations *e* and *f* in fig. 3 (Plate XVI.) assumes the form W′ W′ W′. The concentrated weights (such as the armoured bulk-head, the armoured pilot-tower, the engines, and the mainmast) are in the calculations made for W′ W′ W′ put very nearly into the spaces they really occupy, instead of being spread over 20-feet spaces as in the curve W W; and a further approximation to accuracy might be made by taking the planes of division still closer together. In practice, however, this is quite unnecessary; and it will be seen from the diagram in fig. 3 that even in this exceptional case no very considerable error results from taking the curve W W as a fair representation of the distribution of the weight. This follows from the fact that the curve W W averages, so to speak, the inequalities of W′ W′ W′, and, on the length of 40 feet considered, includes a nearly equal area; that is, represents as nearly as possible the same weight. On this account, when considering the strength of a ship as a whole, we may avail ourselves safely of the method previously described, taking the planes of division, say, 20 feet apart in large ships; but excessively concentrated weights necessarily cause severe *local* bending- and shearing-strains, which must be specially provided against. It will suffice here to say that shipbuilders depend chiefly upon longitudinal bearers or keelsons to distribute such concentrated loads over a considerable length, and prevent their exercising a prejudicial effect on the structure.

My fourth example of the distribution of weight and buoyancy in a fully laden ship floating in still water is taken from the class of broadside ironclads of which the 'Audacious' and 'Invincible' were the first examples. The calculations have been conducted in the same manner as those for the 'Victoria and Albert,' and the results are represented by the curves shown on fig. 4. It is necessary to remark that in these vessels there is an armoured upper-deck battery amidships above the central main-deck battery, and that there is neither a bow nor a stern battery protected by armour, as the fore-and-aft fire, so essential in ironclads, is obtained with the central battery guns on the upper deck. By this means nearly all the principal weights are brought towards the middle of the length, and the extremities are much less heavily burdened than in most other ironclads, not excluding even ships like the 'Bellerophon.' In short, the present case is one of a ship with weights very unusually concentrated at the centre. The curve H H of weight of hull in this case bears a certain resemblance to that of the 'Bellerophon' in fig. 3, only the cusps P, P are more pronounced on account of the armoured bulk-heads of the upper-deck battery. This fact also accounts in part for the difference between the curves of total weight W W for the two ships. It is only necessary to add that the length of the 'Audacious' is 280 feet, and that the same scales have been used for lengths and weights in fig. 4 as in the preceding diagrams. The distribution of weight and buoyancy for this ship may be summarized as follows:—On the 35 feet of length between the bow and the foremost balanced section (R^1 in fig. 4)

the weight exceeds the buoyancy by 115 tons*; between R^1 and R^2, a length of 65 feet, the excess of buoyancy amounts to 220 tons; from R^2 to R^3, 80 feet, the excess of weight is 275 tons; and between R^3 and R^4, on a length of 30 feet, the weight exceeds the buoyancy by 210 tons. These figures furnish another illustration of the advantages resulting from the adoption of moderate length and proportions in iron-clad ships.

In all the preceding examples the ships have been supposed to be fully laden; but there must, of course, occur alterations in the amount and stowage of the weights on board, and consequently in the relative distribution of the weight and buoyancy. Earlier writers on the subject have fully recognized this fact, and nearly all of them concur in stating that the greatest strains experienced by a ship floating in still water are those incidental to the state when all the weights of equipment are removed, but that the character of the strains remained the same as when the ships were laden. In this opinion they were probably correct in so far as the ships of their period were concerned. as the older classes of ships always hogged considerably; but in modern ships when light, the characters as well as the intensities of the strains are often very different from those when they are fully laden. For example, if a paddle-wheel steamer, like the 'Victoria and Albert,' had her heavy weights of engines, boilers, coals, &c. removed, the buoyancy amidships would become greater than the weight, and the ship would be brought into a condition similar to that of the 'Minotaur,' when fully laden, illustrated by fig. 2 (Plate XVI.). A few illustrations will show more clearly that the conclusions drawn by the earlier writers by no means hold for all modern ships, some of which are strained in a very different manner when light and when laden. As extreme cases I will take the 'Minotaur' and the 'Audacious.'

The 'Minotaur's' distribution of weight and buoyancy, when a mere shell, with engines, boilers, &c. all out, as they would be if the ship were undergoing a thorough repair, is shown by fig. 5 (Plate XVI.). The curve H H in this diagram agrees with the curve similarly lettered in fig. 2, and the inner of the two curves marked D D represents the displacement of the various divisions, or, in other words, is the curve of buoyancy for the ship when light. At the extremities the curve H H falls considerably outside the curve D D, the excess of weight before the water-borne section $R^1 R^1$, 90 feet from the bow, being 560 tons, and that abaft the water-borne section $R^2 R^2$ being 500 tons on a length of 80 feet of the stern. When the ship is fully laden, we have seen that the excess of weight forward on a length of 80 feet is 420 tons, and that at the stern there is an excess of 450 tons on a length of 70 feet; so that when this ship is quite light the difference between the distribution of the weight and buoyancy is even greater than it is when the ship is fully laden, and the resulting strains will obviously be more severe.

There is, however, another aspect of this question to which I must briefly refer. It is only when the ship is undergoing a thorough repair that she would be left a mere shell; and usually the engines and boilers are left in place when the other weights are

* The stem of the 'Audacious' is more nearly upright, and the lower water-lines are finer than the 'Bellerophon's.'

removed for the purpose of refit. Supposing this to be so, the engines and boilers must, of course, add greatly to the weight of the middle part of the ship, making the curve of weight in fig. 5 assume the form H E E H. Their effect on the ship's immersion is but slight, and the curve of buoyancy consequently is but little altered, it being represented by the outer one of the two curves marked D D in fig. 5*. In this case it will be observed we have an excess of weight amidships as well as excesses at the extremities. The latter remain nearly the same as in the preceding case; but the change of condition possessing most interest for us is amidships, where we have passed from an excess of buoyancy to an excess of weight amounting to about 80 tons on a length of 55 feet, or thereabouts. In consequence of this there are four water-borne sections (marked $S^1 S^1$, $S^2 S^2$, $S^3 S^3$, $S^4 S^4$) in fig. 5 instead of two only, as there are when the ship is either fully laden or quite light. It would be difficult to give a better illustration of the changes that may take place in the distribution of weight and buoyancy in consequence of changes in the weights on board.

Very few words will suffice respecting the condition of the 'Audacious' under circumstances similar to those of the 'Minotaur.' When quite light, with engines and boilers out, the distribution of weight and buoyancy are shown respectively by the curve H H and the inner of the two curves D D in Plate XVI. fig. 6. There are still four water-borne sections ($R^1 R^1$, $R^2 R^2$, $R^3 R^3$, $R^4 R^4$) in this case; but it will be observed that the excess of weight between $R^2 R^2$ and $R^3 R^3$ is very small, amounting to 65 or 70 tons on a length of 12 feet, so that practically the conditions of strain are almost identical with those of a ship having excesses of weight at the ends only. This is therefore a very interesting change from the condition of the ship when fully laden (as in fig. 4), especially as the excess of weight at the bow is thus increased to about 160 from 115 tons, while that at the stern is about the same as that previously given, 210 tons. A still more interesting change is, however, produced by supposing the engines and boilers to remain in the ship when all the other weights are removed. The curve of weight then assumes the form H E E H in fig. 6, and the outer of the two curves D D represents the buoyancy. By this means it will be seen that six water-borne sections are produced, marked $S^1 S^1$, $S^2 S^2$, &c. in the diagram. At the extremities the excess of weight remains about the same as before, but now between $S^2 S^2$ and $S^3 S^3$ the excess of weight amounts to 155 tons on a length of about 20 feet, and between $S^4 S^4$ and $S^5 S^5$ to 80 tons on a little shorter space; while between $S^1 S^1$ and $S^2 S^2$, a length of about 60 feet, there is an excess of buoyancy of 180 tons, and between $S^5 S^5$ and $S^6 S^6$ there is an excess of buoyancy of 225 tons on a length of about 65 feet. This is by far the most complex case we have yet considered, and I shall revert to it hereafter.

The preceding cases show clearly the effect produced by varying the amount of weight on board a ship; it is also obvious that changes in the stowage of the weights must

* I have preferred marking the two curves of buoyancy in this diagram with one set of letters, as they lie so close together, the weight of engines &c. only giving the ship a little deeper draught in the second case.

produce similar changes in the distribution of the weight and buoyancy, and consequently in the strains brought upon the ship.

Thus far I have dealt only with the actual distribution of weight and buoyancy of ships floating in still water; I have next to investigate the amount of the VERTICAL SHEARING (or racking) and BENDING strains that result from the unequal character of the distribution. By "shearing"-strains are here meant those vertical forces which tend to shear off the part of a ship or girder on one side; and by "bending"-strains are meant those moments resulting from the unequal distribution of weight and buoyancy that tend to alter a ship's longitudinal curvature. In order to give exactitude to my remarks, I will take the case of the 'Victoria and Albert.' The buoyancy and weight of this ship have already been calculated for every 20 feet of the length, and the results have been represented by the curves D D and W W in Plate XVI. fig. 1. It will be obvious that at any station in fig. 1 the length of the ordinate intercepted between those two curves represents the resultant vertical force (that is, the excess of weight over buoyancy, or *vice versâ*) acting on the 20-feet division to which the station corresponds. If the weight be in excess, the resultant force of course acts downward; and if the buoyancy be in excess, the resultant force acts upward. Knowing therefore the amounts, directions, and points of application of all such vertical forces, it is possible to calculate their shearing and bending effect at any transverse section by the simplest mechanical methods.

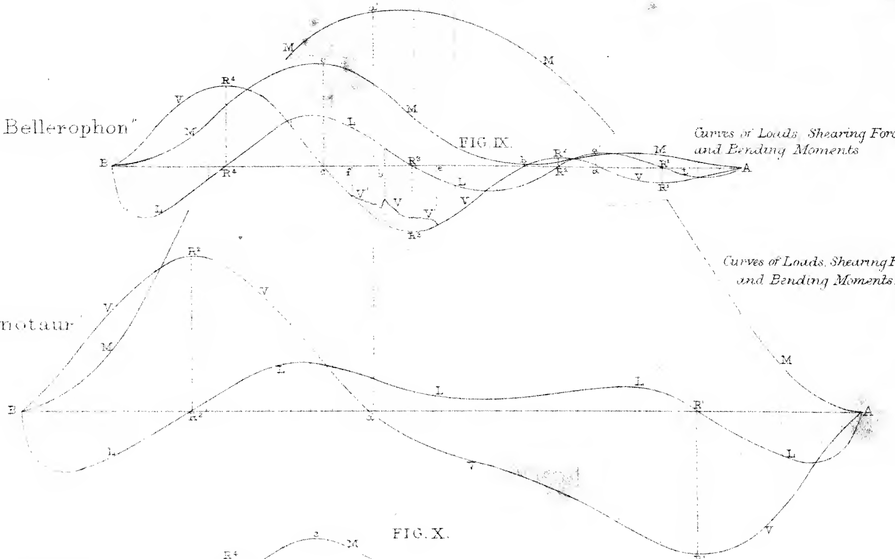
A graphical method may again with advantage be applied to represent these operations; and in Plate XVI. fig. 7 I have given an example of its application to the case of the 'Victoria and Albert.' The base-line A B represents, as in fig. 1, the length of the ship, and the ordinates dotted in fig. 7 correspond to those drawn in fig. 1, while those drawn in fig. 1 correspond to the imaginary transverse planes of division of which the positions are shown midway between the ordinates in fig. 7. On the dotted ordinates lengths are set off representing on a certain scale the distances between the curves D D and W W on the corresponding ordinates in fig. 1, and through the points thus obtained the curve L L in fig. 7 is drawn. This is known as the curve of "loads," or resultant vertical forces, its ordinates representing in direction and position the excesses of weight over buoyancy, and of buoyancy over weight. Where there is an excess of weight, the ordinate representing it is measured downwards below A B, and where there is an excess of buoyancy the ordinate is measured upwards. In this case, as in the curves of weight and buoyancy previously constructed, we may pass from ordinates to areas, and regard the latter as representing the excesses or defects of buoyancy on a certain scale. We must here, however, add the convention that defects of buoyancy shall be represented by areas lying below the line A B, and excesses of buoyancy by areas lying above A B; and in estimating the excess or defect of buoyancy on any part of the ship we must take the *algebraical* sum of (*i. e.* must integrate) the areas of the loops of the curve L L L corresponding to that part. The scale of areas for this curve is marked on the diagram. It will be obvious, therefore, that the curve of loads L L represents, in a manner more

FIG. VIII.

"Bellerophon"

FIG. IX.

"Minotaur"



Curves of Loads, Shearing Forces, and Bending Moments.

Curves of Loads, Shearing Forces, and Bending Moments.

For ordinates of curves VVV. 1 inch = 100 tons

For ordinates of curves MMM. 1 inch = 10,000 foot tons

For lengths along line AB, 5 inches = 200 feet.

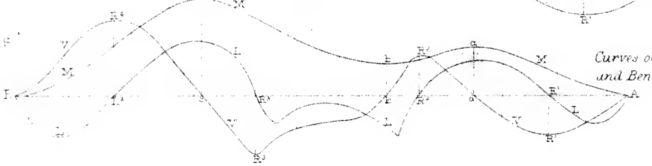
Scales in diag.m XVI.

For areas of curves, 3 square inches = 3000 tons

For lengths along line AB, 3 inches = 200 feet.

FIG. X.

"Audacious"

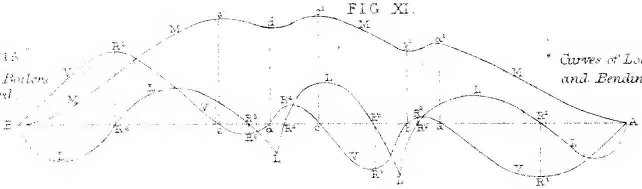


Curves of Loads, Shearing Forces, and Bending Moments.

FIG. XI.

"Audacious"

with Engines & Boilers only on board.

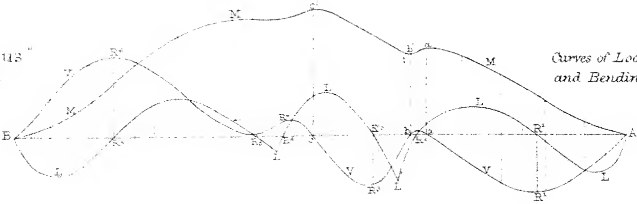


Curves of Loads, Shearing Forces, and Bending Moments.

FIG. XII.

"Audacious"

modified.

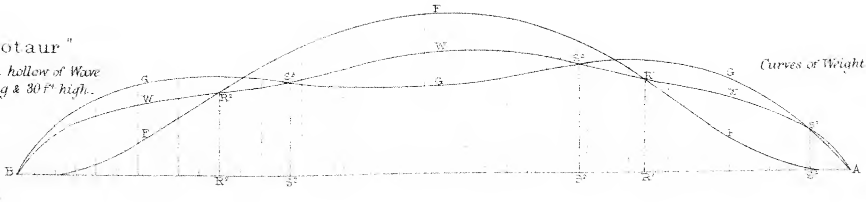


Curves of Loads, Shearing Forces, and Bending Moments.

FIG. XVI.

"Minotaur"

on Crest & in hollow of Wave 600 ft. long & 30 ft. high.



Curves of Weight and Buoyancy.

W. H. Wesley sc.

W. West & Co. imp.

readily understood, the results that would be arrived at by comparing the curves of weight and buoyancy previously constructed for this ship; and I shall show immediately how to pass from it to similar representations of shearing-strains. In Plate XVII. figs. 8, 9, 10, and 11 the same method has been applied to the cases of the 'Minotaur,' 'Bellerophon,' and 'Audacious' (to the last ship both laden and light), the curves of loads being marked L L in each diagram. In order that the hydrostatical conditions of equilibrium may be fulfilled, the joint area of the loops of the curve L L lying above the line A B must, of course, equal the joint area of the loops lying below A B; and the moments of the areas above and of those below A B, about any line perpendicular to it, must be equal. The points R^1, R^2, &c., where the curves of loads cross the axis A B, obviously correspond to what have been previously termed water-borne or balanced sections, where the weight equals the buoyancy.

Next, as to the construction of the curves of shearing-forces. From what has been previously said, it will appear that the shearing-force at any transverse section equals the resultant upward or downward force, measuring the excess or defect of buoyancy on either of the two parts into which the ship is divided by the transverse section. Hence it follows that, to construct a curve of shearing-forces, we have only to integrate the curve of loads (or to obtain the algebraical sums of the areas of the loops of that curve) up to certain stations, and to use the results of these integrations as measures, on a certain scale, of the lengths of ordinates to be set upwards or downwards at the stations according as the areas above or those below the axis are in excess. In performing the integrations we may start from either end. As an example I will take the case of the 'Bellerophon' in fig. 9 for that purpose, remarking the fact that in Plate XVI. fig. 7 and Plate XVII. figs. 8 and 9 the curves V V represent the result of these operations for the 'Victoria and Albert,' 'Minotaur,' and 'Bellerophon' respectively, and that the similarly marked curves in figs. 10 and 11 represent the shearing-forces experienced by the 'Audacious' when fully laden and when she has only her engines and boilers on board respectively.

Turning to fig. 9, it is necessary to state that A has been chosen as the starting-point for the integration of the curve of loads L L, and that the stations up to which the integrations have been carried, in order to determine the ordinates of the curve of shearing-forces V V, are those (drawn midway between the dotted ordinates of the curve of loads) corresponding to the imaginary planes of division, 20 feet apart, with which we started. The ordinate at any section, say, R^3, R^3, is determined as follows. The area of the loops of L L lying below A B and between A and R^3 R^3 is found, as is also the area of the loop lying above A B between the same limits; and the difference between the areas, in this case in favour of the downward forces, is set off on a certain scale of tons per inch (marked on the diagram) on the ordinate R^3 R^3. Through points thus determined the curve V V is then drawn. Had the point B been taken as the starting place, or origin, of the integrations, we should obviously have obtained an equal value for the ordinate R^3 R^3, only its direction would have been opposite to that we have found; and generally we may

that, by starting from B instead of A, the same curves would be obtained, only the ps which now lie above the axis A B would lie below it, and *vice versâ*. This would be of the least consequence, however, since any ordinate of the curve V V simply ws the *amount* of the shearing-force at that station; and the question of *direction* nmaterial, as the same effect is produced whether the part before the station moves ⁄ard or downward relatively to that abaft it.

'here are one or two features of interest in the curve V V, Plate XVII. fig. 9, to which f reference may be made before passing on. Its ordinates have continually increasing ative values between A and the foremost water-borne section R^1, while beyond that ion they gradually decrease (on account of the fact that the curve of loads has sed the axis A B, and has part of its area above as well as part below that line) l at the station $a\,a'$ they pass through a zero value, the area of the part of the curve ›ads below the axis being there equal to the area of the part above the axis. The .on $a\,a'$ therefore divides the ship in such a manner as to render each of the parts re and abaft it separately water-borne, and I shall in my future remarks term such ons "sections of water-borne division"*. At the sections b and $c\,c'$ there are two other values of the shearing-force; each of these is also a section of water-borne division, in passing through them the shearing-force changes sign. At the extremities there f course, no shearing-force. The water-borne section R^1 has been shown to be a ɔn of maximum shearing, and all the other points where the curve of loads crosses ιxis (R^2, R^3, R^4) are also positions where the curve V V has maximum ordinates.

the curves of weight and buoyancy previously laid down in Plate XVI. fig. 3 were .tely accurate, the curves of loads and of shearing-forces in Plate XVII. fig. 9 would ɔe accurate; but as this is not the case, it becomes necessary to examine how great ror is introduced into the curve V V by the errors existing in the curves of weight ›uoyancy, and consequently in the curve of loads. We should naturally look for the mum error in that part of a ship where weights which are really concentrated have spread out over a considerable space in the construction of the curve of weights; .he 'Bellerophon,' in wake of the armoured bulk-heads, affords, as we have seen, a exceptional and extreme illustration of the kind. This has been made use of in to determine what may be fairly assumed to represent the limiting value of the introduced into the curves of shearing-force. For this purpose the corrected curve ·ights W′ W′ in fig. 3 has been employed instead of W W in estimating the shearing-s between the stations e and f, and the result has been graphically represented by between the corresponding stations e and f in fig. 9. It has been assumed here the curve V V gives the correct shearing-force at e; but this is not strictly true, and, the difference between the moments about the station e of the areas included by ʳ and W′ W′ between the ordinates e and f in fig. 3, it appears that the curve V V ıtes too small a shearing-force at e. The connective curve V′ V′ ought therefore to ›mewhat further out from the axis A B, and should cross the curve V V at some

* "Sections of water-borne division" must not be confused with "water-borne sections."

points and lie partly outside it. The maximum correction at the station g would therefore be less than that indicated by the space between V V and V′ V′. At the station g the ordinate of V V represents a shearing-force of 165 tons, that of V′ V′ represents a force of 100 tons, and the difference between the ordinates represents 65 tons. This last amount undoubtedly exceeds the real error in the curve V V for the reasons set forth; and although it would be necessary to construct the curve V V from A up to e in order to determine what its actual ordinate at e would be, we shall be within the mark in assuming that in this very exceptional case the error involved by using the fair curve V V falls below one third of the actual amount of the shearing-force thus obtained. Proportionately this correction is very considerable, but in no ordinary ship would such an extreme example of concentration be met with, and the error would in nearly all cases be much less. It may be added that in practice the exact valuation of the shearing-force is not required, and a sufficiently good approximation can be obtained by the graphical method. Where weights are very concentrated that method will be most in error, and the preceding illustration exemplifies the necessity for great care in its application in such cases in order to ensure thoroughly trustworthy results. In all, or nearly all, ships there is an ample reserve of strength to resist vertical shearing-forces of the most severe character, and on this account comparatively little interest attaches to the determination of the maximum values of the shearing-forces experienced by ships afloat in still water. Even in the case of the 'Minotaur,' which is certainly one that may be expected to give a limiting value, the shearing-force for still water nowhere exceeds one twenty-second part of the total weight; and in the 'Bellerophon' the maximum value does not exceed one thirty-third part, having been reduced so much by means of the better distribution of the weight and buoyancy at the extremities. Without going in detail through the investigation of the shearing-forces experienced by the typical ships previously chosen, it will suffice therefore to indicate a few general properties of these curves of shearing, and to illustrate them by reference to the different ships.

First, as to the determination of the sections of maximum shearing-strain in a ship. The reasoning on which the method of constructing the curves is based obviously leads to the conclusion that these sections must coincide with the balanced or water-borne sections at which the weight exactly equals the buoyancy. It may be summed up in the statements that in proceeding from one end of a ship towards the other the resultant vertical forces, or loads, between the end and the water-borne section all act in the same direction, and that their sum represents the shearing-force at that section; while between any two water-borne sections, such as R^1 and R^2 in fig. 9, the same law holds respecting the resultant vertical forces; and since their sum reaches a maximum at the bounding water-borne sections, the shearing-force, which equals their sum minus a constant quantity, is also, in most cases, a maximum at those sections*. The number of water-borne sections varies, of course, with the distribution of the weight and buoyancy. In the

* Minimum values of the shearing-force sometimes occur at balanced sections, as I shall show almost immediately.

notaur' (as shown in Plate XVI. fig. 2 and Plate XVII. fig. 8) there are only two sec-s, and this is the simplest case, besides being that which is most frequently met with. he 'Victoria and Albert' (as shown in figs. 1 & 7) and in the 'Bellerophon' (as shown ?late XVI. fig. 3 and Plate XVII. fig. 9) there are four water-borne sections, and sequently four sections of maximum shearing. In the 'Audacious,' when fully laden, re are also four such sections (see Plate XVI. figs. 4 and Plate XVII. 10); and this case it is evident that, by transposing 40 or 50 tons weight from amidships ards the extremities, keeping the ship's trim unaltered, the curve of weights might made to cross the curve of buoyancy in two other points, thus bringing up the number vater-borne sections to six. When the 'Audacious' has only her engines and boilers board, she is, in fact, in the conditions here referred to, the balanced or water-borne tions being marked $S^1 S^1$, $S^2 S^2$, &c. in Plate XVI. fig. 6, and $R^1 R^1$. $R^2 R^2$, &c. in Plate 'II. fig. 11. We can also imagine a ship to be so loaded that there would be even a ater number of water-borne sections than this; and it appears, at first sight, difficult determine what law regulates the number of these sections. On an inspection of all cases we have given, however, it will be seen that the number of water-borne sections ılways *even*; and it appears probable that, at least in the greater number of ships ring an excess of weight at the extremities, this law will be conformed to. The ıson is that the excesses of weight and buoyancy for all parts of the ship must balance, d that at the extremities of such ships there are always two loops of the curve of loads ng below the axis.

It will be remarked that we have not fixed the position of the absolute or true ıximum shearing-force as yet, and it will be necessary to determine it, either by cal-lation or by means of the curve of shearing-forces, the latter being the simpler plan. different ships the section of absolute maximum shearing-force is found in various ›sitions. For example, in the 'Minotaur' it is at the aftermost water-borne section R^2 (Plate XVII. fig. 8), while in the 'Victoria and Albert' it is at the water-borne ction $R^2 R^2$ (Plate XVI. fig. 7). In screw steam-ships, as a rule, the greatest shearing-rce is experienced at the aftermost water-borne section, on account of the, to some ktent, unavoidable excess of weight at the stern. The 'Bellerophon' and 'Audacious' e cases in point (see Plate XVII. figs. 9 & 10). The case of the 'Minotaur' is excep-onal in one respect, viz. that there is only a small difference between the shearing-forces t the two water-borne sections; in most cases the absolute maximum would probably e greater relatively to the other maximum shearing-forces.

Next, as to the sections where the shearing-forces have zero values. The extremities f a ship obviously have no shearing-force to resist, and we have seen, in the case of the Bellerophon' (fig. 9), that at the sections of water-borne division the shearing-force is ılso zero. The condition established in that ship holds for all the other ships I have aken, and in fact for all ships when floating in still water. In the 'Minotaur,' which s the simplest possible case, there is (neglecting the extremities) only one such section ($a\,a'$ in fig. 8); in the other three ships, when fully laden, there are three such sections,

where the curves of shearing-forces (V V, Plate XVI. fig. 7, Plate XVII. figs. 9 & 10) cross the axis A B; and in the 'Audacious,' when she has only her engines and boilers on board, there are no less than five such sections (see Plate XVII. fig. 11). All these cases, therefore, give an *odd* number of sections of water-borne division and of zero shearing-forces; and the number of these sections is one less than the number of water borne sections where the curves of loads cross the axis and the shearing-forces are maxima. It may be assumed that in most ships having an excess of weight at the extremities, when floating in still water, the number of sections of no shearing-forces will be odd.

The rules here laid down for the number of sections of maximum and zero shearing-force must not be regarded, however, as universally binding, nor necessarily true; they are rether indications of what may be expected to be conformed to in very many ships. To render this clear we will again take the case of the 'Audacious' when she has only her engines and boilers on board, her curve of loads corresponding to that condition being L L in Plate XVII. fig. 11. Between the water-borne sections R^4 and R^5 there is an excess of weight of 80 tons only, but within those limits there falls a section of water-borne division (*d d'*), and the ordinary rule is conformed to as regards the number of sections of water-borne division being one less than that of the balanced sections. Now suppose 40 tons out of the 80 tons were removed from this space, 25 tons being placed 90 feet abaft its present position, between the stern and the water-borne section R^6, and 15 tons being placed 150 feet before its present position on the fore side of the water-borne section R^1. The ship's trim would obviously remain unaltered, since the moments produced by the alteration of the stowage balance each other; and the only alteration in the curve of loads would be that the areas of the foremost and aftermost loops would be increased, while the area of the loop between R^4 and R^5 would be diminished. These changes are indicated in the curve L L, Plate XVII. fig. 12. Although the number of the water-borne sections and their positions remain unchanged, this transposition of weight has a remarkable effect upon the form and character of the curve of shearing-forces, as will be seen by comparing the curves marked V V in figs. 11 & 12. Starting from the point A, we find the shearing-force at $R^1 R^1$ (fig. 11) to be 175 tons, while that at the corresponding station in fig. 12 is 160 tons. On account of this increase in the shearing-force, the foremost section of water-borne division (*a a'*) falls somewhat nearer to the water-borne or balanced section $R^2 R^2$ in fig. 12 than it does in fig. 11, and the shearing-force at R^2 is only 5 tons instead of 20 tons. The next section of water-borne division (*b b'* in fig. 12) also falls closer to R^2 than it did before; in fact the small loop of the curve V V between *a a'* and *b b'* in fig. 11 almost disappears in fig. 12. After crossing the axis at *b* the curve V V has continually increasing negative values up to the balanced section $R^3 R^3$, where the shearing-force is 150 tons instead of 135 tons as in fig. 11; and this causes the next section of water-borne division (*c c'*) to lie nearer the balanced section $R^4 R^4$ than it did before. Up to this point, therefore, the curves V V for the two cases have very similar characteristics, although they differ in some respects;

but between $R^4 R^4$ and the stern they are altogether different. At $R^4 R^4$ in fig. 12 the upward or positive shearing-force has a maximum value of 45 tons; and as the excess of weight between $R^4 R^4$ and $R^5 R^5$ has been reduced to 40 tons, the shearing-force at $R^5 R^5$ will still act upward, and have there a maximum value of 5 tons. After passing through this minimum value, the shearing-force gradually increases again until it passes through a maximum value at $R^6 R^6$ of 230 tons, and then decreases to its zero value at B. Hence it will be seen that the chief differences between the curves of shearing in Plate XVII. figs. 11 & 12 consist in the facts that the latter has only *three* instead of *five* sections of water-borne division, and that at $R^5 R^5$ a minimum positive shearing-force takes the place of a small maximum negative shearing-force.

It is interesting to remark, further, that by another slight alteration in the distribution of the weight we can convert the small positive maximum shearing-force at R^2 (fig. 12) into a small negative minimum shearing-force, doing away at the same time with the two sections of water-borne division $a\,a'$ and $b\,b'$. For example, if the excess of weight (40 tons) still remaining between $R^4 R^4$ and $R^5 R^5$ be diminished by one half, and 8 tons be placed before $R^1 R^1$ while 12 tons are placed abaft $R_6 R_6$ in such a manner as to produce equal moments and keep the trim unaltered, we shall have the following values for the shearing-forces at the balanced sections:—at $R^1 R^1$ a force of 183 tons acting downwards, being a maximum value, at R_2 a minimum shearing-force of 3 tons, also acting downwards, at $R^3 R^3$ another maximum value (158 tons acting downwards), at $R^4 R^4$ a maximum force acting upwards of 37 tons, at $R^5 R^5$ a minimum upward force of 17 tons, and at $R^6 R^6$ a maximum upward force of 242 tons. The only section of water-borne division, where the shearing-force is zero and the curve crosses the axis, will obviously be a little nearer to $R^4 R^4$ than $c\,c'$ is in fig. 12.

From these examples, then, it will be seen that by slight changes in the distribution of the weights in a ship we may, while keeping the same total weight and the same number of balanced sections, have different numbers of sections of water-borne division and of zero shearing-force; and at the same time it is evident that these changes may turn a maximum value of the shearing-force in one direction into a minimum value in the opposite direction. It becomes necessary, therefore, to add to our previous rules for the sections of maximum and zero shearing-forces. Before doing this it will be convenient to repeat what was said respecting the number of balanced sections in a ship floating in still water. The general laws deduced from various cases are as follows:—if there be, as there generally is, an excess of weight at both the extremities, the curve of loads will cross the axis in an *even* number of points, and consequently the number of balanced sections is *even*. Hence it follows that the number of sections of maximum or minimum shearing must also be *even*; and it is obvious that between two maximum ordinates of the curve of shearing, both of which lie upon the same side of the axis, there must occur either a minimum ordinate also lying upon the same side, or a maximum ordinate lying upon the other side of the axis. Since these conditions hold, and we know in addition that the curve of shearing must cross the axis in at least one point, on account of the fact

that there is an excess of weight at both ends of the ship, it follows that if it crosses the axis in any other points, these additional sections of zero shearing must be either two in number or some multiple of two. In other words, the whole number of sections of zero shearing, which we have shown to be also sections of water-borne division, must be *odd* in ships having excesses of weight at the bow and stern.

Before concluding these remarks on shearing-forces, it may be well to compare the condition of the 'Minotaur,' 'Bellerophon,' and 'Audacious' when fully laden and floating in still water, as they represent different classes of ironclads. At the foremost balanced section (R^1 in Plate XVII. fig. 8) of the 'Minotaur' the shearing-force is no less than 420 tons; at the corresponding station (R^1 in Plate XVII. fig. 9) in the 'Bellerophon' it is only 45 tons, and at that in the 'Audacious' (R^1, in Plate XVII. fig. 10) it is 120 tons. The great difference between the 'Minotaur' and the other two ships is partly due to the fact that she is completely protected with armour, while they have armoured central batteries and water-line belts; and partly to the very fine entrance and V-formed sections which her designers considered desirable. It has already been explained that the pronounced U-form of transverse sections, and the fall-back contour of stem in the 'Bellerophon' have much to do with her small excess of weight at the bow; and it may here be added that in the 'Audacious' the excess of weight over buoyancy, though somewhat greater than in the 'Bellerophon,' is still very moderate. At the aftermost balanced sections the shearing-forces are for the 'Minotaur' about 450 tons, for the 'Bellerophon' 210 tons, and for the 'Audacious' 202 tons. When speaking of the weight and buoyancy we gave the reasons for the difference existing between the 'Minotaur' and the other ships; they are, principally, the armour plating of the stern and the extreme length and fineness of the run. The 'Bellerophon' and the 'Audacious' each have, as we have seen, two other sections of maximum shearing (R^2, R^3 in figs. 9 & 10). At the foremost of these the shearing-force is, for the 'Bellerophon' 50 tons, and for the 'Audacious' 96 tons; while at the aftermost the shearing-force is 200 and 168 tons respectively. These figures show that the concentration of weights amidships, in ships with central batteries and armour belts, while it renders the conditions of strain more complicated, reduces the actual shearing-strains far below those experienced by completely protected ships. There is doubtless a great reserve of strength in all ships against shearing-strains, so that this fact has not much practical weight; but, as I shall show hereafter, the severer *bending*-strains are similarly reduced when the weights are concentrated, and this is a much more important feature.

The Bending moments resulting from the action of the vertical forces on a ship floating in still water next claim attention. I have already indicated the method by which these moments may be estimated when the distribution of the weight and buoyancy are known, and will therefore proceed at once to the graphical method of recording them.

As before stated, it is necessary, in calculating the bending-moment at any transverse section of a ship, to consider the part of the ship on one side of that section as fixed,

and to take account of the moments of the resultant vertical forces representing the excess and defect of buoyancy of the part on the other side of that section. The algebraical sum of these moments represents the bending-moment required. We have in the curves of loads, previously constructed, graphical representations of these resultant vertical forces, both in magnitude and direction, as well as in position, and consequently can find with ease the bending-moments at various stations. The operation simply consists in finding the moments about the various stations of those parts of the curve of loads which lie on one side. I will again take the case of the 'Victoria and Albert,' for which the curve of loads is shown by L L in Plate XVI. fig. 7. It will appear on consideration that the most convenient stations at which to calculate the bending-moments are those (midway between the dotted ordinates of the curve of loads) which have already been used as ordinates of the curve of shearing. Let a (fig. 7) be the station at which the bending-moment is to be determined; then starting from A (which is nearer a than B is, and which is therefore more convenient) the vertical pressure represented by each loop of the curve of loads between A and a must be multiplied by the distance of the centre of gravity of that loop from a; and the difference between the sums of the moments of the upward forces and the sums of the moments of the downward forces will equal the bending-moment at a. At this station the moments of the downward forces are greater than those of the upward forces, and the bending-moment consequently tends to produce hogging, to represent which a length $a\,a'$ is set off *above* A B, showing, on a certain scale of foot-tons per inch, the hogging-moment at a. A similar method is followed at all the other stations, and where (as at b, fig. 7) the resultant moment tends to produce sagging, the ordinate representing its amount is set off below A B. As the result of this process, a series of ordinates is determined for the curve of bending-moments M M, the approximation to accuracy being sufficient for all practical purposes. We have previously seen that for curves of weight, buoyancy, loads, and shearing-forces the graphical method does not cause any important error even in extreme cases, such as the 'Bellerophon's;' and I may add here that for bending-moments the errors resulting from distributing weights that are really concentrated are very much less in proportion than they are in the cases previously considered.

There are one or two matters of practical interest connected with the construction of the curves of moments to which I may briefly refer. The first has already been mentioned, viz. that in calculating the moments it is always better to start from the end of the ship nearer to the section, although the same value would obviously be obtained by starting from the other end, this being one of the hydrostatical conditions of the ship's equilibrium. Another point is, that in working from one end of the ship to the other, the moment at the end towards which we are working should always be zero, since there can be no bending-moment at either end; this constitutes a check on the accuracy of the calculation.

This graphical method of representing bending-moments has also been applied to the 'Minotaur,' 'Bellerophon,' and 'Audacious' (the latter when laden and when light), the

curves for which ships are marked M M in Plate XVII. figs. 8, 9, 10, & 11 respectively. In all cases the same scales for lengths and for moments have been used, so that a fair idea of the relative conditions of strain can be obtained from a comparison of the diagrams, and the scales employed are marked on the various plates. Hereafter I shall refer at some length to the comparative strains of these ships, but will first refer to a few general considerations respecting vertical bending-strains, using the preceding cases as illustrations.

The simplest case is obviously that of a ship which, like the 'Minotaur,' is subject to hogging-moments at every transverse section, these moments gradually increasing in amount from the extremities towards the middle of the length, and attaining their maximum value near the midship section, as shown by the curve M M in Plate XVII. fig. 8. Earlier writers, as I have shown, regarded this as the only case which deserved attention; and recent works on the strains of ships devote greater attention to it than to any other case of strain, doubtless with good reason, since it is that which is most commonly met with. Modern writers, however, have also clearly pointed out the possibility of sagging- as well as hogging-strains being experienced by ships when floating in still water, and have laid down the conditions which must be fulfilled in such cases. But while this is true, it is no less true that a very general belief exists among shipbuilders and others that where an excess of weight over buoyancy exists at the middle of a ship's length there must necessarily be sagging-moments; and in some published works on the subject this is laid down as a general rule. No better illustration of the error of this belief could, I think, be given than that afforded by the 'Bellerophon,' which has an excess of weight over buoyancy amounting to 250 tons at the middle, and yet no sagging-strains at any portion of her length. The curve M M in Plate XVII. fig. 9 shows that this is so; and at the section of minimum hogging-moment, *b*, there is a strain of about 100 foot-tons. The case of the 'Audacious' in Plate XVII. fig. 10 furnishes another illustration of the same kind. In her the excess of weight amidships. when fully laden, amounts to 265 tons; but the bending-moment, so far from becoming a sagging-strain, never falls below a hogging-strain of 3400 foot-tons.

Cases do undoubtedly exist, however, in which great excesses of weight amidships produce sagging-moments in still water, and of these we have an example in the 'Victoria and Albert.' On reference to Plate XVI. fig. 7 it will be seen that in this ship the central part of the curve M M falls below the axis, thus indicating the fact that about 30 feet of the midships length of the ship is subjected to a very small sagging-moment, of which the maximum value does not exceed 170 foot-tons, although the excess of weight over buoyancy amounts to 210 tons. A very small deduction from this excess of weight would suffice also to convert this sagging-moment into a small hogging-moment. For example, suppose 10 tons only to be taken from the excess, and a weight of 4 tons to be placed 150 feet before the position from which the 10 tons were taken, while the remaining 6 tons are placed 100 feet abaft it. The trim of the ship would remain unaltered, and the distribution of buoyancy would therefore be unchanged; but the removal

of the weights would cause a hogging-moment of 600 foot-tons, which, combined with the sagging-moment previously existing of 170 foot-tons, would leave as a final result a hogging-moment of 430 foot-tons at the station where sagging-strains previously existed.

These examples will serve to show the error of supposing that an excess of weight amidships, or at any other part of a ship, necessarily causes sagging, as well as the necessity for taking into account the effect of all the forces on one side of the section for which the bending-moment is being calculated. Unless this is done it is almost impossible to determine whether sagging will take place or not; for in some cases the hogging-moment due to unsupported weights at the extremities will more than counterbalance the sagging-moment due to the excess of weight amidships, as is the case in the 'Bellerophon' and the 'Audacious;' or the reverse may be true, as in the 'Victoria and Albert.'

The effect which alterations in the weights carried by a ship have upon the bending-strains is well illustrated by the comparison of the curves M M in Plate XVII. figs. 10 & 11. By removing all the weights except the engines and boilers, and altering the curve of loads in the manner previously described, the curve of moments in fig. 11 is made to assume a very different form from that in fig. 10. Hereafter I shall have occasion to refer to the principal points of difference; for the present it will suffice to say that in the latter the greatest bending-moment falls nearly amidships, instead of in the after body as before, notwithstanding the fact that at some parts of the middle body there is an excess of weight. The amount of the greatest bending-moment when the ship is light is also somewhat greater than that when she is fully laden; and the curve M M in fig. 11 approximates more nearly in form to that of the similarly marked curve for the 'Minotaur' (fig. 8) than it does in fig. 10, although striking points of difference still exist. Another illustration of the variation in bending-moments which variation in the amount of the weights on board a ship may produce is afforded by comparing figs. 10 & 12 (Plate XVII.); and here we see also how, by reducing the excess of weight between the stations R^4 R^4 and R^5 R^5, we have rendered the curve M M in fig. 12 more nearly continuous in its concavity towards the axis than is the similar curve in fig. 11. The differences between these last two curves have a special interest, on account of the fact that they are entirely due to alterations in the *stowage* of the weights on board, the total amount of weight carried being the same in both cases. The greatest bending-moment in fig. 12 is found at the station $c\,c'$, and amounts to 13,800 foot-tons, or about one-eighth more than it is at the corresponding station in fig. 11, this increase being due to the transposition of weights to the extremities.

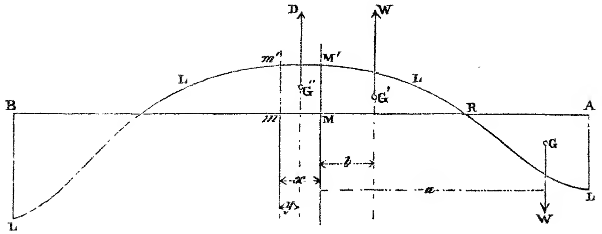
It may be well here to revert briefly to the fact that no ship having an excess of weight over buoyancy at the extremities (as all, or nearly all, ships have) can sag throughout her length when afloat in still water, although she may hog and not sag. The cases of the 'Minotaur,' 'Bellerophon,' and 'Audacious' prove the possibility of the latter condition, and no more need be said respecting it. That sagging alone cannot take place

follows from the fact, that in a ship which sags there must be an excess of weight amidships as well as at the extremities, and there must consequently be some intermediate portions at which there is an excess of buoyancy, and at which hogging-strains will result from the moment of the unsupported weights at the bow and stern. The curve of moments (M M in fig. 7) for the 'Victoria and Albert' illustrates these remarks; in the fore-and-after bodies we find considerable hogging-moments, while amidships, as I have shown, there are sagging-strains. While sagging alone cannot take place in still water, it may, however, occur at sea, or in exceptional positions ashore.

The various conditions of strain of ships floating in still water may, I think, be grouped under the following types:—First, the 'Minotaur' type, including the greater number of vessels, in which the weights are pretty evenly distributed, and the buoyancy is in defect at the extremities only. In some vessels which might be included in this class the weight and buoyancy are equal for a considerable length of middle body; but it will be obvious that in such cases the bending-moment is of uniform amount throughout the middle body, and that the length of middle body might be increased or diminished without affecting the vertical bending-strains. Second, the 'Bellerophon' or 'Audacious' type, in which there is a defect of buoyancy amidships as well as at the extremities, but the bending-moments throughout the length produce hogging-strains, having a minimum value amidships. Third, the 'Victoria and Albert' type, which has a greater proportionate defect of buoyancy amidships, and is there subject to sagging-strains, while in the fore-and-after bodies hogging-strains are experienced. Besides these there may be, and doubtless are, many special cases, wherein, to revert to the graphical method, the curve of loads would have a greater number of loops than in any of the ships we have considered; but the preceding classifications will, as I have said, probably include by far the greater number of ships.

The determination of the positions of those sections of a ship at which the maximum and minimum bending-strains are experienced has been satisfactorily performed by DUPIN and later writers. Among these later writers I may particularly refer to Professor RANKINE, who has done much to advance the application of scientific principles to the determination of the strength and strains of ships. At page 136 of his 'Shipbuilding, Theoretical and Practical,' are given a mathematical demonstration and a graphical representation of the theorem which DUPIN first established, viz. that maximum and minimum bending-moments are experienced by, what I have termed previously, sections of water-borne division. Those who are desirous of following out these investigations will be repaid by a study of Professor RANKINE's method; but that sections of water-borne division do possess this property may be shown by the following simple method.

Fig. 13.



Let L L in diagram fig. 13 represent the curve of loads of a ship that tends to hog simply, and let M M′ be the section of water-borne division. Then the area A L R must equal the area M′ M R, since the weight and buoyancy of each of the parts into which M M′ divides the ship must be balanced. Suppose W = weight in tons represented by each of the areas A L R, M′ M R. We may, as far as bending is concerned, consider the weight W to be acting through G, the centre of gravity of the area A L R, at a known distance a from the section M M′, and the buoyancy W to be acting through G′, the centre of gravity of M′ M R, at a known distance b from that section. Then we shall have, obviously,

$$\text{Hogging-moment at M M}' = \text{W}(a-b).$$

Next, take any section $m\,m'$ on the side of M M′ further from A, and let D tons be the buoyancy represented by the area M′ m' m M. Then if G″ be the centre of gravity of the area M′ m' m M, the force D may be considered to act through G″ at a known distance y from $m\,m'$. Hence if x = distance M m, we shall have

$$\begin{aligned}\text{Hogging-moment at } m\,m' &= \text{W}(a+x) - \text{W}(b+x) - \text{D}y \\ &= \text{W}(a-b) - \text{D}y \\ &= \text{Hogging-moment at M M}' - \text{D}y.\end{aligned}$$

That is to say, the bending-moment at $m\,m'$ is less than that at M M′. We shall obtain similar results if we suppose the section $m\,m'$ to be situated on the side of M M′ nearer to A; for in that case, to obtain the moment of the buoyancy of the area R $m\,m'$ about $m\,m'$, we must add the moment of M M′ $m\,m'$ about that station to the moment of the whole area R M M′ about it. Hence, as $m\,m'$ is now $-x$ distant from M M′, we shall have

$$\begin{aligned}\text{Hogging-moment} &= \text{W}(a-x) - \text{W}(b-x) - \text{D}y \\ &= \text{W}(a-b) - \text{D}y,\end{aligned}$$

as before. Hence it follows that for this distribution of weight and buoyancy the section of water-borne division is also the section of maximum bending-moment. For the more complex case, where there is an excess of weight amidships as well as at the extremities, it can be shown, by a method similar to the foregoing, that the central section of water-borne division is a section of minimum bending-moment, and that in the fore and after bodies there will be two sections of maximum bending-moment if there are two sections

of water-borne division. By a simple extension of the method, the cases similar to those illustrated by Plate XVII. figs. 10 & 11 can also be shown to come under this rule, and the theorem stated above may thus be established generally*.

It has already been shown that sections of water-borne division coincide with sections of zero shearing, and that the number of such sections must be *odd*; hence we have this rule, that the number of sections of maximum and minimum bending-moment must be *odd*. This rule will be seen to hold in all the preceding cases. For example, in the 'Minotaur' there is one section of maximum hogging-moment ($a\,a'$ in Plate XVII. fig. 8); in the 'Bellerophon,' and in the 'Audacious' when fully laden, there are three sections of maximum and minimum hogging-moment, the section of minimum moment being nearly amidships ($a\,a'$, $b\,b'$, $c\,c'$ in Plate XVII. figs. 9 & 10); in the 'Audacious,' when light, with her engines and boilers only on board, there are five sections of maximum and minimum hogging-moment ($a\,a'$, $b\,b'$, &c. in Plate XVII. fig. 11); and in the 'Victoria and Albert' there are two sections of maximum hogging-moment and one of maximum sagging-moment ($a\,a'$, $c\,c'$, and $b\,b'$ in Plate XVI. fig. 7). In the hypothetical case of Plate XVII. fig. 12, based upon the 'Audacious,' there are also three sections of maximum and minimum hogging-moment. If sagging be regarded as a negative phase of hogging, it appears from all these cases that sections of maximum and minimum bending-moment occur alternately, although it must not be forgotten that in its *absolute amount* the sagging-moment, which is termed a minimum, may exceed some of the so-called maximum hogging-moments. What is really meant may perhaps be better expressed as follows:—Between two sections of maximum hogging-moment there must fall either a section of minimum hogging-moment or a section of maximum sagging-moment. In the ships with which DUPIN was acquainted no sagging-moments were experienced, and hence we find him laying down the law, which I believe he was the first to observe, that sections of maximum and minimum bending-moment occur alternately. My statement is simply an extension of the same principle to more complex cases, such as are met with in modern ships. DUPIN also indicates a simple method of determining whether a section of water-borne division is one of maximum or minimum bending-moment. It is as follows:—If the resultant vertical force immediately adjacent to the section of water-borne division acts upward, that section is one of maximum hogging-moment; if downward, it is one of minimum hogging-moment. In order to extend this so as to embrace all the ships we have considered I must add, if the resultant vertical force acts downward, the adjacent section of water-borne division is either one of minimum hogging-moment or of maximum sagging-moment, and *vice versâ*.

All that has been said respecting the variations in the number of sections of zero shearing, or water-borne division, which may be produced by alterations in the stowage

* In the case previously considered, of ships with a long middle body of which the weight balances the buoyancy, this general law also holds; for any transverse section within the limits of the middle body will obviously fulfil the conditions of a section of water-borne division, and of one where the bending-moment is a maximum or a minimum.

of the weights such as keep the number of balanced sections constant, of course applies with equal force to the sections of maximum and minimum bending-moment. In both fig. 11 and fig. 12 (Plate XVII.), for example, we have six balanced sections ($R^1 R^1$, $R^2 R^2$, &c.), but only three sections of maximum and minimum bending-moment in the latter instead of five as before; while we have seen that another slight transposition of weight would leave only one section of water-borne division and maximum bending-moment. A comparison of the curves of bending and shearing in these two cases possesses further interest, on account of the fact that when the curve V V in fig. 12 has a minimum positive ordinate at $R^5 R^5$ instead of a maximum negative ordinate as in fig. 11, the maximum and minimum ordinates of the curve M M at $e\,e'$ and $d\,d'$ respectively in fig. 11 disappear, and we have instead of them a point of contrary flexure in the curve M M of fig. 12. Similarly, if the other transposition were made which does away with the two sections of water-borne division $a\,a'$ and $b\,b'$ in fig. 12, and turns the small positive maximum shearing-force at $R^2 R^2$ into a small negative minimum shearing-force, we should have a point of contrary flexure at $R^2 R^2$ instead of the maximum and minimum ordinates at $a\,a'$ and $b\,b'$ of the curve M M. In fact, from the relations which exist between the curves V V and M M, it is obvious that at the balanced sections, where the shearing-force has maximum or minimum values, the curve of moments has either points of contrary flexure (as in all the illustrations we have given) or singular points where there is a change of curvature. It may be added that we have by our construction supposed the points of contrary flexure in the curve of loads to lie at the balanced sections where the curve crosses the axis; and hence we may say that at the stations where the curves L L have maximum ordinates the curves V V have points of contrary flexure; in other words, the curve V V bears a relation to the curve L L similar to that which the curve M M bears to it*. The broad practical deduction to be drawn from the cases represented by figs. 11 & 12 is, however, simply this, that by transposing weights from the centre to the extremities we render the curve of moments M M less tortuous, which is a matter of no consequence; but we at the same time increase the maximum bending-moment which the ship has to resist, and which may thus become raised to a very undesirable amount. We are thus again reminded of the reduction of the strains of ironclads produced by adopting the belt-and-battery system instead of that of complete protection.

In all ships there must obviously be a section or sections of absolute maximum bending-

* Expressed in mathematical symbols, this relation stands as follows:—Suppose B to be the origin of coordinates in fig. 12, and let the distance of any station from B be called x. Then if y=resultant vertical force, or ordinate of the curve of loads at that station, S=shearing-force, or ordinate of the curve of shearing, and M=bending-moment, or ordinate of the curve of moments, we have by our method of construction, as previously explained,

$$S=\int_0^x y\,.\,dx,$$

$$M=\int_0^x y\,.\,xdx=\int_0^x S\,.\,dx.$$

That is to say, we obtain the ordinates of the curve of shearing by integrating the areas of the curve of loads up to various stations; and obtain the ordinates of the curve of moments, either by integrating for the moments of the curve of loads or for the areas of the curve of shearing.

strain, no matter what number of sections there may be at which that strain has a maximum value. For example, in the 'Victoria and Albert' there are two sections of maximum hogging-moment ($c\,c'$ and $a\,a'$ in Plate XVI. fig. 7), but the absolute maximum is found in the fore body; while in the 'Bellerophon' and 'Audacious' (see Plate XVII. figs. 9 & 10) the absolute maximum is found in the after body. In cases such as the 'Minotaur's,' where there is only one section of water-borne division, that is, of course the section of maximum bending. Attention has previously been drawn to the fact that in such cases the sections of maximum bending (such as $a\,a'$ in Plate XVII. fig. 8) usually lie very near the middle of the length; and many writers on the subject, considering the distribution of weight and buoyancy too exclusively, refer to the midship section as being most severely strained. For still water, however, I have shown above that this is by no means true in many ships; and it is only proper to add that some authors of standing have recently put forward views on this point which are not borne out by my calculations. For example, even Professor RANKINE, speaking of a ship that tends to hog in still water, says*:—"Let M be a transverse section which divides the ship into two parts, each separately water-borne (and which is seldom far from the midship section); then at the section M ..... the bending-moment is a maximum;" and further on he adds, "in the case of a ship that tends to sag amidships the greatest bending-moment is still at M ....., being a sagging-moment instead of a hogging-moment." Having proved above that the "greatest bending-moment" in ships which "tend to sag amidships" falls not at M but in the fore or after bodies, I need not dwell at length on the too hasty inference here made. It will suffice to say that the section of greatest sagging does in such cases sometimes come near the middle of the ship; but in Plate XVI. fig. 7 we have an illustration of the fact that it may come considerably abaft the middle; and there are doubtless cases in which it comes before the middle, its actual position being determined, as Professor RANKINE himself states, by the position of the central section of water-borne division. This sagging-moment, however, is by no means "the greatest bending-moment" experienced by the ship, the distribution of weight and buoyancy determining whether the true maximum bending-moment is experienced by some section in the fore or after body. I may add, that without accurate calculation it is generally possible to tell which body has to bear the greatest strain, by simply observing at which end the excess of weight is greater. For example, in the 'Victoria and Albert' the excess of weight over buoyancy is greater at the bow than at the stern, and the section of absolute maximum bending-strain falls in the fore body; had the reverse been true, the section would have fallen in the after body, as it does in the cases of the 'Bellerophon' and 'Audacious.' Professor RANKINE, while recognizing the existence of these sections of maximum hogging-moment, does not seem to have noticed the fact that the strains at them would generally be more severe than those at the section of maximum sagging-moment.

With respect to the actual amounts of the bending-moments experienced by the typical

* 'Shipbuilding, Theoretical and Practical,' p. 151.

ships I have chosen, when floating in still water, but few remarks are necessary, since these moments are much less severe than those experienced by the same ships when at sea, or when placed in exceptional positions ashore. The 'Minotaur,' on account of her great length, very fine form, and heavily burdened extremities, may be regarded as a limiting case in the amount of her hogging-moment, the maximum value of which (at $a\,a'$ in Plate XVII. fig. 8) is about 45,000 foot-tons, equalling the product of the displacement in tons by about one eighty-eighth of the ship's length. The 'Bellerophon' may be taken as an opposite limiting case, in so far as strain on the midship part is concerned, for the hogging-moment there does not exceed 100 foot-tons. Even at the station of absolute maximum hogging-moment ($c\,c'$ in Plate XVII. fig. 9) for this ship the strain only reaches 12,000 foot-tons, equalling the product of the displacement by about the hundred-and-seventy-sixth ($\frac{1}{176}$) part of the length. It will thus be seen that the changes made in the 'Bellerophon' from the 'Minotaur' have had the effect of rendering the maximum bending-moment about one fourth what it would have been if the long fine type had been conformed to. I may add that the concentration of weights amidships, due to the adoption of the central battery-and-belt system, has had much to do with this; while it has been shown that sagging-moments do not result from the excess of weights amidships. This fact adds one more to the numerous advantages previously shown to be possessed by this system, as compared with the system of complete protection exemplified in the 'Minotaur.' The case of the 'Audacious' (illustrated by Plate XVII. fig. 10) gives further support to this view. The hogging-moment amidships does not exceed 3400 foot-tons, and that at the section of absolute maximum strain in the after body is only 11,000 foot-tons,—that is, equals the product of the displacement by about the hundred-and-fiftieth ($\frac{1}{150}$) part of the length. With respect to the type represented by the 'Victoria and Albert,' it will be sufficient to state that the maximum hogging-moment (at $a\,a'$, Plate XVI. fig. 7) is about 5080 foot-tons, equalling the product of the displacement by the hundred-and-fortieth ($\frac{1}{140}$) part of the length of this ship. Hence it follows that in certain classes of unarmoured ships, with excess of weight amidships, moderate bending-strains may be anticipated, and that when sagging-strains exist their amount is comparatively small.

In this connexion it may also be proper to revert to the tendency which the over-weighted ends of a ship have to break off from the midship part, to which I have already alluded in general terms. The fore bodies of the 'Bellerophon' and 'Minotaur' afford excellent illustrations of the effect which changes in the form of the immersed portions of a ship have upon this tendency. At the foremost water-borne section of the 'Minotaur' (R' in fig. 8, Plate XVII.) the bending-moment of about 19,000 foot-tons is produced by the unsupported weight (420 tons) of the part before it; at the corresponding section of the 'Bellerophon' (R' in fig. 9) the bending-moment is less than 1500 foot-tons, the excess of weight producing it being only 45 tons. Some part of the difference between these bending-moments is, as I have said, due to the complete protection of the 'Minotaur;' but a considerable part is due to the difference between the length and

fineness of the entrances of the two ships, and the change from the V-form to the U-form of transverse section. At the sterns of these ships there is also a remarkable contrast between the bending-moments tending to break off the overweighted parts, although not so great as that at the bows, on account of the absolute necessity for fineness of form in the run of the 'Minotaur;' the bending-moment at the aftermost water-borne section (R^2 in Plate XVII. fig. 8) is upwards of 20,000 foot-tons, while at the corresponding station in the 'Bellerophon' it is only about 7000 foot-tons. These strains on the stern, unavoidable as they are, to some extent often develope weakness in screw steam-ships. With respect to the bending-moments experienced by the foremost and aftermost water-borne sections of the 'Victoria and Albert' and of the 'Invincible' nothing need be said; the curves of moments for these ships are constructed on the same scale as those for the other two, and the lengths of the ordinates afford the means of comparing the strains at various parts.

Before concluding my remarks on the still-water strains of ships, I must refer to another cause of bending to which brief allusion has already been made, viz. the horizontal longitudinal fluid pressure on the immersed part of a ship. The most recent writers on the subject have not taken account of this cause of bending, doubtless because they considered its effect out of all comparison with the effect produced by the vertical forces, an opinion which the greater number of earlier writers also entertained. EULER and Dr. YOUNG were exceptions, as I have shown; and the following results will prove that they were justified in attaching importance to the effect of the fluid pressure, although they did not correctly estimate it. Without criticizing their methods, however, I will proceed to indicate a simple plan for estimating the amount, and the bending-moment, of this pressure at any transverse section of a ship afloat in still water.

According to a well-established hydrostatical law, the resultant fluid pressure, in a horizontal direction, on a solid immersed in it equals the pressure of the fluid on the projection of the surface of the solid upon a plane at right angles to that direction; and this resultant acts through the centre of pressure of the plane area. Applying this principle to a ship, we see that the longitudinal pressure upon any part bounded by a transverse section equals the pressure upon the immersed area of that transverse section, and acts through the centre of pressure of the immersed area. By this means, therefore, we can determine the amount and the line of action of the resultant longitudinal pressure upon the parts of a ship, either towards the bow or towards the stern, cut off by a transverse section; and knowing these two features, we can determine the moment of the pressure about any horizontal line in the transverse section. For our present purpose it will suffice to say that the line about which moments must be taken in order to determine the bending effect produced by the pressure at the section coincides with the centre of gravity of the section*. Assuming that we know its position, and knowing

* It may be interesting to state here that EULER's mistake respecting the action of this longitudinal pressure arose from the fact that he considered the lower side of the keel as the "fulcrum," as it was then termed, about which moments should be taken.

the amount and the line of action of the resultant fluid pressure, the determination of the bending-moment due to it is very simple.

I will now consider the cases of some actual ships, in order to give a more definite idea of the amounts of the bending-moments due to the longitudinal pressure, and the proportions they may bear to the bending-moments resulting from the action of vertical forces. Taking first the 'Minotaur,' it has been found by calculation that the fluid pressure on the midship section amounts to 405 tons. The total depth of the ship being 41 feet, the centre of gravity of the section has been taken 20 feet above the keel, and the centre of pressure of the immersed midship section has been found to be about $9\frac{1}{3}$ feet below it; so that the bending-moment due to longitudinal pressure has in this ship a maximum value of 3780 foot-tons. The bending-moment due to vertical forces has been fixed at 45,000 foot-tons, and is therefore twelve times as great; in other words, the bending-moment due to the longitudinal pressure is only one twelfth of that due to the vertical forces. This seems a very small proportion, but it is obtained from a ship in which the overburdening at the ends and the length are both excessively great, so that the bending-moment due to vertical forces is very large. In ships of more moderate dimensions, having a less excess of weight over buoyancy at the extremities, and of buoyancy over weight amidships, the proportion of the horizontal force is much greater. Taking the case of the 'Bellerophon,' for example, the fluid pressure on her midship section is a little over 350 tons, and the distance between the centre of pressure and the centre of gravity of the section is about $8\frac{8}{10}$ feet, so that the bending-moment due to this pressure equals 3120 foot-tons. This is the maximum value occurring at the midship section, where we have seen the hogging-moment due to the vertical forces to be 100 foot-tons only, so that at that section the former bending-moment is that which virtually fixes the limit of strain. Greater interest attaches, however, to the comparison between the two maximum values of the bending-moment. The absolute maximum. moment produced by the vertical forces has been found to be 12,000 foot-tons, and the moment produced by the longitudinal pressure is therefore a little more than one fourth of this amount. As compared with the 'Minotaur,' we find, then, that the bending-moment due to longitudinal pressure is very much less in actual amount for the 'Bellerophon;' but that, in proportion to the moment resulting from the unequal distribution of the weight and buoyancy, it is much larger in the shorter ship. This fact prepares us for the conclusion arrived at by Dr. Young in his Report on the diagonal system,—that in the short full ships of war in use at the commencement of this century the moment due to longitudinal pressure sometimes amounted to more than one third of the maximum moment produced by vertical forces. Dr. Young's method was incorrect, but his estimate of the relative magnitude of the bending-moments is probably not very far from the truth.

One other example must suffice for this branch of the subject. In the 'Audacious,' when afloat in still water, the pressure on the midship section has been found to be about 295 tons, and the distance between the centre of pressure and the centre of gravity

of the section is approximately a little more than $9\frac{1}{4}$ feet, the bending-moment due to the pressure being about 2740 foot-tons. This is, as nearly as possible, one fourth of the maximum bending-moment (11,000 foot-tons) produced by vertical forces in this ship; and in this respect the 'Audacious' resembles the 'Bellerophon,' although, being a smaller ship, the actual amount of the bending-moment due to the longitudinal pressure is less than that in the 'Bellerophon.'

On a review of these investigations, therefore, it appears that the limiting proportions of the bending-moments due to the pressure, and those due to the vertical forces, might be taken at one twelfth for very long fine ships, and one fourth for ships of moderate length and proportions, having their principal weights concentrated amidships. The cases which fall outside these limits may be considered exceptional amongst modern ships.

I have only determined the maximum values of the bending-moments due to the longitudinal pressure in the preceding examples, because they are the only values having much practical interest, and because my chief aim has been to show that it is not proper to omit all consideration of the bending effect of this pressure in all cases, as is usually done. In long fine ships, such as the 'Minotaur,' the error introduced by omitting it would be comparatively small for still-water strains; but in ships of more moderate proportions this is not the case. Were it at all desirable we might obviously determine the amount, and the moment, about the neutral axis, of the pressure on any transverse section, in a manner similar to that employed above for the midship section, and might represent the results graphically by curves of moments constructed similarly to those for the moments of vertical forces. In fact, by using the same scales of foot-tons, we might combine the curves of moments for longitudinal pressure and for vertical forces, and obtain a single curve which should represent their joint effect. This need not be done, however; for, as I have before remarked, the still-water strains experienced by a ship are not those which regulate the provision of strength; and we shall see hereafter that it is only for still water that the longitudinal has such an effect in proportion to the vertical forces as to require quantitative consideration.

The preceding considerations respecting the still-water strains resulting from the action of vertical and horizontal forces on a ship have been given at considerable length, because they furnish many extensions and corrections of existing knowledge, and also because methods similar to those here employed will hereafter be applied to the severer strains experienced by ships when at sea or when ashore. Besides this they have a special interest and importance themselves. They do not, it is true, exercise very much immediate influence on the distribution of material and the provision of strength in a ship, owing to the greater magnitude of the sea-strains; but they constitute what may be termed the *permanent* strains on the structure, and we have seen how very widely these permanent strains differ under different methods of distributing the weights and the buoyancy. In view of the illustrations chosen, there cannot fail to result a fuller appreciation of this fact than could result from any general statement; and the

importance of careful stowage must be fully realized, as well as the connexion which should subsist between the ship's form and the distribution of her weights. Some existing views have been shown to be erroneous,—notably those respecting the effect of an excess of weight amidships, and the position of the section of maximum strain. The classification I have adopted for ships is confessedly imperfect; but no general laws can possibly be laid down to include the very varied characters of the distribution of weight and buoyancy in all ships, and the types I have chosen have at least the merit of including a very large proportion of the cases met with, besides permitting a generality of investigation such as has not been previously attempted.

In attempting to approximate to the shearing- and bending-strains of *ships at sea*, we meet with a problem of great difficulty, and one which in the present state of our knowledge does not admit of complete or exact solution; in fact it may be doubted whether the very varied and rapidly changing conditions of strain in ships so situated will ever be completely expressed in mathematical language, and brought within the range of accurate calculation, in the same way as still-water strains have here been treated. It is, however, possible to distinguish the principal causes of straining in ships at sea, and in some cases to make approximations to what may be considered as their *limiting* values, as I shall show further on; but the dynamical aspect of the question, although the most important of all, is at present in some respects beyond our power, so far as its expression in *quantitative form* is concerned. That this is the case will be obvious on the most cursory glance at the condition of a ship in a sea-way. Neglecting, for the sake of simplicity, all consideration of rolling motion, and supposing the ship to lie directly bow on to the waves, the passage of each wave along her length establishes, or tends to establish, a vertical motion in the ship as a whole (except under certain special conditions), and a rotatory or pitching motion about some transverse axis, besides producing continual changes in the relative distribution of the weight and buoyancy all along the length. The ship's motion in pitching and ascending can be readily explained in general terms; but to express accurately the speed of that motion, and the corresponding accelerating forces, as well as the straining effect of the percussive shocks that are nearly certain to be caused by it, is an undertaking I shall not attempt. Even if this could be done, it would still be necessary to consider the heaving or vertical motion of the ship and the rapidly varying nature of the wave supports, both of which are causes of important straining-actions; and, in addition, to deal with the effect of the passage of a succession of waves (far from being of uniform dimensions and periods) as well as with the influence of the ship's onward motion. Altogether, therefore, we have before us a most complex question, which can only be touched, as it were, by some approximative method such as that I am about to describe.

From what has just been said, it will appear that there are three principal causes of increase in the longitudinal strains of ships at sea, as compared with their still-water strains,—the vertical or heaving motion of the ship as a whole, which is nearly sure to result from the wave motion, because ships share to some extent the motion of the waves;

the rapid changes in the positions occupied by ships relatively to the crests and hollows of the waves, by which most unequal distributions of the weight and buoyancy are, under some circumstances, produced and the straining forces are increased; and the pitching and ascending motions of ships, which further modify the strains, both by means of the accelerating forces thus developed, and by the percussive forces, shocks, and resistance of the water. These causes I propose to notice very briefly; it is to the quantitative investigation of the second, however, that I have given special attention, with detailed results, which I will presently record.

First, as to the modification of strain that may result from the heaving or vertical motion of a ship floating among waves. It is obvious that such motion must produce changes of strain; for when a vessel in moving downwards receives a check, the effect is to increase the straining-forces acting upon her; and when, in moving upwards, she reaches her highest position, and is for the moment partially abandoned by the water support, the strains upon the hull will be diminished. If the abandonment were total (that is, in the hypothetical case when the ship is left up in the air), all bending- and shearing-strains would, in fact, disappear; for then every particle in the ship would, for the moment, have impressed upon it the accelerating force of gravity, acting equally and in parallel directions throughout her.

Mr. W. FROUDE and Professor RANKINE have both referred to this subject when dealing with the strains of ships at sea; and the latter gentleman has attempted to fix the *limiting* maximum increase of strain produced by vertical motion. This he considers to be about *one fourth* of the still-water strains (for both shearing and bending), this estimate being based upon two or three assumptions (see 'Shipbuilding, Theoretical and Practical,' pages 151, 152). The fundamental assumption made is that the ship may be considered so small in proportion to the waves as to closely accompany their motion, just as a float would do. This obviously differs from the condition likely to be fulfilled by any actual ship. A ship cannot be expected to closely accompany the wave-motion, and her heaving cannot be regarded as the result of the passage of one wave only, but of a succession of waves differing, in all probability, in sizes and forms. For these reasons I cannot adopt Professor RANKINE's estimate (which doubtless has, however, a certain theoretic value) as a basis of practical calculation, nor am I prepared to substitute one of my own.

The practical deduction which should be kept in mind is the general one, that the heaving motion in ships at sea will, under some circumstances, produce increased strains. It seems probable that these strains are not so severe, in most cases, as those which result from variations in the wave supports and from pitching motions; but it must not be forgotten that all three causes may be operating simultaneously, and that their combined effect measures the actual strain on the structure.

Next, as to the additional strains resulting from the changing wave supports on a ship at sea. In dealing with this question I shall consider the two extreme positions of support illustrated by figs. 14 & 15. In the first of these a ship is supported on a single

Fig. 14.

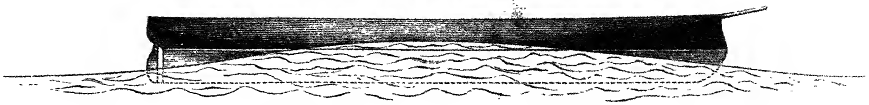
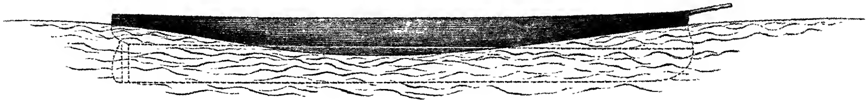


Fig. 15.



wave-crest, at or about the middle of her length; in the second she is situated in a wave-hollow, and has her extremities immersed in the adjacent wave-slopes. The first position obviously tends to develop hogging-strains of considerable amount, and the second to develop sagging-strains. It becomes necessary, therefore, to attempt some approximation to the amounts of these strains in various classes of ships, in order to compare them with the permanent still-water strains previously calculated; and for this purpose two or three assumptions must be made:—

(1) That for the moment the effect of the ship's vertical motion may be neglected.

(2) That for the moment the ship may be regarded as occupying a position of hydrostatical equilibrium.

(3) That the methods of calculating bending- and shearing-strains previously used for still water may be employed here also in order to approximate to the momentary strains.

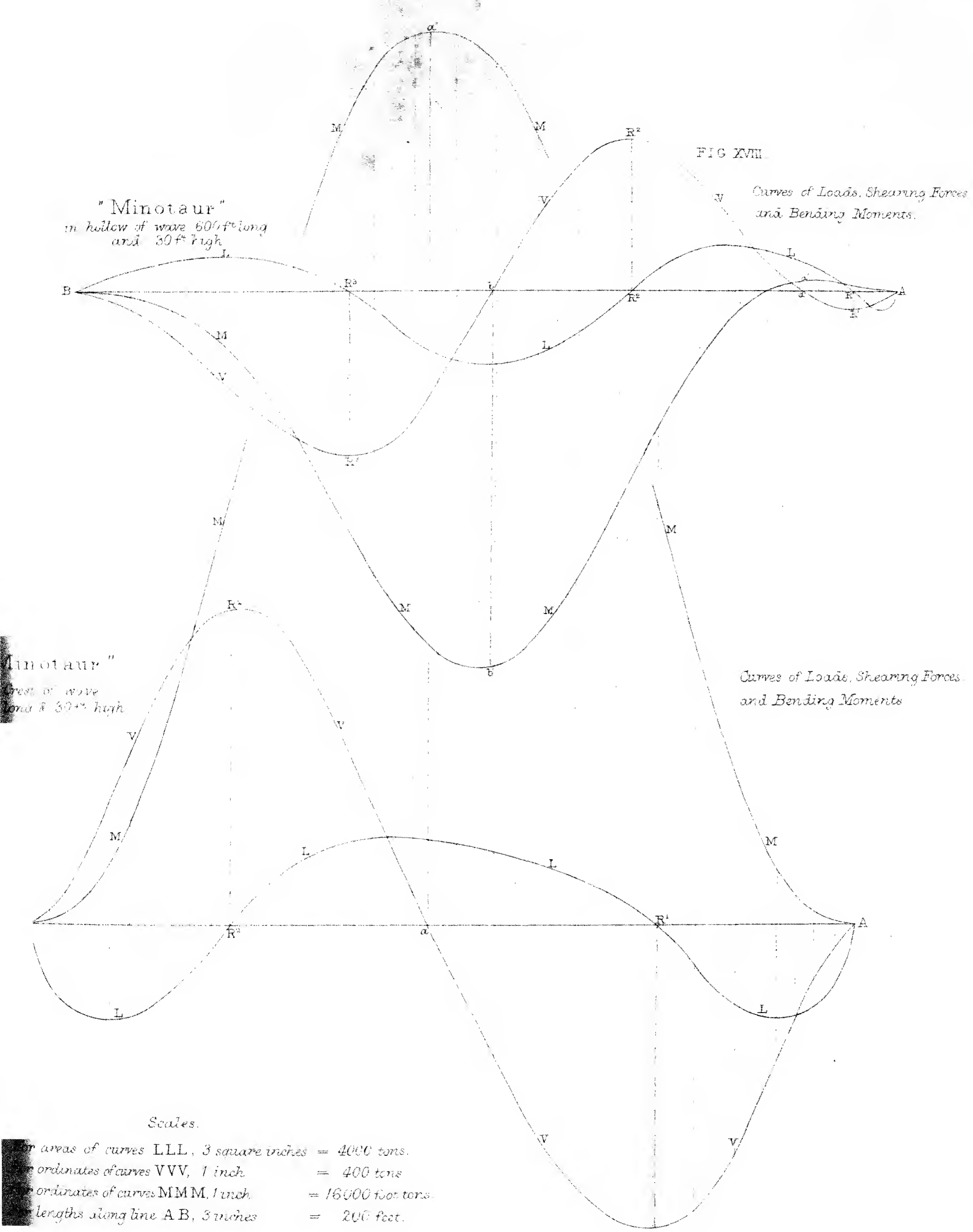
Investigations based on these assumptions, although confessedly imperfect, will give us a better idea of the limits between which the strains produced by changing wave supports lie than we could otherwise obtain.

I have referred to the positions of support illustrated by figs. 14 & 15 as *extreme*, and it may be proper to state briefly my reasons for doing so. I will take the case of fig. 15, and suppose the ship to be moved forward some distance so that her bow comes nearer to the wave-crests, and her stern moves further down into the wave-hollow. The effect of this change will obviously be that a greater length of the vessel will be supported on one wave-slope than on the other, and that the ship as a whole will rest upon a *flatter* portion of the wave-form than it did before, as the curvature is much greater in the central part of the hollow than it is further up the slope. Hence, when the vessel has taken up her new position of equilibrium, it seems certain that the water-level will not sink so much below the height amidships corresponding to still water as it does in fig. 15, nor rise so much above that height at the bow and stern; consequently the bending-strains will be less. Similar considerations render it evident that the position shown in fig. 14 is the other extreme.

Without further preface I shall proceed to consider the strains brought upon our three

FIG. XVII.

FIG. XVIII.



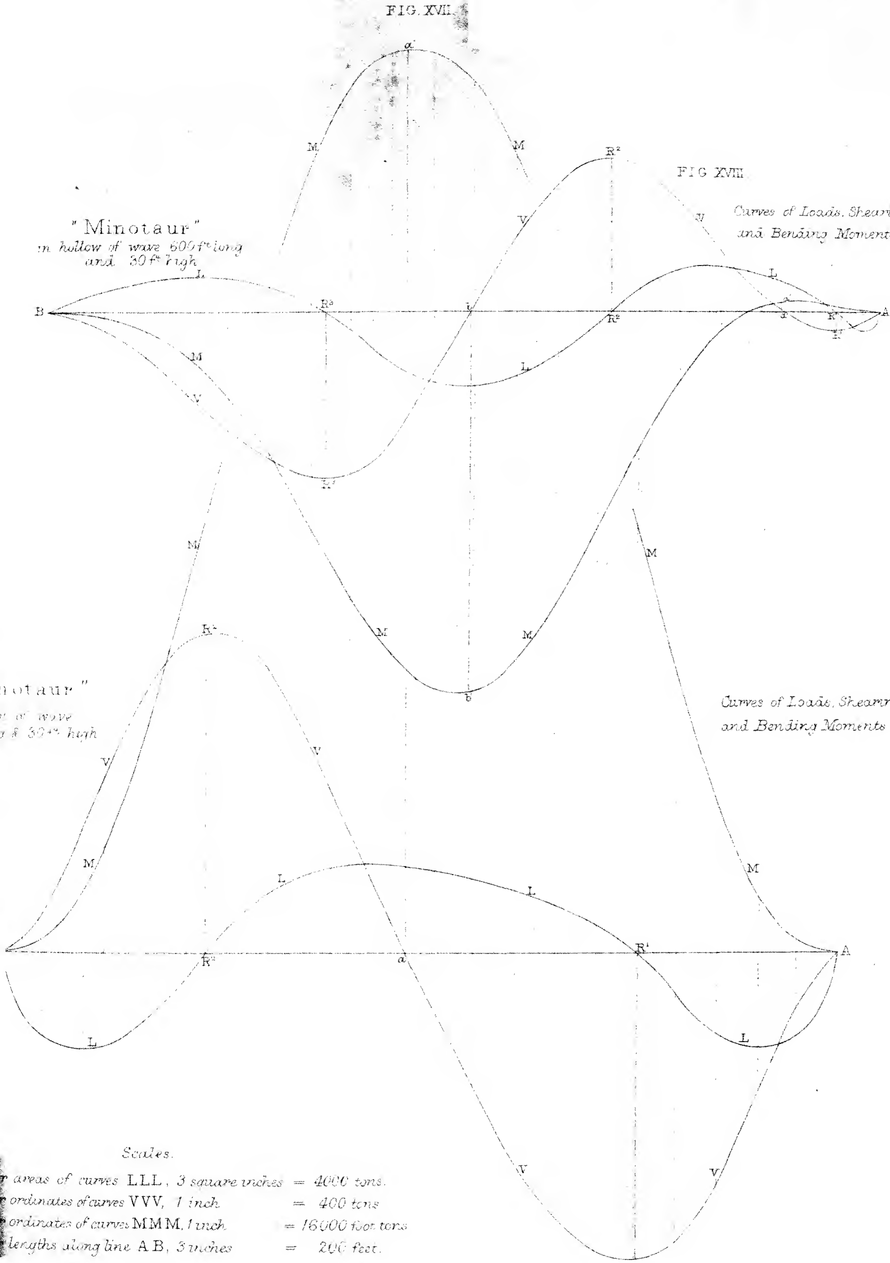
typical ships, the 'Minotaur,' 'Victoria and Albert,' and 'Bellerophon,' when supported on waves under the above-stated conditions. In still water the 'Minotaur,' as we have seen, has excesses of weight at the extremities only, and is subject to hogging-strains throughout her length; while the 'Victoria and Albert' has a large excess of weight amidships as well as at the extremities, and is subjected to sagging-strains in her middle body; and the 'Bellerophon,' although resembling somewhat the 'Victoria and Albert' in the distribution of the weight and buoyancy, has a maximum hogging-moment amidships instead of a sagging-moment.

When the 'Minotaur' floats on a wave-crest, it is clear she must be subjected to greater hogging-strains than are experienced by her in still water, because the excesses of buoyancy midships, and of weight forward and aft, will then be greater. On the other hand, when she floats in a wave-hollow, with her bow and stern deeply immersed in the wave-slopes, we naturally look for excesses of buoyancy at the extremities, and of weight amidships, because there the water-level has been lowered; hence it is reasonable to assume that in a wave-hollow such a ship will be subjected to severe sagging-moments *throughout her length*—a state of things that I previously showed could not exist in any ship, with excess of weight at the extremities, when floating in still water. These general considerations are confirmed by a closer examination and by actual calculation. Plate XVII. fig. 16 exhibits the relation between the weight and buoyancy of the 'Minotaur' when she is balanced on waves 600 feet long and 30 feet high, F F representing the buoyancy on the wave-crest (as in fig. 14), subject to the three conditions just laid down, and G G representing the buoyancy in the wave-hollow. The diagrams of the curves of loads, shearing-forces, and bending-moments are respectively shown for the wave-crest and wave-hollow in Plate XVIII. figs. 17 & 18; and the same scales have been employed in constructing them as were used for the corresponding curves representing still-water strains. It has been found by calculation that the excesses of weight at the bow and stern on the wave-crest become as nearly as possible *double* of those existing in still water, and that the shearing-strains become increased in about the same proportion, having a maximum value of 925 tons instead of 450 tons; while the maximum bending-moment amounts to more than 105,000 foot-tons instead of 45,000 foot-tons as in still water. The maximum shearing-force on the wave-crest is experienced by the station about 90 feet from the stern, and the maximum hogging-moment by a station not far from the middle of the length. In this case, therefore, the *character* of the strains remains practically unchanged as compared with those in still water, but their *intensity* is, roughly speaking, doubled. When the ship floats in the hollow of waves of the same dimensions (as in fig. 15), a very different state of things is met with, and one which presents some special points of interest. The deeper immersion of the stern in the wave-slope has the effect (figs. 16 & 18) of producing an excess of buoyancy of 410 tons on the first 110 feet of length, and the lowering of the water-level in the hollow causes a *defect* of buoyancy of no less than 880 tons in the middle portion of the ship, where in still water the *excess* of buoyancy is nearly as great. At the bow, however, even more striking results

are met with. Its transverse sections are so fine, and its weight is so great, that the increased immersion does not suffice to entirely do away with the excess of weight existing in still water; in fact on the first 20 feet of length there is a small excess of weight (about 50 tons) still remaining; and we thus meet for the first time with a case where the conditions of the two extremities of a ship differ, an excess of buoyancy existing at one end simultaneously with an excess of weight at the other. In actual ships at sea this must often be so, since the pitching and ascending motions are sure to produce a deep immersion at one extremity simultaneously with the emersion of the other extremity. Reverting to the 'Minotaur' in the wave-hollow, I need only state that the maximum shearing-force is found to be almost the same as that for still water (470 instead of 450 tons), and that the maximum bending-moment is also of nearly identical amount with that for still water (44,000 instead of 45,000 foot-tons), only it is a sagging-strain instead of a hogging-strain. In fact for five-sixths of her length the 'Minotaur' is subjected to sagging-strains when in the wave-hollow; and as the small excess of weight at her bow can only be expected to exist in armoured ships of her extreme length and fineness, under such circumstances we are quite warranted in assuming that, as a rule, ships of this type will sag *throughout the whole length* when floating in wave-hollows. In such ships also, so far as this example enables us to judge, the sagging-strain in a wave-hollow is likely to fall below one half of the hogging-strain on a wave-crest.

Within certain limits the strains of a ship of the 'Minotaur' type will be increased when the lengths of the waves become decreased and their steepness increased. For instance, I have previously supposed the 'Minotaur' to be balanced on waves 600 feet long, while her own length is only 400 feet; but if the waves were decreased in length, the strains would usually be increased so long as the decrease is not sufficient to cause the extremities of the ship to be immersed in the slopes of adjacent waves. The latter consideration roughly fixes the limit of decrease in the length of the wave by the condition that the wave and the ship shall be of equal lengths; and consequently to find the limiting values of the bending- and shearing-strains corresponding to the extreme positions of support, we will suppose our typical ships to be balanced on waves of their own length, and of such steepness as is likely to be met with in ocean-waves. The results of calculations made on these bases, and with the foregoing assumptions, I shall now briefly describe, as well as those for the corresponding wave-hollows.

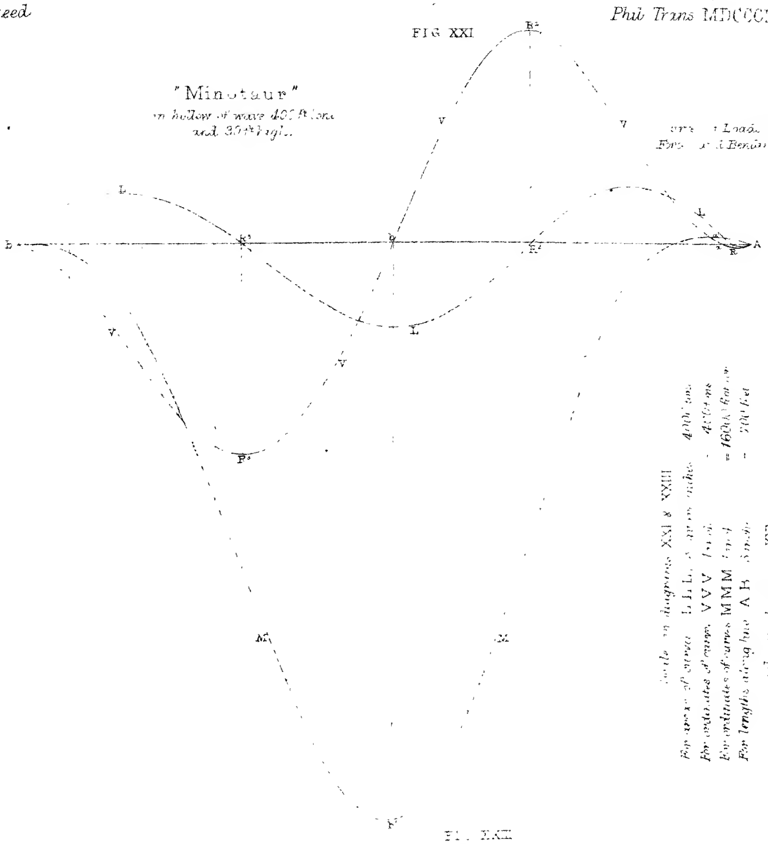
First I will take the case of the 'Minotaur' on a wave 400 feet long and 25 feet high, instead of 600 feet long and 30 feet high. The results are graphically recorded in Plate XIX. figs. 19 & 20, and Plate XX. fig. 21. On the crest of such a wave the excesses of weight at the bow and stern respectively are found to be no less than 1275 tons and 1365 tons. The maximum shearing-force is 1365 tons, and the maximum hogging-moment is 140,300 foot-tons. By changing the dimensions of the wave, therefore, the maximum shearing-force has been increased by 435 tons, about one half, and the maximum hogging-moment by 35,000 foot-tons. The latter now equals $3\frac{1}{8}$ times the



Scales.

r areas of curves LLL, 3 square inches = 4000 tons.
r ordinates of curves VVV, 1 inch = 400 tons
ordinates of curves MMM, 1 inch = 16000 foot tons
lengths along line A.B, 3 inches = 200 feet.

FIG XXI



Scale in diagrams XXI & XXIII
For areas of curves L L L, 1 square inch = 4000 tons
For ordinates of curves V V V 1 inch = 4000 tons
For ordinates of curves M M M 1 inch = 160000 foot tons
For lengths along line A B 1 inch = 200 feet
Scales in diagram XXV
For areas of curve 1 square inch = 2000 tons
For lengths along line A B 1 inch = 100 feet

FIG XXII

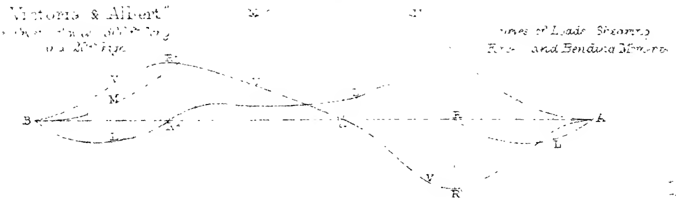
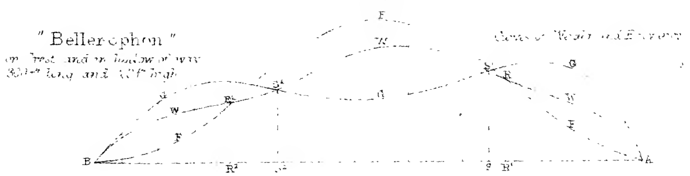


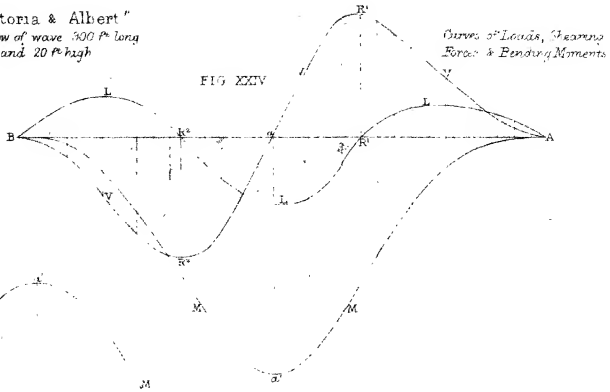
FIG XXV



"Victoria & Albert"
in hollow of wave 300 ft. long
and 20 ft. high

FIG. XXIV

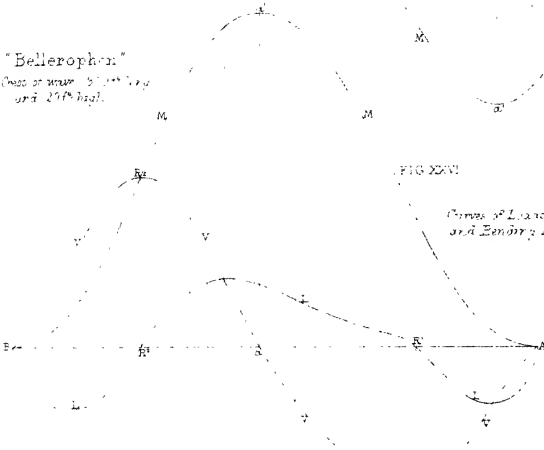
Curves of Loads, Shearing Forces & Bending Moments



"Bellerophon"
on crest of wave 3[illegible] ft. long
and 20 ft. high.

FIG. XXVI

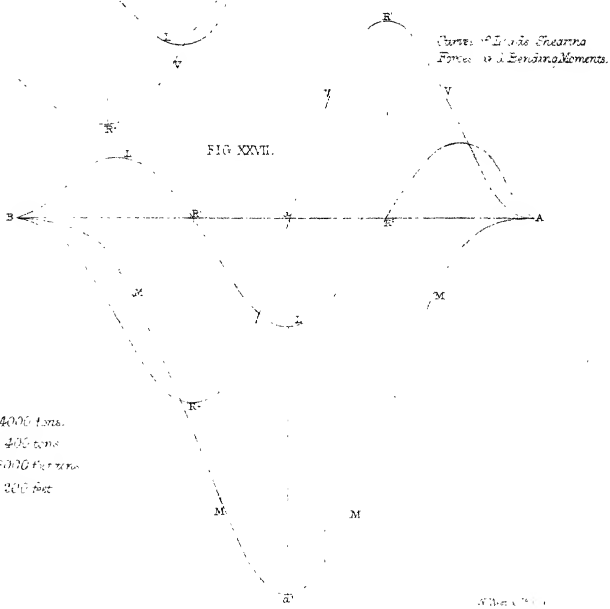
Curves of Loads, Shearing Forces and Bending Moments



"Bellerophon"
in hollow of wave 3[illegible] ft. long
and 20 ft. high.

FIG. XXVII.

Curves of Loads, Shearing Forces and Bending Moments.



Scale in diag.ms XXIV, XXVI & XXVII

For areas of curves L L L, 3 square inches = 4000 tons.

For ordinates of curves V V V, 1 inch = 400 tons

For ordinates of curves M M M, 1 inch = 16000 foot tons

For lengths along line A B, 5 inches = 200 feet

maximum hogging-moment in still water. In the wave-hollow the deep immersion of the extremities and the less immersion of the middle leads to the following distribution of the weight and buoyancy. For the first 120 feet of the stern an excess of buoyancy of 695 tons, for the next 160 feet amidships a defect of buoyancy of 1380 tons, for the next 110 feet an excess of buoyancy of 695 tons, and right forward a very small excess of weight (10 tons on about 10 feet). The remarks previously made respecting the similar distribution on the 600-feet wave apply to this case also, and need not be repeated. Our chief interest centres in the determination of the maximum shearing-force and sagging-moment. The former amounts to 695 tons, and the latter to 74,800 foot-tons. As in the previous case, the section of maximum bending-moment falls nearly amidships, and the figures show that the sagging-strain (fig. 21) is a little more than one half the hogging-strain incidental to support on the wave-crest (fig. 20), while it is considerably greater than the still-water strain. These comparative results may be summarized approximately as follows.—

'Minotaur' type. Strains under various conditions.

	Still Water.	On Wave-crest.	In Wave-hollow.
Maximum shearing-force ÷ displacement.	$\frac{1}{22}$	$\frac{1}{7}$	$\frac{1}{14}$
Maximum bending-moment ÷ displacement × length.	$\frac{1}{88}$ (hogging)	$\frac{1}{28}$ (hogging)	$\frac{1}{53}$ (sagging)

By means of these proportions it is possible to approximate to the amounts of the bending-moments and shearing-forces in other ships of the type when their displacements and lengths are known, only it is necessary to remember that in many cases the results obtained must be regarded as strictly limiting values, on account of the fact that in the 'Minotaur' the excesses of weight at the extremities for still water and the fineness of form are both extremely great.

Passing on to the second typical ship, the 'Victoria and Albert,' we will suppose her to be balanced on the crest of a wave of her own length (300 feet) and 20 feet high from hollow to crest. This height has been taken because it bears nearly the same proportion to the length as that of the 400-feet wave on which we supposed the 'Minotaur' to be balanced, and fairness of comparison between the conditions of strain in the two types is thus ensured. Under these circumstances careful calculations, of which the results are recorded in Plate XIX. fig. 22, Plate XX. fig. 23, and Plate XXI. fig. 24, show that the additional immersion of the middle body does away altogether with the excess of weight existing there in still water, while the decreased immersion of the ends leads, of course, to an increase in their excesses of weight. In short, on the wave-crest the condition of this ship becomes similar to that of the 'Minotaur' in still water. At the bow the excess of weight amounts to 220 tons, and at the stern to 185 tons, while

between these parts, on a length of about 160 feet amidships, the buoyancy is 405 tons in excess. The resulting maximum strains are as follows:—Shearing-force 220 tons, bending-moment (hogging) 16,400 foot-tons. In still water the maximum shearing-force is 140 tons, and the maximum hogging-moment about 5080 foot-tons, this moment being experienced by a section considerably before the middle of the length, whereas on the wave-crest the section of maximum moment is comparatively near the middle.

This leads me to mention a point of some importance in the regulation of the longitudinal strength of ships. The ordinary assumption is that for both still-water strains and the strains of ships at sea, the midship section is that which has to bear the maximum moment, and that in moving from that section out towards the extremities we find the bending-moments continually decreasing in amount. I have already shown that this view is not correct so far as still-water strains are concerned, and that the position of the section of maximum moment may lie far away from the middle of the length—as it actually does in the 'Victoria and Albert' and in the 'Bellerophon.' The question arises, therefore, is the section of maximum moment generally near the middle when a ship is supported on a wave-crest or in a wave-hollow? or may that section be ordinarily expected to lie away from the middle in ships not of the 'Minotaur' type? So far as these investigations go, the answer to the question is, that the midship section, or some section comparatively near it, has to sustain the greatest bending-moment in all three types and in both the extreme positions of support. Hence it follows that the popular view, although incorrect in some respects, leads on the whole to conclusions that very properly influence practice; and general experience confirms the soundness of the opinion that, after keeping up uniformity of strength throughout a considerable length of the middle body, it is advantageous, from this point of view, to reduce the ship's scantlings as we proceed towards the extremities. What laws should regulate the reductions is a question upon which I shall not now enter.

When the 'Victoria and Albert' floats in the hollow of waves of the dimensions just stated, her condition of strain is, of course, entirely different from that in still water and from that on the wave-crest. The strains brought upon her are very remarkable. The excess of weight amidships (210 tons) existing in still water becomes exaggerated to 785 tons, and instead of having excesses of weight at the extremities we have considerable excesses of buoyancy, no less than 390 tons forward and 395 tons aft. The result of these changes is to produce sagging-moments throughout the ship's length, gradually increasing in amount as we proceed from the extremities towards the middle, and reaching their maximum at a station near the midship section. The maximum moment under these circumstances amounts to 31,000 foot-tons, and the maximum shearing-force to 395 tons.

From these figures we obtain the following approximate summary:—

'Victoria and Albert' type. Strains under various conditions.

	Still Water.	On Wave-crest.	In Wave-hollow.
Maximum shearing-force ÷ displacement.	$\frac{1}{16}$	$\frac{1}{11}$	$\frac{1}{6}$
Maximum bending-moment ÷ displacement × length.	$\frac{1}{139}$ (hogging)	$\frac{1}{43}$ (hogging)	$\frac{1}{23}$ (sagging)

The variation in the amount and character of the bending-moments is the most interesting feature in this summary. On the wave-crest the maximum hogging-moment is somewhat, more than three times as great as the maximum hogging-moment, for still water; and in this respect there is a clear resemblance to the similar proportions for the 'Minotaur,' although, in proportion to the products of the lengths and displacements, the 'Victoria and Albert' is much less severely strained than the 'Minotaur.' For the wave-hollow, however, the case is widely different. In the 'Minotaur' the bending- (sagging-) moment there has a maximum value less than *twice* as great as the still-water hogging-moment; but in the 'Victoria and Albert' the sagging-moment is about *six times* as great as the still-water hogging-moment. The 'Minotaur,' therefore, experiences her absolute maximum bending-moment when suspended on a wave-crest, while that for the 'Victoria and Albert' is incidental to flotation in a wave-hollow. This result would naturally be anticipated, when the conditions of strain previously shown to exist when the ships are floating in still water are taken in connexion with the distribution of the buoyancy in the extreme positions of support among waves. Hence follows the practical deduction that ships of the 'Minotaur' type are likely to be subjected to hogging-strains severer in their character than any sagging-strains that will be brought upon them; while vessels of the 'Victoria and Albert' type are likely to be most strained by sagging. Although, as I have said previously, the latter type only includes, at present, ships intended for special services, such as yachts, blockade-runners, &c., its investigation leads to a valuable extension of the ordinary theory respecting bending-strains, and appears to have been overlooked even by the most recent writers. For instance, Professor Rankine (at page 153 of 'Shipbuilding, Theoretical and Practical') says of the sagging-moment incidental to support in a wave-hollow, that "in all cases of ordinary occurrence in practice, the sagging-moment thus produced is less severe than the hogging-moment produced when the ship is balanced on the crest of a wave." This statement is undoubtedly true of many ships; but it would apparently lead to very erroneous results if it were supposed to hold universally, and I have therefore drawn attention to it. The method which in my opinion ought to be followed in approximating to the probable strains that a ship will experience, is first to determine, from her form and stowage, to what type she is likely to belong, and then to consider whether or not it is necessary to take the sagging- as well as the hogging-strains into account in calculating the strength required in her structure.

The third typical ship, the 'Bellerophon,' next claims attention; and I shall suppose her to occupy the two extreme positions of support amongst waves of the same dimensions as those taken in the 'Victoria and Albert,' 300 feet long and 20 feet high. The results of the calculations are graphically recorded in Plate XX. fig. 25 and Plate XXI. figs. 26 & 27. It has been previously shown that the still-water strains of the 'Bellerophon' bear much resemblance to those of the 'Victoria and Albert,' and the same thing is true of their strains on the wave-crests and in the wave-hollows. The fact that the excess of weight amidships in still water is proportionately less in the 'Bellerophon' than it is in the 'Victoria and Albert' prepares us, however, for the result which calculation develops, viz. that the hogging-moment of the 'Bellerophon' on the wave-crest bears a larger proportion to the sagging-moment in the wave-hollow than the corresponding hogging-moment of the 'Victoria and Albert' bears to the corresponding sagging-moment. On the wave-crest the excess of buoyancy amidships in the 'Bellerophon' amounts to 1000 tons (a striking change from the 250 tons *defect* existing in still water), while the excesses of weight at the bow and stern amount to 445 and 555 tons respectively; the resulting strains,—maximum shearing-force 555 tons, maximum hogging-moment 43,600 foot-tons. The latter is about $3\frac{2}{3}$ times the maximum hogging-moment for still water; and it is experienced by a section very near the middle of the length, although the section of maximum still-water strain is considerably abaft the midship section. In the wave-hollow the conditions of strain are exactly reversed. We then have an excess of weight of 1240 tons amidships, and excesses of buoyancy amounting to no less than 640 and 600 tons at the bow and stern respectively; the maximum shearing-force becomes 640 tons, and the maximum sagging-moment is 48,800 foot-tons, about four times the maximum hogging-moment in still water, and about 5200 foot-tons greater than the maximum hogging-moment on the wave-crest. Put into the form of a summary, these results stand as follows:—

'Bellerophon' type. Strains under various conditions.

	Still Water.	On Wave-crest.	In Wave-hollow.
Maximum shearing-force ÷ displacement.	$\frac{1}{33}$	$\frac{1}{13}$	$\frac{1}{11}$
Maximum bending-moment ÷ displacement × length.	$\frac{1}{176}$ (hogging)	$\frac{1}{48}$ (hogging)	$\frac{1}{43}$ (sagging)

Here, then, we have another case in which the maximum sagging-strain exceeds (although not largely) the maximum hogging-strain, and which confirms the opinion previously expressed respecting the impropriety of considering hogging as the only strain which need be considered when arranging the details of a ship's structure.

Some interest also attaches to a comparison between the strains of the two typical ironclads 'Minotaur' and 'Bellerophon,' viewed merely as types of form and distribution

of weight. In still water the bending-strain (in proportion to the products of the displacements and the lengths) is about one half as severe in the 'Bellerophon' as it is in the 'Minotaur,' on a wave-crest it is less than *two-thirds* as severe, and in a wave-hollow it is proportionately about one and one-fourth as great in the 'Bellerophon' type as it is in the 'Minotaur.' Another point of contrast is found in the fact that the maximum hogging- and sagging-strains are practically about equal in the 'Bellerophon' type, whereas in the 'Minotaur' the hogging-strain is very nearly double the sagging-strain. The longer ship must have, therefore, a very great reserve of strength against sagging if she is strong enough to resist hogging; and this fact prevents the material from being so well disposed as it might be if there were not so great a difference between the two extremes of strain. This follows from the consideration that an iron-built ship, such as the 'Minotaur,' is comparatively strong against sagging, because (if the longitudinal strength of the keel and keelson-work is kept up, and the bottom plates are properly butt-strapped) there is sure to be a reserve of tensile strength in the bottom, and because under compressive strains the wood upper deck becomes very effective; whereas against hogging the upper part of the ship is usually much weaker, both absolutely and as compared with the lower part. In the case of an armoured ship like the 'Minotaur,' the power to resist sagging is also added to very greatly by the resistance offered to compression by the upper strakes of armour, which contribute besides, in a considerable, although a much less degree, to her power to resist hogging.

The preceding comparison, as I have said, is limited to the consideration of the 'Minotaur' and 'Bellerophon' as *types*; but there remains to be considered the important question of the actual strains experienced by these two vessels when in their extreme positions of support among waves. These are as follows:—

On Wave-crest.

	Minotaur.	Bellerophon.
Excess of weight forward	1,275 tons.	445 tons.
Excess of weight aft	1,365 ,,	555 ,,
Excess of buoyancy amidships ...	2,640 ,,	1,000 ,,
Maximum shearing-strain	1,365 ,,	555 ,,
Maximum bending-moment	140,300 foot-tons.	43,600 foot-tons.

In Wave-hollow.

	Minotaur.	Bellerophon.
Excess of buoyancy forward	685 tons.	640 tons.
Excess of buoyancy aft	695 ,,	600 ,,
Excess of weight amidships	1,380 ,,	1,240 ,,
Maximum shearing-strain	695 ,,	640 ,,
Maximum bending-moment	74,800 foot-tons.	48,800 foot-tons.

The only matter to which attention requires to be specially drawn is the comparative amounts of the bending-strains, because against shearing-strains even the 'Minotaur' may be expected to have a large reserve of strength. On the wave-crest, then, we find the 'Minotaur's' bending-moment more than three times as great as the 'Bellerophon's;' and in the wave-hollow it is about half as much again as the 'Bellerophon's.' But we must not stop here. In the 'Bellerophon' the material at the midship section available against bending-strains is, to say the least, equal in efficiency to that of the 'Minotaur;' and consequently I shall be within the truth in saying that, in proportion to the strains previously calculated, the 'Bellerophon' is about three times as strong as the 'Minotaur' is against hogging, and about once and a half as strong against sagging.

From the preceding investigations it appears that the *statical* strains resulting from extreme positions of support of a ship floating among waves may, under the assumed conditions, reach amounts varying between *three* and *six* times the still-water bending-strains. (These results, of course, do not in any way represent the dynamical strains due to pitching and heaving motions.) This being so, it appears that some idea of the relative strengths of ships may be obtained by using the approximate values of the maximum bending-moments found above for the different types under different circumstances. Experience alone can enable us to judge whether the *absolute* strength of a ship is likely to prove sufficient against both dynamical and statical strains; and in order to do so we should have to take some ship which had answered and compare her construction with the proposed design. At the same time the investigations to which I have just drawn attention serve to indicate two features in which the strains of ships at sea differ from still-water strains,—the first, and most obvious, being their much greater severity, and the second their great and rapid variation in both intensity and character. On the latter feature a few additional remarks may be made.

Between the two extreme positions of support we have considered, others may be imagined, and undoubtedly occur, in which a ship is so circumstanced as to be subject to hogging-strains at some portions of her length and to sagging-strains at other portions. For instance, when the crest of a wave is intermediate between the bow and the

middle of a ship's length, and the after body is lying across the hollow, we should expect to find hogging-strains forward and sagging-strains aft; and as the wave-crest moved aft relatively to the ship, the character and intensity of the bending-strains must be continually varying in passing from one extreme to the other. Of the extent to which this variation reaches we already have some idea; but we must also introduce the idea of its *rapidity* before we can realize any thing like its full effect.

In order to make my remarks as definite as possible, I will again refer to the three typical ships on waves of the dimensions previously assigned. The time of transit of waves 400 feet long is rather less than nine seconds (really 8·83 seconds), and in half that time, or $4\frac{1}{2}$ seconds, we may suppose the 'Minotaur' to have passed from the wave-crest to the wave-hollow; while her maximum bending-strain has changed from 140,000 foot-tons of hogging- to 74,000 foot-tons of sagging-moment; and every transverse section has been subjected to similar changes in the character and amount of the strains brought upon it. The great rapidity in the variation of the strain is so apparent from this brief statement of facts that I need not dwell upon it.

The 'Victoria and Albert' and the 'Bellerophon' have been supposed to float on waves 300 feet long, for which the whole time of transit is less than 8 seconds (really 7·65 seconds). In half that time, therefore, or in less than 4 seconds, the ship may have passed from crest to hollow, and the bending-strains have passed through successive phases from one extreme to the other. On the crest the maximum hogging-moment of the 'Victoria and Albert' has been found to be about 16,400 foot-tons; in less than 4 seconds, in the wave-hollow, the maximum sagging-moment may have reached 31,000 foot-tons, at the same section of the ship which previously had to resist the hogging-strain. This is, proportionately, the greatest change which we have met with. In the 'Bellerophon' the change in the same brief interval, although not so great in proportion, is very striking; for on the crest the hogging-moment amidships is 43,600 foot-tons, and in the wave-hollow the sagging-moment is 48,800 foot-tons.

I give these figures merely as indications of what may be expected to happen in the changes of strain in ships at sea; and they probably fall much below the truth, since, as I have just said, no account has been taken of the effect of violent pitching-motions, which must lead to still more abrupt and violent changes. Enough has been said, however, to show how important this feature of the subject is; and I will simply add that a very convenient way of expressing the effect I have been attempting to describe is afforded by the supposition that the ship is fixed, and that what may be termed "waves of strain" roll through her structure. The introduction of this idea will help us to understand more clearly how *changes* in strain affect a structure; for a very small strain (considered statically), which would not affect a comparatively weak structure sensibly if it were constantly acting in one direction, will suffice to destroy a far stronger structure if its direction is continually and rapidly changed.

This subject has not escaped the attention of preceding writers; and Mr. Fairbairn has made some interesting remarks upon it, at page 13 of his work on 'Iron Ship-

building,' where he refers to the results of a series of experiments on the endurance of iron-jointed beams when subjected to changes in the loads put upon them. He says "the joints of an iron-rivetted beam sustained upwards of three million changes of *one fourth* the weight that would break it, without any apparent injury to its ultimate powers of resistance. It broke, however, with 313,000 additional changes when loaded to *one third* the breaking-weight, evidently showing that the construction is not safe when tested with alternate changes of a load equivalent to *one third* the weight that would break it." In the case of ships, however, Mr. FAIRBAIRN thinks the strain brought upon the material should not exceed one fourth or one fifth its ultimate strength, on account of the fact that the changes of strain are not merely effective as regards its *amount*, but also as regards its *direction*. His final conclusion is that in iron ships "it seems highly probable that a strain of 5 tons per square inch on the material acting alternately in opposite directions would at least injure, if it did not ultimately fracture, the material after a great number of alterations." Professor RANKINE, I may add, also considers that the strain on the material in iron ships should not exceed one fifth of the ultimate strength, and thus provides for the changes of strain of which I have been speaking.

It has sometimes been taken for granted, by writers on this subject, that while the straining-actions in smooth water become greater as the figure of the vessel becomes finer and sharper, the additional straining-actions produced by waves become less, and that these two opposite changes in a rough way compensate for each other. This conclusion is based, however, upon hypothetical cases, and not upon actual ships. On turning to our typical ships we find that it is by no means always correct. Take, for instance, the cases of the 'Minotaur' and 'Bellerophon,' one a long fine ship, and the other a comparatively short ship with fuller water-lines. In still water we have seen that the maximum hogging-moment in the 'Minotaur' is about $\frac{1}{88}$ of the product of the displacement by the length, while the corresponding moment in the 'Bellerophon' is about $\frac{1}{176}$ of the corresponding product—that is to say, the hogging-moment in the short ship is about one half as great in proportion as that in the long ship. This is, so far, quite in accordance with the view that the bending-actions in smooth water become greater as the figure of the vessel becomes finer and sharper; but when we pass to the case of the wave-crests, we find that the long fine ship is still much more severely strained than the short one. The maximum hogging-moment in the 'Minotaur' then equals $\frac{1}{28}$ of the product of the displacement by the length; that in the 'Bellerophon' is less than $\frac{1}{48}$ of the corresponding product—that is to say, is about $\frac{7}{12}$ as great, in proportion, as the 'Minotaur's' maximum strain. We see, then, that in no sense is there such a compensation for the additional strains due to wave-support as has been supposed. In the two ships I have here taken the distributions of the weight are, of course, very different, as well as the forms of their immersed bodies, and this helps to make the strains less severe in the shorter ships; but even after allowing for this it appears that the adoption of finer lines does not produce the mitigating effect in waves which has been supposed.

I must next refer briefly to the subject of the bending-strains resulting from the

horizontal fluid pressure in a ship at sea. We have already seen that this pressure has an effect too considerable to be neglected in certain classes of ships floating in still water; and that in a ship like the 'Bellerophon' the effect might rise to one fourth of the bending-moment due to vertical forces. I have also stated that in ships at sea the conditions of strain are very different, and that no great error would be introduced by neglecting the effect of the horizontal fluid pressure, and considering only the vertical forces. This I shall now proceed to show, availing myself once more of the two extreme positions of support illustrated by figs. 14 & 15 (p. 446).

First, let us consider the case of a ship on a wave-crest. We have already seen that the vertical forces incidental to that position had the effect of making the maximum bending-moment from three to four times as great as it was in still water; but it will be evident that the effect of the horizontal fluid pressure does not increase in that ratio, if it increases at all. A glance at fig. 14 shows that the depression of the water-level at the bow and stern must have the effect of uncovering, so to speak, a large portion of the midship section at the middle, and so greatly reducing the pressure on that midship section below what it would be if the water-level at that section were preserved throughout the length*. In addition to this, at a wave-crest the tension and pressure of the fluid are reduced by means of the vertical motion; so that both these causes, acting conjointly, produce such a reduction of the bending effect of the horizontal pressure as to render it, in all probability, no greater than, even if so great as, that of the corresponding pressure in still water. Consequently in proportion to the bending-moment due to the vertical forces, that produced by the horizontal pressure may be safely neglected for this position.

The same thing is true for the wave-hollow, only for a different reason. In that position, as shown by fig. 15 (p. 446), the water-level at the extremities of the ship is higher, and that in the middle lower, than in still water. Hence it is obvious that, for some portion of her depth lying *above* the centre of gravity of the midship section, the ship will be subjected to compressive strains, as well as for that portion lying *below* the centre of gravity. The effect of this will evidently be to reduce the bending-moment of the horizontal pressure; and this reduction will in all probability more than counterbalance the effect of the increased tension of the water in the wave-hollow. On this account we may assume that the effect of the horizontal pressure may be neglected in comparison with that of the vertical forces, the latter being, as we have seen, from three to six times as great in the wave-hollow as it is in still water.

These general considerations are confirmed by roughly approximate calculations made for the 'Bellerophon' when on the crest and in the hollow of waves of her own length. From what has just been said, it will appear that in making such calculations we shall obtain results considerably exceeding the true ones if we assume that the pressure is always due to the maximum height of the water-level on the ship; so that if these

* It is true that an additional area of midship section is covered at the sides by the rise of the wave-crest, but, as the next sentence mentions, the tension and pressure of this wave-crest are small.

results may be neglected in comparison with the effect of the vertical forces, much more may the true bending effect of the fluid pressure be neglected. For the wave-crest I find that the water-level at the midship section is about $7\frac{1}{2}$ feet above that in still water; and that if the level were uniform throughout the ship's length (which it by no means is) this immersion would give a pressure of about 620 tons, the bending effect of which would be about 3700 foot-tons. This is certainly much greater than the true bending effect, yet it only reaches about *one twelfth* of the bending-moment due to vertical forces on the wave-crest. In the wave-hollow similar results are obtained. The water-level at the bow and stern then rises about 11 feet above that in still water; and supposing this level to be uniform throughout the length, the pressure on the midship section would be about 770 tons, producing a bending-moment of about 3450 foot-tons—that is to say, about one fourteenth of the bending-moment due to vertical forces in the wave-hollow. The true bending-moment of the pressure would be less than this, and might consequently be neglected. Hence we are confirmed in the opinion that for ships at sea it is only necessary to consider vertical forces.

I must now pass on to notice briefly the longitudinal strains of ships in exceptional positions—such as those produced by launching, grounding, and other causes. Although exceptional, these strains undoubtedly occur; and their effect may be, as I shall show, more severe than that of any of the statical strains to which approximations have been made. Preceding writers have recognized this fact to some extent, but no author has done so much as Mr. FAIRBAIRN towards giving quantitative expressions to these exceptional strains.

BOUGUER, ROMME, and other early writers had a very clear conception of the principal causes of launching-strains. They call attention to the fact that in all, or nearly all, launches the ship's bow or stern is water-borne before the other end has left the launching-ways, and that this cannot be prevented altogether even when the ways are extended further out into the water than is customary. Some of them go so far as to urge the policy of building ships in dock in order to avoid these strains, and support their opinion by statements of the large amount of "breakage" that takes place when a ship is set afloat for the first time from a slip-way. Others combat this opinion, and show by actual examples that ships built in dock also break when they are floated, although the breakage is not always so considerable as when ships are launched; so that the severity of the strains due to launching cannot be estimated simply by the amount of breakage. The latter opinion is undoubtedly the correct one; and the breakage recorded for wood ships when launched may be regarded as due in part to what is termed the ship's "settling" in her new position afloat—that is, to her reaching such a condition as to make her powers of resistance balance the bending-moments due to the unequal distribution of the weight and buoyancy. In part, however, the breakage is undoubtedly due to the dynamical strains connected with launching; and although we cannot separate the effects produced by these two causes, we may very properly regard the amount of the breakage as to some extent a measure of the relation between the strength of the structure and

the strains brought upon it. I will pass over the facts respecting the breakage in the older classes of wood ships, and confine attention to modern wood ships. Speaking of merchant vessels, Mr. GRANTHAM says*, "It is the general custom with builders to leave the gangways of the bulwarks in modern ships unfinished, lest the hull should so much alter in form by settling in launching that the rails would not again fit their places; and no builder would willingly copper a vessel when new, but rather allow her first to find her own position in the water, as she would then be less liable to wrinkle the sheets." In wood-built ships of war there is also a considerable amount of breakage, as the following facts will show. Our finest screw line-of-battle ships, of which the length was about 260 feet, broke, on the average, about 2 inches, some ships proving weaker than others, and the breakage in one case (that of the 'Gibraltar') amounting to 4 inches on a length of 200 feet. Our finest screw frigates, which are 300 feet long, broke from 3 to 4 inches; the 'Galatea,' 280 feet long, broke 3 inches. Shorter ships, of course, usually broke less than these long fine ships. Iron ships, I need hardly say, display very little change of form or breakage when launched, the character of the materials and fastenings used in their construction being so much less yielding than those employed in wood ships, so that they resist more successfully strains of equal intensity.

The severest strains connected with the launching of ships are, however, those which occasionally result from partial launches. The well-known case of the early iron ship 'Prince of Wales' illustrates this statement; as, owing to an accident to the launching-gear, she was left for some time with her bow resting on the edge of a wharf and her stern supported by the water—in fact suspended by the extremities; but although so severely strained no breakage took place. Another case in point is found in the wood line-of-battle ship 'Cæsar,' which stopped on the launching-ways at Pembroke Yard in 1853, and remained for seventeen days with about 64 feet of the stern unsupported by ground-ways. The result was that the stern drooped about 2 feet in a length of 90 feet.

A similar but more recent case of stoppage in launching is that of the iron-clad frigate 'Northumberland,' which in March 1866 stopped with about 52 feet of the after part unsupported, and remained in that position for thirty days. The weight of the unsupported part may be roughly estimated at 440 tons, and the moment of this weight about the aftermost point of support at 11,700 foot-tons. At the corresponding station in the 'Minotaur,' when afloat in still water, the bending-moment is about 9200 foot-tons, and when supported on the crest of a wave of her own length, the bending-moment is about 12,000 foot-tons. Now the 'Northumberland,' although a sister ship to the 'Minotaur,' has had her disposition of armour altered to the central-battery-and-belt system, and by this means rather more than 100 tons weight of armour and backing have been removed on the 52 feet of length from the stern forwards. The effect of this may be fairly assumed to be a reduction of the still-water bending-moments at the station in question to about 7000 foot-tons, and of the bending-moments on the wave to 10,000 foot-tons; and hence it follows that the strains resulting from the stoppage in launching in her

* At page 92 of his work on 'Iron Shipbuilding.'

case were slightly severer than even the strains corresponding to the exceptional position afloat which we have considered. It is proper to add, however, that they were purely statical strains and acted only in one direction, whereas the strains of waves are, as we have seen, constantly changing in character and intensity, and are therefore very much more trying to the structure. The result of careful observations showed, however, that there was scarcely any change of form in this iron-built ship, the maximum amount of breakage being $\frac{7}{16}$ inch only in a length of 342 feet, and this becoming reduced to $\frac{5}{16}$ inch when the ship was floated.

The very various positions which ships occupy when they ground may all be supposed to lie between the position where the only support is found at the middle of the length, and that where there are supports only at the extremities. Mr. FAIRBAIRN has chosen these extreme positions as those by which the provision of longitudinal strength in a ship should be regulated; and while I cannot entirely agree with this choice, on account of the fact that such positions are never occupied except by a few ships, and by them only in consequence of accidents, I am prepared to admit that there are cases on record which show that such positions may be occupied. Ships have, for example, grounded on rocky bottoms and on causeways, and have been left by the tide with their ends unsupported; and others have grounded in such a manner as to be supported at the extremities only. Under these circumstances it may be well, therefore, to attempt an approximation to the limiting values of the strains incidental to the extreme positions of support ashore, and to compare them with the statical strains which have been calculated for ships at sea.

The most severe strains to which a ship aground can be subjected are those incidental to support at the bow and stern only. In this position the heavy-weighted amidship portion, of course, tends to make the ship sag; and as it is a very simple mechanical problem to determine the amounts of the upward pressures at the points of support, as well as the weights of the various portions of the ship between the points of support, it is possible to calculate the shearing-forces and the bending-moments at various stations by a method similar to that previously used for ships afloat. The graphical method of representing shearing-forces and bending-moments might also be applied to this case were it considered necessary; but this has not been done, on account of the fact that we are principally interested in determining an approximate value for the maximum bending-strain, and therefore care but little about the other values, which have no practical importance. Before giving quantitative examples, taken from actual ships, of the maximum sagging-strains that may occur in this position, it will be only proper to call attention also to the obviously great increase in the maximum shearing-forces which a ship then experiences. Roughly speaking, we may say that one half a ship's weight is taken at the bow, and the other half at the stern when the middle is unsupported; so that near the points of support the shearing-force is approximately one half the ship's weight. In preceding investigations it has been shown that for still water the shearing-force has an approximate maximum value of $\frac{1}{22}$ part of the total weight; and that for

support on a wave-crest the shearing-force probably does not exceed $\frac{1}{7}$ part of the total weight; hence the above-mentioned fact, as to the increase of shearing-strains in a ship ashore, becomes obvious.

The quantitative examples which have been chosen to illustrate the maximum strains incidental to suspension by the ends only are based upon the three typical ships previously considered, viz. the 'Minotaur,' the 'Bellerophon,' and the 'Victoria and Albert.' As the result of calculations made for the 'Minotaur,' it has been found that the centre of gravity of the ship, when fully laden, is 210 feet from the bow, and that when she rests upon the extremities the upward pressure at the bow equals 4925 tons, while that at the stern equals 4975 tons. These quantities obviously give the limiting values for the shearing-forces experienced by the ship. Next, as to the maximum bending-moment which will be experienced by the transverse section of the ship that contains her centre of gravity. Other calculations have been made which show that the centres of gravity of the two parts into which the ship is divided by the transverse section of maximum bending-moment are respectively 85 feet before and 81 feet abaft this section. Hence, starting from the fore end, we have—

Maximum bending-moment = moment of upward pressure at the bow minus the moment of the weight of the fore part of the ship
= 4925 tons × 210 ft. − 4830 tons × 85 ft.
= 579,300 foot-tons (in round numbers); or about
= displacement in tons × $\frac{1}{7}$ of the length in feet.

In preceding investigations it has been shown that the approximate maximum bending- (hogging-) moments for this ship are for still water 45,000 foot-tons, and for support on a wave-crest 140,300 foot-tons, while in a wave-hollow the maximum sagging-moment amounts to 74,800 foot-tons. These figures speak for themselves, and illustrate the large increase in the bending-moments caused by the exceptional supports. The increase in the maximum shearing-force, from 450 tons in still water, 1365 tons on the wave-crest, and 695 tons in a wave-hollow, to 4975 tons when the ship is ashore, is no less striking.

The 'Bellerophon' furnishes our second example. By means of calculations similar to those made for the 'Minotaur,' it has been found that the transverse section passing through the centre of gravity of the whole ship is about 144 feet from the bow and 156 feet from the stern, and that the centres of gravity of the parts of the ship before and abaft this section are distant from it 54 feet and 64 feet respectively. The weights of these parts are respectively 3825 tons and 3225 tons, and the upward pressures at the bow and stern being respectively 3666 tons and 3384 tons, we have

Maximum bending-moment = 3666 tons × 144 ft. − 3825 tons × 54 ft.
= 321,400 foot-tons (in round numbers); or about
= displacement in tons × $\frac{1}{7}$ of the length in feet.

It has been previously shown that the approximate maximum bending- (hogging-)

moments experienced by this vessel are 12,000 foot-tons for still water, and 43,600 foot-tons for a wave-crest, while the maximum sagging-moment in a wave-hollow is 48,800 foot-tons. The shearing-forces have also been found to have the following approximate maximum values:—In still water 210 tons, on a wave-crest 555 tons, and in a wave-hollow 640 tons, while aground it equals 3666 tons. The comparative conditions of strain of the ship under these different circumstances can be fairly determined by comparing these figures. It is also interesting to remark the different manner in which changes of the attendant circumstances affect the strains of the two classes of ironclads represented by the 'Minotaur' and the 'Bellerophon.' The reader can trace these for himself; I would simply call attention to the fact that when aground and supported at the extremities the maximum bending-moments bear very nearly the same proportions to the products of the lengths and displacements, notwithstanding the very different distribution of the weight in the two ships. This similarity is mainly due to the fact that the centres of gravity of the fore and after bodies in the two ships are very nearly the same part of the length of these bodies distant from the bow and stern. For example, in the 'Minotaur' the centre of gravity of the fore body is 116 feet distant from the bow, and the fore body is 201 feet long, these quantities being very nearly in the ratio of 3 to 5; while in the 'Bellerophon' the centre of gravity is 90 feet from the bow, and the length of the fore body is 144 feet, the ratio between these quantities being nearly the same as before.

Our third example is drawn from the 'Victoria and Albert,' in which ship the centre of gravity is about 157 feet from the bow and 143 feet from the stern. When supported at the extremities only, the bow sustains a pressure of about 1120 tons, and the stern of 1230 tons; and the centres of gravity of the two parts into which the ship is divided by the transverse section containing her centre of gravity are respectively 60 feet before, and 46 feet abaft this section. Hence we obtain by the same method as before:—

$$
\begin{aligned}
\text{Maximum bending-moment} &= 1120 \text{ tons} \times 157 \text{ ft.} - 1020 \text{ tons} \times 60 \text{ ft.} \\
&= 114{,}700 \text{ foot-tons (in round numbers); or about} \\
&= \text{displacement in tons} \times \tfrac{1}{6} \text{ of the length in feet.}
\end{aligned}
$$

The approximate maximum bending-moments previously determined for this ship are, for still water 5080 foot-tons, on a wave-crest 16,400 foot-tons, in a wave-hollow 31,000 foot-tons; the approximate maximum shearing-forces under the same circumstances are respectively 140 tons, 220 tons, and 395 tons. When ashore the maximum shearing-force is, of course, 1230 tons. By comparing this ship with the 'Bellerophon' and 'Minotaur,' it will be seen that the maximum sagging-moment bears a larger ratio to the product of the displacement by the length than it does in the two ironclads. This is due to the fact, already illustrated, of the greater concentration of weights at the centre of the paddle-wheel steamship.

These three examples afford us the means of approximating to the limiting values of the shearing-forces and bending-moments of ships supported at the extremities only,

and from them we obtain the following limits. For shearing-forces the maximum lies between one half and three fifths of the displacement, and for bending-moments between one seventh and one sixth of the product of the displacement by the length. These are, of course, to be regarded simply as *limiting* values; in practice they can scarcely occur, because no ship is likely to rest at the extremities only without having a moderate base of support. The amount of the strains actually experienced would depend, obviously, upon the length of the base of support and its greater or less nearness to the bow and stern.

When a ship ashore rests upon a middle support and has her ends unsupported, which in the other extreme position she can occupy, she is, as I have said, less severely strained than when resting on the ends only, the reason for this fact being that the heavy weights carried in the amidship portion are comparatively close to the point of support in one case, whereas in the other they are much more distant. The greatest difference in strains should consequently be looked for in ships having very concentrated weights amidships; and it is interesting to remark that while such a concentration has been shown to be beneficial in reducing most of the principal strains experienced by ships afloat, it is the cause of the increase in sagging-strains in ships ashore. A ship having her weights uniformly distributed throughout the length would be subject, when supported at the middle only, to hogging-moments equalling in amount the sagging-moments incidental to support at the extremities. Actual ships, however, have not any thing like a uniform distribution of weights, and the greatest weights are usually found near the middle. In spite of the increase in the severest exceptional strains thus caused, however, it cannot be doubted that the ordinary distribution of the weight is beyond comparison better than uniform distribution would be; for its beneficial effect, in reducing strains in ships afloat, is continually called into play, while the other effect is seldom, if ever, produced.

In order to show the relative magnitude of the classes of exceptional strains in ships ashore, I will again take the three typical ships and give a few quantitative results. The only explanation required of the method I shall follow is that when a ship is supported on a single point, vertically below her centre of gravity, the maximum shearing-force will equal the weight of either the fore or the after body, and the maximum bending-moment will equal the product of the weight of one of these bodies by the distance of its centre of gravity from the point of support. The previous investigations therefore supply all the *data* required for the further calculations.

In the 'Minotaur' the after body is the heavier, and its weight, 5070 tons, constitutes the maximum shearing-force. The product of this weight by the distance of the centre of gravity of the after body from the transverse plane passing through the centre of gravity of the whole ship, 81 feet, gives us:

$$\begin{aligned}\text{Maximum hogging-moment} &= 5070 \text{ tons} \times 81 \text{ ft.}\\ &= 410{,}600 \text{ foot-tons (in round numbers); or about}\\ &= \text{displacement in tons} \times \tfrac{1}{10} \text{ of the length in feet.}\end{aligned}$$

The maximum shearing-force is a little greater in this case than in the case of support at the ends; and the maximum hogging-moment is considerably over two thirds the maximum sagging-moment. It may also be interesting to compare these exceptional hogging-strains with the strains incidental to support on a wave-crest. These are:— hogging-moment 140,300 foot-tons, shearing-force 1365 tons.

The 'Bellerophon' is divided into two parts by the transverse plane containing her centre of gravity, such that the foremost or heavier part weighs 3825 tons, and has its centre of gravity 54 feet before the plane of division. The maximum shearing-force for support at the middle is therefore 3825 tons, a little greater than in the previous position, and we have

$$\begin{aligned}\text{Maximum hogging-moment} &= 3825 \text{ tons} \times 54 \text{ ft.}\\ &= 206{,}500 \text{ foot-tons (in round numbers); or about}\\ &= \text{displacement} \times \tfrac{1}{10} \text{ of the length.}\end{aligned}$$

This moment is considerably under two thirds of the maximum sagging-moment previously found for this ship; and we have in this fact an illustration of the statement made above respecting the difference existing between the exceptional hogging- and sagging-strains in ships with concentrated weights amidships as compared with other ships. In the 'Victoria and Albert' we should expect to find a more striking illustration, and we really do so. Her after body is heavier than the fore body, and its weight, 1330 tons, constitutes the maximum shearing-force, while the centre of gravity is 46 feet abaft the centre of gravity of the whole ship. Hence we have

$$\begin{aligned}\text{Maximum hogging-moment} &= 1330 \text{ tons} \times 46 \text{ ft.}\\ &= 61{,}200 \text{ foot-tons (in round numbers); or about}\\ &= \text{displacement} \times \tfrac{1}{12} \text{ of the length.}\end{aligned}$$

The maximum sagging-moment in this ship is therefore not very much less than double the maximum hogging-moment experienced by the ship ashore; when compared with the 'Minotaur,' the case of the 'Victoria and Albert' appears still more striking.

These three examples lead to the conclusion that the limiting maximum values of the strains experienced by ships supported only at the middle may be fixed as follows:— for shearing-forces between one half and two thirds of the displacement; for bending-moments between one ninth and one eleventh of the product of the displacement by the length. As in the other extreme position of support, these limits can never be approached closely in actual ships, on account of the more or less extended base of support which is pretty certain to be found under ships ashore.

In order to facilitate a comparison between the various strains experienced by the three typical ships under different circumstances, the following tabular statement of the results arrived at in the preceding pages is given.

TABLE of Maximum Bending-moments and Shearing-forces determined for the 'Minotaur,' 'Bellerophon,' and 'Victoria and Albert.'

	Minotaur.		Bellerophon.		Victoria and Albert.	
	$\frac{\text{Shearing-force.}}{\text{Displacement.}}$	$\frac{\text{Bending-moment.}}{\text{Displacement} \times \text{length.}}$	$\frac{\text{Shearing-force.}}{\text{Displacement.}}$	$\frac{\text{Bending-moment.}}{\text{Displacement} \times \text{length.}}$	$\frac{\text{Shearing-force.}}{\text{Displacement.}}$	$\frac{\text{Bending-moment.}}{\text{Displacement} \times \text{length.}}$
In still water . .	$\frac{1}{22}$	$\frac{1}{88}$	$\frac{1}{33}$	$\frac{1}{176}$	$\frac{1}{16}$	$\frac{1}{139}$
On a wave-crest .	$\frac{1}{7}$	$\frac{1}{28}$	$\frac{1}{13}$	$\frac{2}{97}$	$\frac{1}{11}$	$\frac{1}{43}$
In a wave-hollow .	$\frac{1}{14}$	$\frac{1}{53}$	$\frac{1}{11}$	$\frac{1}{43}$	$\frac{1}{6}$	$\frac{1}{23}$
Supported at the extremities . .	$\frac{1}{2}$	$\frac{1}{7}$	$\frac{1}{2}$	$\frac{1}{7}$	$\frac{1}{2}$	$\frac{1}{6}$
Supported at the middle . . .	$\frac{1}{2}$	$\frac{1}{10}$	$\frac{1}{2}$	$\frac{1}{10}$	$\frac{2}{3}$	$\frac{1}{12}$

The values of bending- and shearing-strains here given are not put forward as strictly accurate, although from the detailed calculations on which they rest they may be regarded as very close approximations to the strains experienced by various classes of ships under the assumed conditions. These conditions, as we have seen, do not include all the circumstances in which ships may be and are placed, but for still water the approximation made is doubtless very close; and for the exceptional positions of support ashore the limiting values of the strains are also very close to the truth, although in practice these limits can never be reached. For ships at sea, in the extreme positions of support assumed, the values given for the strains, of course, represent the statical aspect of the question; the attempt to put into figures the straining effects of pitching and ascending has not been made.

The voluminous calculations upon which the statements and tables comprised in this paper are based, together with the diagrams, were made with great care, under my direction, by Mr. W. H. WHITE, Jun., and Mr. JOHN, Fellows of the Royal School of Naval Architecture, and Draughtsmen at the Admiralty. Mr. WHITE also assisted me greatly in the detailed preparation of the Paper.

XVII. *On the Measurement of the Chemical Intensity of Total Daylight made at Catania during the Total Eclipse of Dec. 22nd,* 1870. *By* Henry E. Roscoe, *F.R.S., Professor of Chemistry, Owens College, Manchester, and* T. E. Thorpe, *F.R.S.E., Professor of Chemistry, Andersonian University, Glasgow.*

Received June 15,—Read June 15, 1871.

The following communication contains the results of a series of measurements of photochemical action made at Catania in Sicily, on Dec. 22nd, 1870, during the total solar eclipse of that date, with the primary object of determining experimentally the relation existing between this action and the changes of area in the exposed portion of the sun's disk. The attempt to establish this relation has already been made by one of us from the results of observations carried out by Captain John Herschel, R.E., F.R.S., at Jamkhandi, in India, during the total eclipse of August 18th, 1868*. Unfortunately the weather at Jamkhandi at the time of the eclipse was very unfavourable for observation; the estimated amount of cloud during the time of the eclipse amounted to about 7, the sun occasionally being even completely obscured.

In addition to the errors arising from the unsettled state of the weather, a further element of uncertainty was unavoidably introduced in the subsequent calculation in allowing for the variation in chemical intensity caused by the alteration in the sun's altitude during the progress of the eclipse. It has been shown that the relation between the sun's altitude and the chemical intensity at any given place is represented by the equation

$$CI_a = CI_0 + \text{const.} \times a,$$

where CI_a signifies the chemical intensity at any altitude (a) in circular measure, CI_0 the chemical intensity at 0°, and const. a a number derived from the observations†. Since no special series of observations were made at Jamkhandi (lat. 16° 30′ N.) in order to determine the constant, its approximate value could only be obtained from observations made at Pará, in Brazil (lat. 1° 25′ S.), during a different season of the year.

It appeared from these observations that, as might be expected, the rate of diminution of the chemical intensity of total daylight during the first portion of the eclipse up to the point at which the disk is half obscured is greater than corresponds to the area of darkened solar disk, whilst from this point up to totality the rate of diminution of chemical action is much less than that of the exposed portion of the disk.

The occasion of the recent solar eclipse presented a favourable opportunity for rede-

* Roscoe, Mem. Lit. and Phil. Soc. of Manchester, 1868–69, vol. iv. [3] p. 202.

† Philosophical Transactions, 1867, p. 555; 1870, p. 315.

termining this point, and accordingly we undertook a fresh series of observations at Catania in connexion with the Government Eclipse Expedition. The method of measurement employed in this, as in the above-mentioned series, was that already described by one of us*. It consists in exactly estimating the tint which uniformly sensitive paper coated with a thin film of silver chloride acquires on exposure to the action of daylight for a given time. The observations were made in the Garden of the Benedictine Monastery of San Nicola, at Catania. The position of the observatory was

$$\text{Lat. } 37^\circ\ 30'\ 12''\ \text{N.}$$
$$\text{Long. } 1^\text{h}\ 0^\text{m}\ 18^\text{s}\ \text{E.},$$

as determined by Mr. SCHOTT, of the United States' Coast Survey. By barometric measurements it was found to be about 170 feet above the sea-level, a result confirmed by the independent observations of Mr. SCHOTT. In order to secure as clear an horizon as possible, the insolating instruments were mounted on the roof of a portico in the garden overlooking the bay to the south and west. The place was well adapted to the work, the only intervening object of consideration being the Monastery itself, distant about 100 paces to the east, the dome of which subtended an angle of about 10° with the plane of the paper. To the north lay Etna, about twenty miles distant, the summit of which subtended an angle of rather more than 4°. The influence exerted by these objects in cutting off the diffused solar radiation is far too inconsiderable to affect the results. The sensitive paper was exposed in the plane of the horizon, the insolating instruments being placed upon a tripod stand about 4 feet 6 inches in height. This stand was carefully levelled at the commencement of each series of observations, and the insolators were firmly clamped down after being adjusted due east and west, in which position they remained throughout the entire course of the observations. As we contemplated making consecutive exposures for direct and diffused chemical intensity every ten minutes during the progress of the eclipse from first to last contact, it was necessary to employ greater lengths of sensitive chloride-of-silver paper than we had hitherto used. The brass insolators were therefore proportionately increased in length, so as to allow of at least twenty-five observations to be made on a single strip. We assured ourselves, in the first place, that the paper could be kept for hours in the insolator exposed to the bright sunshine without darkening in the slightest degree, so long, of course, as the moveable brass slide covered the hole. This was doubtless in part due to the good radiating surface of the polished brass insolator, whereby the paper remained unheated even under the influence of direct sunshine; for we have observed that the blackening which silver-chloride paper containing excess of silver nitrate suffers in time, however carefully it may be protected from the light, is promoted by increased temperature; thus a strip of sensitive paper kept in the dark more rapidly undergoes alteration in the tropics than in England.

Two insolating instruments were employed placed side by side; by means of one the chemical action of total daylight was first observed in the ordinary manner, and imme-

* ROSCOE, Bakerian Lecture, Philosophical Transactions, 1865, Part II. p. 605.

diately afterwards the chemical action of the diffused daylight was determined by the other by projecting on the exposed portion of the sensitive paper the shadow of a small blackened ball, so placed that its apparent diameter seen from the surface of the paper was slightly larger than the sun's disk. As the instruments were clamped in a constant position, it was necessary to vary from time to time the position of the blackened ball with reference to the plane of the paper in order to bring its shadow upon the exposed portion. We have, however, shown in a former communication that the height of the blackened ball from the sensitive paper may safely vary between 140 and 200 millimetres without producing any appreciable difference, and in the present experiments its height was maintained between these limits.

The experiments were commenced on Dec. 19th, with the object of determining the value of the constant before referred to. As an example of these observations, 65 in number, we give the first one of the series:—

Observed time $11^h\ 15^m$ A.M.

Duration of exposure for total daylight 5^s.

Duration of exposure for diffused daylight 7^s.

	Readings on calibrated strip.	Mean.
I.	100, 97, 100, 97, 104, 96, 97, 96	=98·3
II.	93, 93, 97, 94, 95, 92, 93, 94, 95	=94·0

The duration of exposures divided into the values in the Table accompanying the graduated fixed strip give:—

Direct Chemical Intensity	0·037
Diffused Chemical Intensity	0·101
Total Chemical Intensity	0·138

The following Tables contain the results of this preliminary series of observations:—

Table I. (*a*).

December 19th, 1870.				
Hour.	Diffused.	Direct.	Total.	Remarks.
h m				
11 15	0·101	0·037	0·138	Sky throughout the day perfectly cloudless and of a pure blue. Gentle breeze from E. Barometer 29·97 inches.
11 45	0·112	0·032	0·144	
12 0	0·113	0·037	0.150	
12 15	0·120	0·026	0·146	
12 30	0·108	0·042	0·150	
12 47	0·106	0·039	0·145	
1 5	0·115	0·025	0·140	
1 15	0·096	0·036	0·132	
1 57	0·091	0·023	0·114	
2 6	0·082	0·028	0·110	
2 22	0·075	0·020	0·095	
2 36	0·062	0·019	0·081	
2 57	0·060	0·022	0·082	
3 6	0·060	0·017	0·077	
3 21	0·050	0·012	0·062	
3 36	0·049	0·011	0·060	

TABLE I. (*b*).

December 20th, 1870				
Hour	Diffused	Direct	Total	Remarks
h m				
9 22	0·070	0·034	0·104	In the early part of the morning the sky was slightly cloudy, particularly on the horizon to the S. and S.W. Strong W. breeze. Bar. at 9 A.M. 29·87 inches. Thin haze throughout the day, but the blue colour of the sky was not much diminished. Bar. at noon 29·81 inches, at 5 30 29·74 inches.
9 52	0·085	0·052	0·137	
10 23	0·096	0·078	0·174	
10 52	0·097	0·077	0·174	
11 37	0·104	0·074	0·178	
11 52	.....	.....	0·172	
12 7	0·106	0·066	0·172	
12 52	0·106	0·078	0·184	
1 23	0·096	0·058	0·154	
1 55	0·086	0·038	0·124	
2 24	0·075	0·028	0·103	
2 54	0·073	0·016	0·089	
3 24	0·057	0·008	0·065	
3 55	0·042	0·005	0·047	
4 34	....	......	0·009	Sun just above horizon, low bank of clouds above S. horizon. Less haze.

TABLE I. (*c*).

December 21st, 1870					
Hour	Diffused	Direct	Total	Cloud	Remarks
h m					
8 24	0·036	0·011	0·047	......	This day was by no means so favourable for observation as the two preceding days, the sky was cloudy throughout the day, particularly to the S. and S.W. W. breeze. Bar. at 8.54, 29·69 inches. do. 11.54, 29·61 „ do. 4.30, 29·50 „
8 54	0·050	0·018	0·068	......	
9 24	0 067	0·020	0·087	.....	
9 54	0·075	0·032	0·107	3	
10 24	0·094	0·044	0·138	3	
10 54	0·091	0·028	0·119	6	
11 24	0·115	0·053	0·168	8	
11 54	0·124	0·056	0·175 ?	8	
12 24	Sun obscured.			9	
12 44	0·119	0·038	0·157	9	
1 24	0·109	0·043	0 152	7	
1 54	0·086	0·051	0·137	5	
2 24	0·083	0·036	0·119	4	
2 54	Sun obscured.		0·072	4–5	
3 24	do.		0·058	8	
3 54	do.		0·042	5	

FIG 1

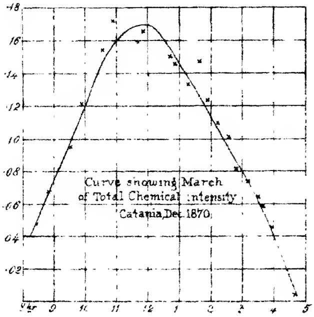


FIG 2

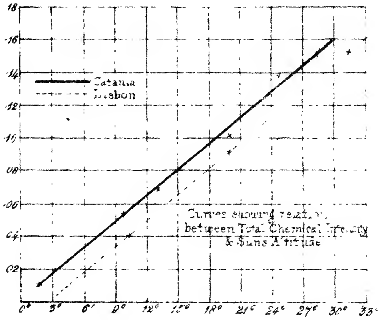


FIG 3

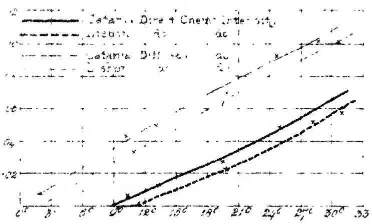


FIG 5

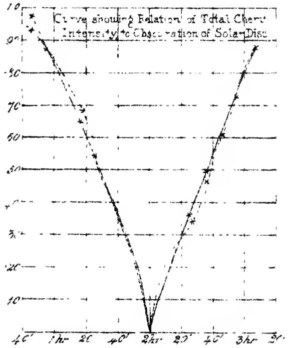


FIG 4

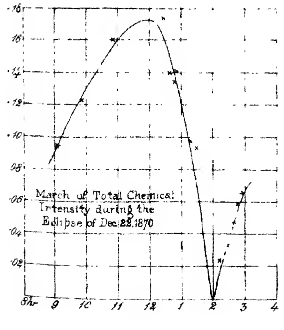


FIG 4a

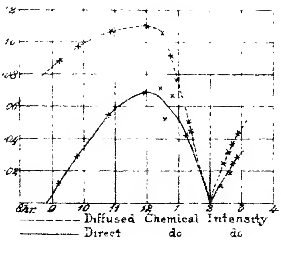
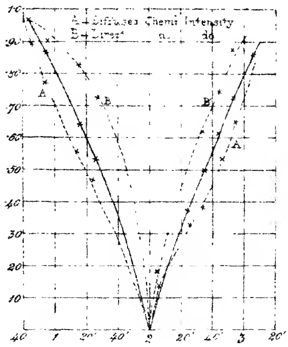


FIG 6



On grouping together the observations taken at about the same hours on the three days, the following mean values are obtained:—

TABLE II.

Hour	Altitude	No of experiments	Chemical intensity		
			Diffused	Direct	Total
h m	° ′ ″				
8 24	11 4 13	1	·036	·011	·047
8 54	15 9 39	1	·050	·018	·068
9 23	19 9 35	2	·068	·027	·095
9 53	22 22 15	2	·080	·042	·122
10 24	25 8 21	2	·095	·061	·156
10 52	26 28 12	1	·097	·077	·174
11 30	28 43 46	4	·108	·050	·158
11 57	29 5 40	2	·118	·044	·162
12 11	29 0 12	2	·113	·046	·159
12 30	28 37 4	1	·108	·042	·150
12 44	28 7 31	2	·110	·039	·149
1 10	26 46 42	2	·105	·031	·136
1 24	25 48 54	2	·103	·050	·153
1 53	23 21 20	3	·088	·037	·125
2 6	22 3 46	1	·082	·028	·110
2 24	20 5 30	3	·078	·028	·106
2 44	17 40 44	4	·062	·018	·080
3 6	14 46 51	1	·060	·017	·077
3 22	12 32 0	3	·053	·010	·063
3 36	10 28 46	1	·049	·011	·060
3 55	7 34 41	2	·042	·005	·047
4 34	1 30 28	1	·009	·000	·009

On again grouping the observations made at hours equidistant from noon, we obtain the mean numbers contained in the following Table:—

TABLE III.

Mean altitude	No of observations	Diffused	Direct	Total
° 1 30 28	1	0·009	0·000	0·009
9 28 10	7	0·044	0·008	0·052
13 9 57	7	0·050	0·014	0·064
19 57 49	12	0·072	0·028	0·100
24 46 12	7	0·095	0·049	0·144
28 24 10	14	0·108	0·047	0·155

These results are graphically represented in Plate XXII. figs. 1, 2 & 3.

The observations on the day of the eclipse (the 22nd) were commenced shortly after 9 o'clock, and up to the time of first contact were made regularly at intervals of about an hour. It will be seen from the remarks contained in the last columns of Table I. that the fine weather we had experienced up to the 19th was gradually drawing to a close; the barometer gradually fell from 29·97 inches at noon on the 19th to 29·50 inches at 4.30 P.M. on the 21st; the wind, too, had veered round, and clouds were slowly

accumulating. During the night of the 21st much rain fell; but shortly after sunrise on the 22nd the clouds in great part disappeared, every trace of haze was dissipated, and the sky was of the purest blue: it will be seen that the measurements of photochemical action made up to the time of first contact are almost absolutely coincident with the mean numbers derived from the observations of the three preceding days. As the eclipse progressed, and the temperature of the air fell, clouds were again formed, and from $1^h\ 40^m$ up to the time of totality it was impossible to make any observations, as the sun was never unclouded for more than a few seconds at a time; indeed just before totality a slight shower of rain fell. As the illuminated portion of the solar disk gradually increased after totality, the clouds rapidly disappeared, the estimated amount falling from 9 (overcast=10) to 3 in about fifteen minutes. The observations were then regularly continued to within a few minutes of last contact.

Although the disk and by far the greater portion of the heavens were completely obscured by clouds during the period of totality, rendering any determination of the photochemical action perfectly valueless in view of our special object, it was yet thought worth while to attempt to estimate the chemical intensity of the feebly diffused light at this time. That it has a certain degree of actinism is of course evident from the fact that photographs can be taken during totality. Immediately, therefore, after the supposed commencement of totality the slit was opened, and the sensitive paper exposed for ninety-five seconds; as the end of totality was very plainly indicated about two or three seconds after the slit had been again closed, it was clear that the paper had been exposed only during the period of totality. Not the slightest action, however, could be detected on the paper; on comparing them in the dark room it appeared considerably lighter than the extreme light end of the graduated strip. The calibration of the fixed strip employed in our measurements was not carried out to the extreme limits of the tinted portion, but on the supposition that it decreased uniformly in shade from end to end; or, in other words, assuming that the curve representing the decrement in intensity of tint maintained its symmetrical character throughout, which is a perfectly legitimate assumption to make when we bear in mind the manner in which these tinted strips are made, it follows that the amount of chemical action during the totality could not exceed 0·003 of the unit which we adopt, and in all probability it is much less.

The results of the day's observations are contained in Table IV.

TABLE IV.

Hour.	Altitude.	Sky.	Sun.	Total.	Cloud.	Remarks.
h m	° ′					
9 3	16 31	0·085	0·011	0·096	2	Bar. 29·27 inches.
9 54	22 33	0·096	0·027	0·123	3	
10 54	26 44	0·105	0·055	0·160	2	Pure blue sky.
12 0	29 09	0·107	0·066	0·173	3–4	
12 24	28 45	0·105	0·070	0·175	3	Large cumuli over Etna and on horizon.
12 34	28 29	0·090	0·050	0·140	—	
12 44	28 08	0·090	0·044	0·134	4	
12 54	27 34	0·077	0·066	0·143	4	Sun close to clouds.
1 4	27 00	......	......	[0·084]	6	Sun overcast.
1 16	26 17	0·053	0·045	0·098	6	Sun unclouded.
1 24	25 49	0·043	0·051	0·094	7	Yellow light strongly marked.
1 40	24 25	......	......	......	......	No exposure: sun overcast.
The disk was overcast during the time of totality by clouds; the paper, however, was exposed for the entire time of totality, but not the slightest action was evident, Alt. 22° 41′–22° 31′.						
2 9	21 43	......	......	0·024		
2 25	20 03	0·023	0·010	0·033	3	Sun unclouded.
2 34	18 56	0·028	0·019	0·047	2–3	”
2 44	17 42	0·035	0·022	0·057	2–3	”
2 54	16 33	0·041	0·023	0·064	2–3	”
3 4	15 14	......	......	0·062	3	”
3 20	......	......	......	......	......	Sun clouded over.

These results are graphically represented in figs. 4 and 4 *a*.

Table V. shows the relation of the total chemical intensity to the area of the solar disk obscured. The following approximate method, which we have employed for determining the relative area of the sun eclipsed at the times of observation, is sufficiently accurate for our purpose. Three large disks were cut from a stout and uniformly thick sheet of paper, two to represent the solar, the third the lunar disk, their radii being in the ratio of the apparent semidiameters of the sun and moon on the day of the eclipse. Each of the sun-cards was accurately weighed on a chemical balance; and on one the ratios of the intervals elapsing between first contact and the various times of observation before totality were pointed off along a diameter, and the edge of the lunar disk advanced successively to these points, and the segments cut off and weighed. The total weight of the solar disk into the weights of the various segments gives approximately the area of the sun covered at the corresponding time of observation. In a similar manner the approximate areas after totality were obtained. This method of procedure, which neglects the influence of the moon's motion and of the earth's motion of rotation, is of course crude; but as the commencement of the eclipse occurred near noon, the error thus introduced may certainly be disregarded for areas covered up to $1^h\ 24^m$, the time of the last observation taken before totality. To the areas thus obtained for the times of observation after totality we have applied the necessary corrections, for which we are indebted to Mr. SEABROKE.

Column I. gives the apparent solar times of observation; column II. the corresponding altitude of the sun. Column III. shows the corresponding total chemical inten-

sity of the uneclipsed sun calculated from the mean curve in fig. 2. Column IV. shows the relation of this intensity to the sun's altitude, the chemical intensity immediately before first contact being taken as unity. Column V. gives the immediate results of the photochemical observations during the eclipse. Column VI. shows these results calculated from the mean curve in fig. 4; column VII. the same corrected for variation in the sun's altitude; column VIII. the same referred to the total chemical intensity immediately before first contact as unity. Lastly, Column IX. shows the magnitude of the eclipse at the times of observation, the unobscured sun being regarded as unity.

TABLE V.

I.	II.	III.	IV.	V.	VI.	VII.	VIII.	IX.
h m	° ′							
12 44	28 08	0·153	0·998	0·134	0·140	0·140	0·915	0·961
12 54	27 34	0·151	0·987	0·143	0·132	0·134	0·876	0·880
1 16	26 17	0·143	0·935	0·098	0·098	0·105	0·686	0·637
1 24	25 49	0·140	0·915	0·094	0·078	0·085	0·555	0·534
2 2	22 35	0·121	0·791	0·000	0·000	0·000	0·000	0·000
2 9	21 43	0·115	0·752	0·024	0·019	0·025	0·165	0·127
2 25	20 03	0·107	0·699	0·033	0·033	0·047	0·307	0·338
2 34	18 56	0·101	0·660	0·047	0·047	0·071	0·464	0·498
2 44	17 42	0·095	0·621	0·057	0·057	0·092	0·601	0·602
2 54	16 23	0·088	0·575	0·064	0·064	0·111	0·725	0·736
3 4	15 14	0·081	0·529	0·062	0·071	0·134	0·876	0·861

The relation of columns VIII. and IX. is graphically represented in fig. 5, the unbroken line representing the magnitude of the eclipse, the abscissæ represent the time, and the ordinates the corresponding chemical intensity and area of exposed disk.

From these observations we deduce the law that the diminution in the total chemical intensity of the sun's light during an eclipse is directly proportional to the magnitude of the obscuration.

We now proceed to investigate the influence respectively exerted by the (1) diffused and (2) direct radiation.

Table VI. contains the results of the determinations of chemical intensity of diffused light.

Column I. gives the apparent solar times of observations; column II. the corresponding solar altitude; column III. the chemical intensity of diffused light calculated from the mean curve in fig. 3 obtained from the observations of Dec. 19, 20, 21. Column IV. gives the relation of this intensity to the sun's altitude, the chemical intensity of the diffused light immediately before first contact being taken as unity. Column V. gives the immediate results of the determinations of chemical intensity of the diffused light during the eclipse. Column VI. shows these results calculated from the mean curve in fig. 5; column VII. the same corrected for variation in altitude; column VIII. the same referred to the chemical intensity of diffused light immediately before first contact as unity. Column IX. shows the magnitude of the eclipse at the time of observation, the unobscured sun being regarded as unity.

TABLE VI.

I.	II.	III.	IV.	V.	VI.	VII.	VIII.	IX.
h m	° ′							
12 44	28 08	0·100	0·980	0·090	0·090	0·092	0·900	0·961
12 54	27 34	0·098	0·961	0·077	0·077	0·080	0·786	0·880
1 16	26 17	0·095	0·931	0·053	0·053	0·057	0·558	0·637
1 24	25 49	0·093	0·912	0·043	0·043	0·047	0·462	0·534
2 2	22 35	0·083	0·814	0·000	0·000	0·000	0·000	0·000
2 9	21 43	0·080	0·784	......	0·012	0·015	0·151	0·127
2 25	20 03	0·074	0·725	0·023	0·023	0·032	0·311	0·338
2 34	18 56	0·072	0·706	0·028	0·028	0·040	0·389	0·498
2 44	17 42	0·067	0·657	0·035	0·035	0·053	0·522	0·602
2 54	16 33	0·064	0·626	0·041	0·041	0·065	0·642	0·736

TABLE VI. (*a*).

I.	II.	III.	IV.	V.	VI.	VII.	VIII.	IX.
h m	° ′							
12 44	28 08	0·053	0·982	0·044	0·056	0·057	0·966	0·961
12 54	27 34	0·052	0·963	0·066	0·051	0·053	0·898	0·880
1 16	26 17	0·048	0·889	0·045	0·045	0·051	0·864	0·637
1 24	25 49	0·047	0·870	0·051	0·037	0·043	0·730	0·534
2 2	22 35	0·038	0·704	0·000	0·000	0·000	0·000	0·000
2 9	21 43	0·035	0·648	......	0·007	0·011	0·186	0·127
2 25	20 03	0·033	0·611	0·010	0·012	0·020	0·339	0·338
2 34	18 56	0·029	0·537	0·019	0·020	0·037	0·627	0·498
2 44	17 42	0·027	0·500	0·022	0·022	0·044	0·745	0·602
2 54	16 33	0·024	0·444	0·023	0·023	0·052	0·881	0·736

The relation of columns VIII. and IX. is graphically represented in fig. 6, the abscissæ and ordinates having the same signification as in fig. 5.

On comparing the curve A, representing the chemical intensity of diffused light, with the curve of the solar obscuration, it is seen that the rate of diminution in the chemical action exerted by the diffused light is up to a certain point greater than corresponds to the portion of sun eclipsed, whilst from this point up to totality the rate of diminution becomes less than that corresponding to the progress of the eclipse. As a consequence of this fact, and of the law that the diminution in the total photochemical action exerted during an eclipse is proportional to the magnitude of the obscuration, it follows that the rate of diminution and increase of intensity of the chemically active rays in direct sunlight is much slower and quicker than corresponds to the changes of area in the exposed portion of the solar disk. This is graphically shown in curve B, fig. 6. The same rapid diminution of the chemical action of the diffused daylight during the former part of the eclipse was observed at Jamkhandi. It is doubtless due to the dark body of the moon cutting off the light from the highly luminous portion of sky lying on one side of the sun's disk.

That peculiar change of colour which it has long been noticed terrestrial objects assume during an eclipse was very strongly marked on the present occasion. Outside the garden of the Monastery, and just below our place of observation, was a profusion of a variety of *Opuntia*, on the flat broad leaves of which the change in colour was admirably depicted. When the obscuration of the solar disk amounted to about one half

it was observed that they acquired a bright olive-green tinge, producing an effect as if they had been seen through yellow glass. This effect, it was remarked at the time, reached its maximum about 1^{h} 30^{m}, at which time it will be seen that the curves representing the diminution of the direct and diffused chemical intensity are most widely separated from the mean curve representing the total chemical action.

On a former occasion we communicated to the Royal Society the results of a series of Observations on the Chemical Intensity of Daylight, made at Moita, near Lisbon, at the level of the sea, during August 1867, from which we deduced the relation between the sun's Altitude and the Photochemical Action of Total Daylight in a Cloudless Sky*. As Catania (lat. 37° 30′) and Moita (lat. 38° 40′) are nearly in the same latitude, it becomes interesting to compare the results obtained in December at the former place with those obtained for similar altitudes in August at the latter. The results of this comparison are seen in the following Table:—

TABLE VII.

Mean altitude.		Number of observations.		Chemical intensity.					
				Sun.		Sky.		Total.	
Lisbon.	Catania	L.	C.	L.	C.	L.	C.	L.	C.
° ′	° ′								
	1 30	......	1	......	0·000	......	0·009	......	0·009
	9 28	......	7	......	0·008	......	0·044	......	0·052
9 51	......	15	......	0·000	......	0·038	......	0·038	
	13 10	......	7	......	0·014	......	0·050	......	0·064
19 41	......	18	......	0·023	......	0·062	......	0·085	
	19 58	......	12	......	0·028	......	0·072	......	0·100
	24 46	......	7	......	0·049	......	0·095	......	0·144
	28 29	......	14	......	0·047	......	0·108	......	0·155
31 14	......	22	......	0·052	......	0·100	......	0·152	

Figs. 2 & 3 give a graphical representation of these relations of chemical intensity as ordinates to the sun's altitude as abscissæ. The unbroken curve in fig. 2 shows the Catania observations, the dotted curve those made at Lisbon. In fig. 3 the observations of direct chemical intensity are represented by the broad lines, those made at Catania being distinguished by the unbroken curve. In all cases the positions of the experimentally determined points are given to show how closely they lie to the curves. In both cases it is evident that the relation between the solar altitude and the total chemical intensity is represented by a straight line, although the Catania observations slightly exceed by a constant difference those made at Moita, in conformity with the slight difference in latitude, and with the fact that the former determinations were made at a greater elevation above the sea-level.

The Catania observations further confirm the fact which we then announced, that for altitudes below 50° the amount of chemical action effected by diffused daylight on a surface placed in the plane of the horizon is greater than that exerted by direct radiation, and also that at low altitudes (9° or 10°) direct sunlight is almost completely robbed of its chemically active rays.

* Philosophical Transactions, 1870, p. 309.

XVIII. *On the Organization of the Fossil Plants of the Coal-measures.*—Part I. *Calamites.* *By* W. C. WILLIAMSON, *F.R.S., Professor of Natural History in Owens College, Manchester.*

Received November 11, 1870,—Read January 26, 1871.

A few preliminary words may be necessary to prevent misunderstanding respecting the claims and objects of the following memoir. When I entered upon the investigation of which it records the results, I found, in the writings of various British and foreign authors, a copious Calamitean literature; but the widest discrepancies prevailed amongst them both as to facts and to inductions. I therefore determined to pursue the study of this group of fossils as if de novo, *to record the facts which I observed, and to draw from those facts alone such inferences as seemed legitimate, both facts and inferences being in a certain sense, and so far as was possible under the circumstances, new and original. But it necessarily follows that some of these facts and inferences are not absolutely new, though many of them, I think, will be found to be additions to our knowledge of the subject; whilst others, though not new, have presented themselves to me in a light different to that in which they have been regarded by my able predecessors in the study. Such being the object of the memoir, I have not deemed it desirable to include in it a record of all the observations made by preceding writers. As a rule I have only referred to them when the discussion of some moot point rendered such a reference necessary. The fundamental aim of the memoir is to demonstrate the unity of type existing amongst the British Calamites. Brongniart, Dawson, and other writers believe that there exist amongst these plants two types of structure, the one Cryptogamic and Equisetacean, the other Exogenous and Gymnospermous; on the other hand, Schimper and Carruthers regard the whole as Equisetaceous, affording an example of the diversity of opinion on fundamental points to which I have already referred. Of course, before arriving at their conclusions, Brongniart, and those who adopt his views, had fully apprehended the exogenous structure of the woody zone of the Calamite, which is further illustrated in this memoir. The separation of each internode into vertical radiating plates of vascular and cellular tissues, arranged alternately, was familiar to Brongniart, Unger, and other early observers. Cotta regarded the cellular tracts (my primary medullary rays) as medullary rays; but this interpretation was rejected by Unger, and the same divergence of view on this point has recurred amongst subsequent writers. Unger also noticed what I have designated secondary medullary rays, but at a much more recent date Mr. Carruthers disputed their existence. In their 'Fossil Flora of Great Britain,' Lindley and Hutton gave very correct illustrations of the position of the roots of Calamites relatively to the stem; and yet for years afterwards some of their figures reappeared in geological text-books in an inverted position, the roots doing duty as leaves; so far was even this elementary point from being settled. The true nature of the common sandstone form of Calamites, viz. that they are inorganic casts of the interior of the woody cylinder from which the pith has been removed, has been alike recognized by Germar, Corda, and Dawes; but they referred the disappearance of the cellular tissues of the pith to inorganic decay which took place subsequently to the death of the plant. It appears to me that the condition in which we find these cellular tissues affords no countenance to this conclusion. They are as perfectly preserved, when present, as any of the other tissues of the plant. Their inner surface, nearest the fistular cavity, presents no appearance of death and decay, but of rupture and absorption, which I conclude has occurred during life,—a different hypothesis from that adopted by my predecessors, and for which my reasons will be assigned in the memoir. The labours of Mr. Binney are referred to in the text. He figured the longitudinal internodal canals, but was disposed to believe that they had merely formed passages for vessels. He gave, however, excellent figures of the woody wedges, the primary medullary rays, and the cellular medulla, with its nodal septa or diaphragms.*

The above may be received as examples of the many discordant views entertained by the various authors who have written upon the subject of Calamites, and may probably be regarded as a justification of the method which I have deemed it best to adopt in this memoir.—Note added July 12, 1871.

NOTWITHSTANDING the large amount of attention which has recently been paid to the

study of the Calamites of the Coal-measures, wide differences of opinion still exist respecting them amongst the highest authorities on the subject. Consequently I have availed myself of the valuable opportunities which the labours of Mr. J. BUTTERWORTH, of Shaw, near Oldham, have brought within my reach to make a very extensive series of observations upon the plant. For this undertaking the materials have been so ample that I feel justified in speaking strongly upon some points that have hitherto been doubtful, and on which some of my fellow-labourers in the field of palæo-phytology entertain views different from my own.

The questions at issue group themselves under several distinct heads:—

1. Do all the well-known plants hitherto designated Calamites belong to one natural family, or are there two groups of these objects—the one Cryptogamic, represented by the true *Calamites*, and the other Phanerogamic, and represented by the *Calamodendra* of BRONGNIART?

2. Are there several genera divisible into numerous species, with well-marked internal characteristics, or are there but few specific types, each of which, though they are all constructed upon one common plan, exhibits a wide range of variability in the details of its internal organization?

3. What are the casts commonly known as Calamites? and what parts of the plants do their varied superficial markings represent?

4. To what living plants are these fossil forms most closely related?

To all the above questions I think my materials suffice to give answers, though I would guard against the error of making the plants of the Lancashire Coal-measures, amongst which I have chiefly laboured, the representatives of all that may be found elsewhere, though the probabilities that we may so regard them are very strong. This lesson has been recently taught to such phytologists as needed it, myself being one of the number, by the history of the Palæozoic Conifera. The only true coniferous wood which I have seen in the Coal-measures of this country is the Sternbergian Dadoxylon, which I described some years ago in the Transactions of the Philosophical Society of Manchester; and even in that example, as I have recently shown*, some of the characteristics of the highest Conifers are wanting. But in New Brunswick Dr. DAWSON has found true coniferous woods in the greatest abundance, not only in the Carboniferous but in the Devonian beds—a fact which shows that even in studying the flora of that early age, when individual types were much more cosmopolitan than now, we require great caution in accepting those of one continent as evidences of what exist in another.

In the volume recently published by the Palæontographical Society† Mr. BINNEY has pointed out many of the more conspicuous features characterizing the Calamites of Lancashire. With some of his descriptions I cordially agree, but from others I have been obliged to differ, as will appear in the following pages. The plant has possessed three

* Monthly Microscopical Journal, August 1869.

† Observations on the Structure of the Fossil Plants found in the Carboniferous strata.

distinct concentric layers of tissues, a central pith, surrounded by a ligneous zone, which in its turn was invested by a thick cortical or epidermal cellular structure. The pith (Plate XXIII. fig. 1, *b* *), cellular and solid in the very young growths, very soon became fistular in the older internodes (1 *a*). The woody zone surrounding the pith closely resembled, in its organization, the first year's shoot of a recent Conifer. It consisted of numerous woody wedges (1 *f*), each one starting at its inner extremity from a narrow canal (1 *e*). These wedges were separated from each other by peculiar prolongations of the pith (1 *c*), to which I would assign the name of primary medullary rays; whilst secondary medullary rays separated the constituent vascular laminæ of each wedge, as in recent Exogens. These wedges, with their intervening primary medullary rays, extended vertically in straight lines from node (1 *i*) to node. At each of the latter points they underwent an entire rearrangement to be described in detail. Investing the woody cylinder was a thick, cellular, cortical layer, in which I have failed to discover any traces of vessels. At each node the cellular pith extended across the entire medullary area, so that the fistular interior of the stem consisted of a linear series of oblong chambers, each one of which corresponded with an entire internode, and was separated from its neighbours by the several transverse medullary diaphragms referred to.

Having thus indicated the *general* features of the most common type of Calamite, we may now proceed to a more detailed examination of the different varieties that I have obtained. The sections represented by the figures from 2 to 10 inclusive belong, I believe, to one variety, though they were not all prepared from the same specimen. This also appears to be the most common form, since a large proportion of the examples which I have examined belong to it. Its various tissues may be described in the order of their superposition, beginning at the centre.

The Pith.—This invariably consists of the common type of cellular parenchyma, though the forms which the cells assume vary according to the direction followed in making the section. When cut transversely (Plate XXIV. fig. 9, *b*) they exhibit the ordinary hexagonal form, though their sides are usually somewhat unequal; and in this section there is no approach to any linear arrangement, such as we find in the vertical sections (Plate XXIII. fig. 8 & Plate XXIV. fig. 10, *b*). In the latter we almost invariably find the cells elongated vertically. This is especially the case with the innermost ones, and with those (Plate XXIV. fig. 11, *b*) forming the inner wall of the longitudinal canals hereafter to be noticed. We also observe that they are arranged in linear vertical rows, their parallel faces being constantly at the upper and lower ends of each cell. The narrow transverse diameters often seen in the innermost of these medullary cells (as in fig. 8, *b'*) do not constitute a primary condition, but are the result of physiological changes to be described. Fig. 10 represents a vertical section through the centre of a node (*i*), and two internodes (*k*). It exhibits the two large fistular cavities of the latter (*a*, *a*) filled

* This figure is an attempted restoration of part of a stem including one node and part of two internodes, portions of the cortical layer and of the woody zone being removed so as to reveal the external Calamitean surface of the pith *b*.

with inorganic matrix, and the transverse diaphragm or septum of cellular tissue marking the node (Plate XXIV. fig. 10, *i*)*. It has been supposed by more than one writer that the fistular cavities did not exist in the living plant, but that they were the result of disorganization and decay. Such, however, is certainly not the case. The sharply defined outlines and exquisite preservation of their innermost cells demonstrate that no such decay has affected these specimens. In its very young state the pith was solid and unbroken—a fact demonstrated by the specimen represented in Table 3, fig. 7 of Mr. BINNEY's Monograph already referred to. But the small size and exceeding rarity of all such specimens demonstrates that the rapid growth of the woody axis caused the pith to become ruptured and fistular at a very early period. The process thus commenced ended finally in a complete absorption of the pith, explaining some points in the history of these plants of which no explanation has hitherto been found. On looking vertically at the cellular diaphragm (fig. 10, *n*), we discover that whilst its more peripheral portions consist of elongated cells like those common to the rest of the pith, in its thinner central part we have a very regular and delicate form of parenchyma (fig. 7). Some writers consider that they have found vascular tissues in the pith. I have not done so.

The Woody Zone.—This, as we have seen, is a zone of variable thickness, consisting of a series of wedges composed of vascular and cellular tissue combined (fig. 1, *f*), separated from each other by cellular prolongations of the pith (fig. 1, *c*), in which the cells have a special permanent arrangement. Every woody wedge commences, at its apex, which is directed inwards towards the centre of the stem, at a narrow canal (fig. 1, *e*) of uniform diameter, and which runs in a direct vertical line from node to node. These several parts requires a separate examination. Each wedge extends vertically in a straight line (fig. 2, *f*), the length of the internode. It consists of a series of laminæ which increase in number as we proceed from within outwards from the intercalation of additional ones; hence the widening of the broad external base of the elongated wedge. Each separate lamina is composed of a linear series of vessels, disposed in a radiating line (Plate XXIII. fig. 9, *f*). In the transverse section, these vessels appear to be nearly square (fig. 6, *g*), whilst in the longitudinal one they exist as elongated, unbranched tubes (fig. 5, *g*), which are sometimes reticulated (fig. 4), and at others transversely barred. Very varied names have been assigned to these tubes by different authors. Mr. BINNEY speaks of them as "*pseudo-vascular*," and Dr. DAWSON and Mr. CARRUTHERS as "*scalariform*" tissue, neither of which terms appears to be exactly appropriate. They are unquestionably modified forms of spiral tissue, though we have every reason to believe that they were incapable of being unrolled; but they are not scalariform, in the sense in which the term is applied to the vessels of ferns and some other living Cryptogams, since they do not exhibit angles thickened by continuous ligneous deposits as is seen in those examples. Still less should they be designated pseudo-vascular, since they are true vessels. The term "barred"

* We have in this specimen a curious illustration of the fact that this pith was fistular whilst the plant stood erect, inasmuch as a number of vegetable spores (fig. 10, *x*) have found their way into the cavity, and now rest upon the diaphragm which has arrested their further descent.

seems an appropriate one to apply to them. In some parts of his Monograph Mr. BINNEY describes them as having "their walls perforated with oval openings." This is certainly not the case, the supposed openings being merely thin places in the tubes where the original cellulose wall has remained unthickened by secondary internal deposits of lignine. These vessels approach very closely to the true annular vessel; but I have never yet seen an example in which individual rings could be traced apart from their neighbours, as is so constantly observable amongst recent annular tissues. As these peculiar vessels are very common amongst the plants of the Coal-measures, occurring in the Sigillariæ and other genera as well as in Calamites, it seems desirable that they should be distinguished by some appropriate name, and the term "barred" appears to answer the purpose. The "reticulate" vessels so common amongst Calamites are but modifications of the same structure, occurring in the same stems with the barred varieties, and not unfrequently the two appear to be convertible. In several instances I have noticed that the vessels at the inner extremity of the wedge were barred, whilst those constituting its peripheral portion were reticulated. In many specimens, though the boundaries of the vessels are well defined, all trace of their internal organization has disappeared. Such examples appear as represented in the vessels (*g*) of fig. 3; but I believe that this is an abnormal condition due to imperfect fossilization, since we can constantly trace the transition from the smooth to the barred and reticulated forms. The vessels at the medullary or inner extremity of each radiating lamina are, as Mr. BINNEY has correctly pointed out, frequently smaller than the peripheral ones, though the difference is not always very marked. The largest vessels in each specimen range from ·003 to ·006 of an inch; the smaller ones are often less than half these diameters. Their number in a linear series varies with the age of the plant and with the part from which the transverse section is taken. In very young plants they are but few, whilst in one specimen I counted 354 in one row; as will be seen shortly, they are more numerous at the nodes than at the internodes. The number of the woody wedges, as well as the distances between the canals whence they spring, varies with age and other circumstances. I have discovered no evidence leading me to believe that the number of the wedges was increased after a young shoot, however minute, was once organized. In some instances I have counted as few as fifteen, and Mr. BINNEY has figured an example with but nine. On the other hand, I have counted as many as eighty in a transverse section of a stem little more than half an inch in diameter; in one old arenaceous cast I found that the outer surface indicated ninety of these wedges. Indications of wedges intercalated after the first growth had begun have only been met with in one example, in which a solitary wedge first appears at a point a little external to the concentric line formed by the inner angles of the rest of the series, and it was devoid of the usual accompanying longitudinal canal. This absence of regularly intercalated wedges has some physiological significance in relation to the age and growth of these stems. The distance intervening between contiguous longitudinal canals practically marks that existing between the centres of contiguous wedges. This often varies somewhat in the same section, though the variation is limited. In the

specimen (figs. 9, 10) where the entire diameter of the branch, bark included, is only ·2 of an inch, the distance from one canal (*e*) to another is from ·03 to ·04. In specimens ranging from half an inch to an inch in diameter these distances increase from ·08 to ·1, whilst at the base of some very large arenaceous casts I have evidence that they increased to ·25. These steady enlargements of the wedges accompanying the general growth of the stem demonstrate a corresponding intercalation of new vascular *laminæ* into the exterior of each wedge as the exogenous development of the woody zone increased its diameter.

Secondary Medullary Rays.—Whether viewed in their transverse or vertical section, we discover that the laminæ composing each woody wedge are separated from each other by vertical cellular films, apparently undistinguishable by any important, definite feature from the medullary rays of the higher Exogenous Plants. Fig. 6 represents portions of four laminæ (*g*) from a transverse section of a wedge. Two of these are in contact, as is not unfrequently the case; the others are separated by lines of cellular tissue, which in most instances can be distinguished, in this section, by their darker hue, and by their cells possessing thinner and less sharply defined walls than is the case with the vascular tissues; but sometimes, in the transverse section of the stem, these cellular lines so closely resemble the vascular laminæ, that they are scarcely capable of being distinguished from each other. Such distinction is easy enough in the vertical sections, whether tangential (fig. 5) or radial (fig. 11). In the former we see that these cells (fig. 5, *d*) are arranged linearly in single vertical series, there being rarely more than two vessels in direct contact with each other. Sometimes we observe one isolated cell; in others several cells are superimposed. In all cases they are compressed laterally, so that their length greatly exceeds their transverse diameter. Their transverse septa vary their direction indefinitely, being sometimes rectangular, in others oblique and overlapping. Plate XXIV. fig. 11 represents one of several sections made with the utmost possible care. The section is a radial one, passing through the longitudinal canal (*e*) at its inner margin, and through the exact centre of the woody wedge at its opposite or cortical end. This exactness was rendered necessary by the circumstance that the two lateral surfaces of each woody wedge are impressed by similar cells belonging to the primary medullary rays; and it has been supposed by one of our authorities in Phytology that I had mistaken the one for the other. There is not now the slightest room for such a supposition. In this section we see the cells of the pith at *b*. At *e* we have the longitudinal canal; the entire range of the section to the left of the canal, in the drawing, consists of a lamina of barred vessels, whilst delicately projected upon the walls of the vessels throughout a great part of the section we have the vertically arranged cells of these secondary medullary rays. They exhibit the strongest tendency to arrange themselves murally and in continuous lines, stretching from the pith to the bark, only instead of representing modern bricks disposed horizontally, as is usual in the muriform medullary rays of Phanerogamous Exogens, they more resemble the long bricks of ancient Rome set upon their narrow ends. Still the extreme regularity of their arrangement justifies me in describing it as *muriform*.

On turning from the internodes to the well-defined nodes of these Calamites, we discover a series of very important modifications in the disposition of their tissues, altering many of their features. If a vertical section be made through the centre of the stem, we have the appearances presented at fig. 10, *i*, whilst fig. 2 represents the arrangement of the tissues in a tangential section made midway between the pith and the bark.

The first feature which arrests attention in the vertical section is the material transverse enlargement of the woody zone (fig. 10, *f*) which takes place at the node (10, *i*). This enlargement is both internal and external. In the former case the woody layer encroaches upon the pith, and in the latter upon the bark. The increment is due to the development of a considerable number of barred or reticulated vessels, but especially the former, which take their rise in contact with the outermost medullary cells above the node, and following an arching course across it, their concavities being directed towards the medulla, again terminate, as they arose from the medullary cells above the node, in those below it. It follows from this arrangement that only the outermost of these nodal vessels are prolonged across the internodes to the adjacent nodes above and below.

In the transverse section we find, as the vertical one would lead us to expect, that the woody wedges at the nodes are much longer from their medullary to their cortical surfaces than at the internodes. The canals from which they respectively take their rise are either wholly wanting here, or so reduced in dimensions as to become quite inconspicuous. The large primary medullary rays (fig. 1, *c*) have also become so restricted as in many cases to be scarcely traceable, bringing the wedges into very near contact, and rendering the resemblance between the section and the similar one of an ordinary Coniferous Exogen still more close than in the case of the internodes. But the most striking peculiarity in the nodal arrangement of the tissues is seen in the tangential section (fig. 2, *i*). In the example represented in the Plate we have part of one node (*i*) and of two internodes (*k k*). We now discover that each woody wedge, as it ascends to the node, divides into two portions, each of which bulges out somewhat, and which proceed obliquely upwards in a divergent manner to contribute their respective quotas to a corresponding but alternating series of wedges in the internode above. On magnifying one of these divergent portions (fig. 2, *o*) more highly, as is done in Plate XXIII. fig. 3, we discover that two changes have taken place compared with what was observed in the internodes:—1st, the vessels (*g*) pursue a more undulatory and divergent course, leaving wider spaces between them; and 2nd, those spaces (*d*) representing the secondary medullary rays, are occupied by irregular groups of cells which are very frequently arranged in double, and not unfrequently in threefold series. These sections also explain the almost complete disappearance of the primary medullary rays already noticed in describing the transverse section of this portion of the Calamite.

But another very important feature occurs in these nodal parts of the organism. In fig. 2, two lenticular spaces (*m*) appear in the diverging portions of the woody wedges, and an examination of large series of specimens demonstrates that these spaces represent branches. When the tangential section is made close to the pith, we find that the

spaces are wholly occupied by cellular tissue, but when made in the centre of the woody zone, we have the structure represented in Plate XXIV. figs. 13 & Plate XXVIII. fig. 38. In fig. 13 the central mass (*m*) is a combination of vascular and cellular tissue. The distinction between the two elements is not seen in this tangential section, which is practically a transverse section of the structure. But in longitudinal sections, like fig. 10, we obtain evidence that a considerable number of vessels, derived from the woody wedges, are suddenly deflected and proceed outwards towards the bark in a compact parallel series. The point specially to be noted in these vascular communications with external branches is their small size. The aggregate of the cells and vessels composing each one is always less than that of the woody wedge which it penetrates. These arrangements will be referred to again when describing fig. 38, which represents another type.

The next structures demanding attention are the primary medullary rays, or radiating cellular masses (fig. 1, *c*), which separate the woody ridges, and which like the latter extend as continuous planes from node to node (fig. 2, *c*), where, as already indicated, they become merged with the woody tissue in the shape of enlarged secondary medullary rays. As the irregular parenchymatous cells of the pith pass outwards between each two contiguous woody wedges, they gradually assume a more regular disposition. Their parallel faces become also parallel with the surface of the pith, throwing them into linear series, which, in the transverse section, radiate from the medulla to the bark. At first these cells are of large size, but they rapidly become less as they pass outwards. This circumstance, combined with the simultaneously increased regularity in the arrangement of the cells and with their more uniform size, causes their appearance, as they approach the bark, to differ very little in the transverse section from the vessels of the woody wedges on each side of them. Indeed in many old stems the line of demarcation is only to be traced with exactness in the tangential sections. These cells seem to vary considerably in size. Sometimes they are comparatively narrow, as in fig. 9; but in others the central row stretches across the ray as a series of long narrow parallelograms, the long axis of each cell extending nearly from one woody wedge to the next. So far as I can ascertain, these variations are merely the result of differences in the rate and conditions of growth, since I often find them exhibiting considerable difference on opposite sides of the same stem. In the tangential section (fig. 2, *c*) these cells do not appear to be arranged in any regular order, merely presenting the aspect of an irregular parenchyma. But the case is different with the radial sections, one of which is represented in fig. 8. At *b* we have the medullary cells assuming the vertically elongated aspect frequently seen in all longitudinal sections of the pith, whether radial or tangential; but all the remainder of the section is composed of more or less regular lines of muriform cells, identical in every point of form, size, and general aspect with those of the secondary medullary rays (fig. 11).

In determining the physiological character of these cellular masses we labour under some difficulties, because of the general aspects of the plants to which they belong. The plants are, as abundant evidence proves, cryptogamic in their fructification and affinities;

and to speak of medullary rays in such examples, appears at the first glance to involve a misappropriation of terms. But, on the other hand, there is no doubt whatever that the woody zone which I have described is truly exogenous in its structure and mode of growth, though belonging to a Cryptogamic plant. Consequently we have in it the elements essential to the idea of one form of medullary ray. We have had, in the first instance, a cellular bud, within which vascular masses were developed; and the portions of the primitive cellular tissue, connecting the inner cellular pith with the outer cellular layer, whether we call the latter bark or epiderm, are essentially the same things, so far as their genesis is concerned, in the Calamite and in a young Dicotyledonous shoot of the first year. The strong disposition shown by the cells of these structures in the Calamite to assume a mural arrangement confirms, though it is not necessary to, the idea of these organs being medullary rays; and as these are the first of this class of organs that appeared in each young Calamite, I have applied to them the term usually employed by our best vegetable physiologists in such cases, viz. *primary medullary rays.* For similar and obvious reasons I assign to the rays separating the component laminæ of the woody wedges the title of *secondary medullary rays.* They are formed subsequently to the others, and, as in the case of Exogens, their number in the Calamite continues to increase as long as the wedges containing them continue to grow. All the facts I have detailed justify the conclusion published in some of my previous memoirs, that in this woody zone of the Calamite we have a true exogenous growth; whether we regard its intimate organization or its genesis, we are brought to this decision. There is a peculiarity in the exterior of this zone not to be overlooked. In his monograph already referred to, Mr. Binney figures a decorticated stem, the decorticated exterior of which is marked by strong longitudinal ribs and furrows, with a thickened node. This configuration of the decorticated Calamite has been common to the type now under consideration, as shown in my restored figure 1. Though subject to considerable variations as to the degree of prominence, the transverse section always presents a more or less undulating outline, of which the exteriors of the woody wedges constitute the projecting portions. When we come to consider the nature of the arenaceous casts usually seen in geological collections, we shall find that the contour referred to forms an unfailing indication whether any specimen under examination represents the exterior or the interior of the woody zone.

The Longitudinal Canals.—M. Brongniart many years ago obtained some silicified stems of Calamites from the Coal-measures of Autun, which showed that the woody wedges took their rise from narrow pores or canals. Mr. Binney has further demonstrated the same fact in his memoir, but was doubtful respecting their nature. He says that Dr. Hooker, "after carefully examining these openings, I believe, came to the conclusion that they were passes for a peculiar kind of tissue which has unfortunately been destroyed, rather than the mere cavities which we now see in the specimens" (*loc. cit.* p. 20). This supposition, however, is certainly not correct. We have the clearest evidence that these are true intercellular canals (Plate XXIV. fig. 11, *e*) running from node

to node, at which points, in the type under consideration, their extremities appear to terminate as cul-de-sacs. Their inner or medullary wall is almost invariably composed of very narrow elongated cells (fig. 11, *b*) with square extremities, being in fact elongated modifications of the ordinary medullary cells. I have not succeeded in discovering any cells in their peripheral wall; this appears to be composed of the innermost and first formed barred vessels of the woody wedge (11, *g*), to which each canal belongs. Their diameter varies from ·03 to ·1 of an inch, their width being uniform throughout their entire length, except at their extremities, where they rapidly contract until they disappear.

The Bark or Epidermis.—In a lecture delivered before the Royal Institution of Great Britain (April 16, 1869), Mr. CARRUTHERS incidentally mentions that the Calamitean stem possessed "a *thin* cortical layer." I am not aware, however, that the true bark has hitherto been either figured or described. The discovery by Mr. BUTTERWORTH, of the small stem represented in figures 9 & 10 removes much of the existing obscurity on this subject. Of all the scores of microscopic sections of stems of which I have either made or examined, it is remarkable that two examples alone exhibited evidences of the existence of a bark, all the rest having been decorticated. The same observation is applicable to those figured by Mr. BINNEY, in none of which is this tissue seen. These facts show that the cohesion between the bark and the woody zone must have been exceedingly slight, at least in the small fragments inclosed in the ironstone nodules from which the majority of our specimens retaining their structure have been derived.

The entire diameter of the small stem referred to is ·2 of an inch. The thickness of the woody zone at the internode is about ·05. At its narrowest part the thickness of the bark is ·025, whilst it becomes very much thicker at the nodes. It has in fact been a rather thick parenchymatous layer, in which the cells were very irregular both in size and distribution. Some few cells of large size appear dispersed amongst others of smaller dimensions. There is no trace of linear or other special arrangement of these cells, either in the transverse (fig. 9) or vertical (fig. 10) sections. Where the bark crosses the node, in the latter section, its surface has become somewhat disintegrated; but though Mr. BUTTERWORTH and myself have prepared several sections of the specimen, I have not been able to discover the slightest trace of vascular bundles, or even of solitary vessels crossing the cellular parenchyma. The peripheral outline of the vascular woody zone at this point is sharp and well defined, and the course of the vessels undisturbed*. The only difference seen between the longitudinal, tangential, and transverse sections, is a slight vertical elongation of the cells in the two former. The cells of the outermost surface differ little if at all from those of the interior, and I detect no trace of a special cuticular layer. It exhibits no indication of being an exogenous bark, no trace of the tripartite division seen in that of the gymnospermous Conifera existing in it.

* Unfortunately the arching vessels of the node exhibit no trace of a branch crossing them at this point, as is frequently the case with parallel sections of other specimens. Consequently we miss a valuable opportunity of seeing the relations of these branches to the bark.

The type which I have thus described in detail appears to be the one occurring most abundantly in the Lancashire Coal-measures; but along with it some others occasionally occur, which may have a specific value, though its exact amount is difficult to determine, since they are most of them rare, and of some only single examples have yet been found. Plate XXIV. fig. 14 represents one of those, of which I have seen several specimens. The entire diameter of the decorticated stem is ·21. Each woody wedge contains from six to eight laminæ (*g*), separated by strongly marked secondary medullary rays (*d*). Each of the principal laminæ consists of a linear series of from nine to twelve vessels, which are remarkable for their large size. We have already seen that the largest vessels usually found in the type already described, range from ·003 to ·006, whilst near the longitudinal canal they are generally much smaller; but in this instance the vessels are as much as ·01 in diameter. The specimen also illustrates very clearly the intercalation of additional laminæ, as at g', g''. The cells of the pith are smaller than usual, and those of the primary medullary rays (*c*) especially so. Not having a longitudinal section of the specimen, I am unable to determine whether the small structures intervening between the inner extremities of the vascular laminæ (*g*) and the longitudinal canals (*e*) are minute vessels or elongated cells. The specimen thus possesses a marked individuality, though I am at present unable to determine whether it represents a state or a species. Plate XXIV. fig. 15 represents an example of which we have both the longitudinal and transverse sections, both of which are very distinct in their peculiarities. The specimen is decorticated, and has a diameter of ·25. The exterior of its woody cylinder has been almost smooth, exhibiting none of the usual flutings arising from the prominence of the woody wedges (*f*). Indeed in the transverse section it is impossible to say where the woody laminæ terminate, and the cellular tissues of the primary medullary layers begin. The medullary cells (*b*) are remarkably large, and the canals (*e*) also rather larger than usual; but the vessels and the cells of the primary medullary rays are not more than ·0025 of an inch in diameter. The suddenness with which the large medullary cells contract to form the small cells of the primary medullary rays constitutes a striking feature of the plant, the general aspect of which is almost exactly the opposite of the one last described in every respect.

Fig. 16 represents the medullary extremities of one of the woody wedges (*f*), and two of the primary medullary rays of another variety possessing great beauty. The woody laminæ are very regular, and the intervening secondary medullary rays very distinct, whilst the manner in which the irregular parenchyma of the medulla (*b*) passes into the linear arrangements of the primary medullary rays (*c*) is very clearly shown. But the most remarkable feature of the specimen is the entire absence, from the apex of every one of the woody wedges, of the longitudinal canal. I at first thought that the section might have traversed a node into which the canals did not extend; but I am convinced that such was not the case. The place of the canal is occupied by firm, strong-walled cells. I think it probable that the example of which a xylograph is given by Mr. Binney in his work (p. 20), and which led Dr. Hooker to surmise that the canals had

been channels for the transmission of special vessels, may have been of this variety. We have only seen two stems of this type*. Plate XXV. fig. 17 represents a transverse section of the half of one of the primary medullary rays (*c*), and the adjacent part of a woody wedge (*f*) of a remarkable form. Fig. 18 is a carefully drawn copy of a tangential section of part of a similar primary medullary ray from the same example. The laminæ of vascular tissue (*g*), instead of terminating at the usual boundaries of the woody wedge (*f*), have continued to be developed externally to it, encroaching upon the primary medullary ray, as exogenous growths increased the diameter of the stem. Practically the result of these additions has been to convert each large primary medullary ray into a series of smaller secondary ones, with but one, two, or three linear rows of cells in each, reducing them to a condition differing little, in the transverse section, from what we find in the secondary medullary rays of the woody wedges. But when we turn to the tangential section (fig. 18), we discover that these vessels (*g*), not having been subjected during growth to the uniform pressure mutually affecting those in the woody wedges, have not only pursued a more tortuous course amongst the large cells, but their form and diameter has been modified by the unequal resistance of those cells, so that whilst in some instances their diameter is ·005, in others they are reduced to a mere thread. I have only met with one example of this remarkable variety.

I have next to call attention to a peculiar form identical in many respects with one that I described in the fourth volume of the third series of the Memoirs of the Literary and Philosophical Society of Manchester, and to which I gave the generic name of *Calamopitus*. Some examples of this type possess the highest interest, because they throw a most important light on the nature of the forms of Calamite so common in the shales and sandstones of the Upper Coal-measures. Fig. 19 is a representation, of the natural size, of a transverse section of a compressed stem. The dark centre has been a large fistular medullary cavity, whilst the walls of the surrounding cylinder have been remarkably thin in proportion to the size of the stem; the diameter of the latter has been about ·66, the entire thickness of the decorticate woody cylinder, including its contained layer of pith-cells, not having been more than ·08—a condition of things most favourable to that flattening of the stem so frequently seen in the fossils which are laid horizontally in their matrix. Though this cylinder is so thin, it contains about eighty distinct woody wedges. Plate XXV. fig. 20 represents two of these woody wedges more highly magnified. Two features alone require to be noted in this section. One is the well-marked crenulated outline (*x*, *x*) separating the pith from the persistent woody zone. This line is especially remarkable for its distinctness where it crosses the primary medullary rays (*c*). In this feature the specimen resembles my previously described *Calamopitus*, as well as fig. 15 of the present memoir; only its crenulations are much deeper, approaching less towards a straight line, than in the latter figure. The second point referred to is the remarkable prominence exhibited by the external base of each woody wedge (*f*), which is

* Since the above description was written I have met with a third specimen, and think it possible that a new genus may be required for its reception, since it lacks some Calamitean features.

more marked than in any other specimen that I have examined. It will also be observed that, as in fig. 15, the transverse section exhibits little or no difference between the sections of the vessels of the woody layer and the cells of the primary medullary rays.

Plate XXVI. fig. 21 represents a fragment taken from the upper part of fig. 19, opposite to the small star, and which is in the highest degree instructive: in addition to the transverse section, this specimen exhibits, at its inferior half, the free longitudinal surface of the fossil running at right angles to the section. *a* is part of the central fistular cavity filled with dark ironstone, *b* is the cellular pith corresponding with fig. 20, *b*; and in like manner the crenulated line *x*, *x* corresponds with the similar line in fig. 20. In the left-hand portion the woody zone is retained *in situ*; but to the right this has been detached from the pith (*b*), the separation taking place at a vertical surface corresponding with the crenulated line *x*, *x*, and leaving behind it a fluted surface identical in every respect with that of the ordinary Calamites with which we have so long been familiar. The sharply defined longitudinal grooves (*e'*, *e'*), separating the concave ridges, are clearly seen to be identical in position with the longitudinal canals (*e*); whilst the prominent ridges (*f*, *f*), or exteriors of the woody wedges, occupy a position more peripheral, but radially vertical to that of the parallel grooves (*e'*)—arrangements which throw a flood of light upon the ordinary structureless examples of Calamites.

Plate XXVI. fig. 22 is a tangential section of this specimen made in the plane of the longitudinal canals, or a little more internal than the crenulated line (figs. 20 & 21, *x*, *x*); but since the section fortunately crosses a node at the line *i*, *i*, we obtain evidence that the arrangement of the vertical canals (*e*, *e*) at this portion of the plant differs from what exists in the types previously described. Instead of terminating near the node as cul-de-sacs, in the present example they enlarge into triangular spaces, and then, dividing right and left, they pass downwards to form the canals of the internode next below*. The cells of the pith describe at this point a series of Roman arches (*b'*, *b'*), which form the inferior boundaries of the canals at their bifurcation; the rest of the section is principally occupied by the ordinary parenchymatous medullary cells, as they present themselves opposite to the primary medullary rays, though here and there a few bundles of barred vessels demonstrate that it has dipped into the woody zone. This is especially the case at the node and near the centre of the specimen, where an elliptical figure bounded by vessels marks the position of a young branch such as is represented in fig. 13. But the section under consideration reveals some other important features. Within and below each of the Roman arches (*b'*, *b'*) we observe that the cellular tissue is ruptured longitudinally for a short space. In some instances this rupture (*l*, *l*) is so slight as to be scarcely appreciable, and might indeed have been deemed accidental but from its constant relations to some other features yet to be considered.

In Plate XXV. figs. 23 & 24 we have two radial longitudinal sections, both of them being so made that the superior half of each above the node (*i i*) has passed vertically

* In fig. 37 we have another radial section which intersects one of the horizontal portions of a canal (*c*) passing from one triangular space to another.

through the longitudinal canal (*e*). As in the other figures, *a* is the fistular cavity, *b* the medullary cells, *g* the vessels of the woody wedges which, in fig. 24, are seen to pursue their usual arching course as they cross the node (*i*). At *e* we have the longitudinal canals of the superior of the two internodes intersected in each of the sections.

In fig. 24 this canal passes out of the line of the section, but in fig. 23 it dilates, at the node, into one of the large triangular spaces seen in fig. 22. These, and similar sections which I possess from the same specimen, seem to indicate that, in it, the cellular medullary nodal diaphragm was not complete, but that the various internodal fistular cavities communicated freely with each other.

The most important feature of these sections is seen in fig. 23, *l*, where we find that the longitudinal fissures seen in fig. 22, *l*, *l*, are the inlets to well-marked oblong passages filled with the same amorphous matrix as occupies the fistular cavities of the pith, and which are thus proved to have been open canals when the plant was entombed. In the memoir on *Calamopitus* already referred to, I entered at considerable length into the history of these canals, applying to them the name of verticillate medullary *radii*. But at the time when that memoir was written my imperfect specimens had not enabled me to discover either the fistular character of these calamitean stems or the peculiar process of medullary absorption which has taken place in them. Hence I stated that "these radii appear to have been composed of the same tissue as the medulla itself, judging from the circumstance that the inorganic material with which they are filled is identical with that replacing the pith. They have most probably united the pith with the bark" (*loc. cit.* p. 163). It is now evident that the account which I then gave of the position of these curious rays in relation to the rest of the tissues, and which I further illustrated by a diagrammatic figure (*loc. cit.* tab. 5. fig. 17), is in all respects correct, with the exception that they prove to have been *canals* formed, first by a rupture, and afterwards by an absorption of the cells of the primary medullary rays, and were not special prolongations of the medullary parenchyma through those rays, as I deemed probable. Various attempts have been made to show that these canals originally transmitted special vascular bundles either to roots or to branches, which vessels were supposed to have become decomposed, admitting the inorganic matrix into the cavities formed by their disappearance. But all my specimens negative this idea. In the first place, these canals are not planted upon the node from which such branches would spring, but a little below it; whilst immediately above them, and in their right place, directly over the node, we find the true lenticular bundles going to such branches, springing from the innermost vessels of the ligneous zone. This is demonstrated in fig. 25, which represents a tangential section like fig. 22, only passing through the external instead of the internal surface of the woody zone; *c*, *c* indicate cellular primary medullary rays, *f* the woody wedges, *i*, *i* the node, with a verticil of lenticular structures (*m*) marking the position of the branches, whilst at *l*, *passing through the cellular tissue, below and altogether disconnected from the vascular structures*, are the disputed organs. The sketch (fig. 25)

represents the section viewed as an opaque object, in which the white crystallized carbonate of lime with which the fistular pith is filled, and which has also occupied these canals, causes them to stand out with remarkable clearness. In the example of *Calamopitus* described in my previous memoir, I figured some sections of the vascular bundles going off to the branches where the vascular tissues were so conspicuous and had so remarkable an arrangement as conclusively to demonstrate their true character; and though in the specimen under consideration the similar structures (fig. 25, *m*) are not so highly developed, there is no question that in both examples their nature is the same. These facts do away with the possibility of the canals in question having any direct relationship with the branches.

At the Meeting of the British Association at Liverpool, Mr. CARRUTHERS suggested that these organs were points from which roots had been given off. Such, however, cannot have been the case, since in the specimens in which they occur they exist throughout the entire length of the stem from its base to its summit. Moreover they are located in the centre of the *cellular* tissue of the *upper* part of each primary medullary ray, whilst, as I shall immediately demonstrate, there is abundant evidence proving that the roots were given off from the *lower* part of each internode. The position of these canals in relation to each internode of the stem—their isolation, internally amid the cellular tissue of the medulla, and externally in that of the primary medullary ray—the obvious mode of their formation, 1st, by the rupture, and, 2nd, by the absorption of those cellular tissues—the entire absence from every example yet examined of all trace of vascular tissue, either in the medulla from which they spring, or in that accompanying them in their outward course—and finally, the circumstance that they are always filled with the same inorganic material as that which occupies the fistular cavities of the pith, are facts pointing irresistibly to a common conclusion, viz. that these organs were narrow canals, arranged in a verticil immediately below the transverse medullary septum of each node, and that they formed channels of communication *through* the woody zone, between the uppermost part of each internodal fistular cavity and the inner surface of the bark*. Being fully convinced that such is their true nature, I propose to recognize them by the name of "infranodal canals" instead of "verticillate medullary radii," which I formerly applied to them.

In the tangential section now figured, these canals exhibit an *oblong* contour; in all those described in my preceding memoir on the subject they were *round*. It will be seen that these differences correspond with what we find in the more ordinary, structureless Calamites of this class occurring so frequently in beds of shale and sandstone.

Having described the more important varieties of structure seen in our specimens, I will now endeavour to correlate these with the common forms so frequently seen in cabinets.

* I have represented one of these canals at *l* in the restored fig. 1, but having no conception what effect they had upon the bark, I have represented the latter as extended across their open extremities. This arrangement of the bark is purely hypothetical; the canals may have penetrated it as well as the ligneous zone.

Hitherto nearly all phytologists, with the exception of Mr. BINNEY and Mr. CARRUTHERS, have endeavoured to infer the structure of Calamites from the appearances presented by specimens from which all structure has disappeared. A comparison of their inferences with what are now positively ascertained facts, demonstrates the danger of this mode of procedure, and confirms an opinion I have elsewhere expressed, that no determinations respecting fossil plants can have much absolute value save such as rest upon internal organization; that is the basis upon which all scientific recent botany rests, and no mere external appearances can outweigh the positive testimony of organization in fossil types. But whilst thus insisting upon the supreme value of structure as a guide, I am not blind to the importance of a mass of evidence that has been derived from the study of external forms, and especially that for which we are indebted to my two friends, Dr. DAWSON, of Montreal, and M. GRAND' EURY, of Saint Etienne, in France. Whilst many of the specimens which I have described have unquestionably been aërial *stems*, the minute size of others makes it exceedingly probable that they were small branches; nevertheless there is no variation in the structure of these two classes of organs, beyond what has arisen from the gradual absorption of the pith as the plant increased in age. But we have in this difference the first clue to the history of the common fossil forms. It has often been observed that very small Calamites were exceedingly rare, leading to innumerable surmises as to the cause of this fact.

The specimens of Calamites usually seen are casts of the interior of the wood-cylinder of the stem, either composed of sandstone or of shale, and are generally covered with a thin homogeneous layer of carbonaceous matter. The early writers almost invariably turned these specimens upside down, believing their obtuse or conical bases to be the uppermost extremities of the stems. These observers experienced further difficulties in the circumstance that whilst the internal casts were fluted longitudinally and marked at intervals by transverse constrictions, similar features were exhibited, more or less strongly, by the external surface of the carbonaceous covering of each specimen. Hence the apparent probability naturally suggested itself, that in the living plant one calamitean structure had existed in the interior of another. But it was invariably believed that the internal Calamite represented a solid pith that had disappeared from the inorganic decomposition of its tissues after death, the place of the lost tissue having been supplied by sand or mud, according to the nature of the sediment under which the plant became buried. At an early period of my recent observations I became satisfied that the living plant had possessed a fistular medulla, and M. GRAND' EURY, studying a very different class of specimens to mine, arrived at the same conclusion*.

* Observations sur les Calamites et les Asterophyllites, par M. GRAND' EURY, Comptes Rendus, tom. lxviii. p. 705. That author correctly says, "la présence fréquente aux jointures de cloisons plus ou moins entières est une évidence complète que ces tiges étaient fistuleuses." But in his definition of the genus *Calamites*, and in many of his general observations embodied in the same memoir, this careful observer has been seriously but almost inevitably misled, arriving at several conclusions diametrically opposed to the demonstrations which my better preserved specimens have afforded.

In young stems and branches these fistular cavities coexisted with a surrounding pith, consequently when they became filled with sand or mud, the latter materials would harden into a permanent cast of each hollow cavity. But though such casts might exhibit the nodal constrictions due to the transverse medullary diaphragms, they would show none of the longitudinal ridges and furrows characteristic of Calamites. The reason for this would obviously be the intervention of the pith between its central cavities and the inner surfaces of the woody wedges to which such furrows were due. Hence no one, finding such a cast, would recognize it as belonging to a Calamite. But as each stem increased in age and dimensions the pith gradually disappeared, not by mere decay, but by a vital process of rupture and absorption. As in the living Exogens, the existence of the pith seems to have been correlated with the first formation of the circle of woody wedges; and when the growth of these was fairly started, it seems to have been no longer needful to the plant except at each node, where it continued to exist either as a complete or, what is more probable, as a partial diaphragm. After this disappearance of the pith, successive additions continued to be made to the exterior of each woody wedge, as in true Exogens, and also to that of the primary and secondary medullary rays. No definite traces of concentric rings indicative of interrupted growth appear in the transverse sections*. The pith does not appear to have been necessary to the plant whilst these external additions were being made to its woody tissues. Fortunately one specimen has been found by Mr. BUTTERWORTH in which I have discovered remarkable evidence of what has taken place as the pith disappeared. Plate XXVII. fig. 26 represents three woody wedges (*f*) and two primary medullary rays (*c*, *c'*) from a stem which must at least have had a diameter of 1 inch. The wedges and medullary rays present the usual features, but the pith is everywhere breaking up into large spaces, which, opposite the primary medullary rays, assume the definite rounded form represented at fig. 26, *c'*, *c'*. The shrivelled, half absorbed cells at *b'* have nearly disappeared, and within this line they are wholly gone. At *c'*, *c'* the absorbent action has reached its limits; it has touched the boundary line indicated by fig. 21, producing that undulating outline of the medullary cavity which gives to its common arenaceous casts their calamitean form, and which here, as elsewhere, is only found in stems of notable dimensions. In the specimen of *Calamopitus*, which I have previously referred to as described in the Transactions of the Manchester Literary and Philosophical Society, the absorbent action has gone yet further. Almost every trace of the pith has disappeared through its operation. The inorganic cast of the medullary cavity exhibits its sharply defined Calamitean outline in immediate contact with the persistent vascular and cellular tissues of the woody zone; and it is an interesting fact that the specimen which thus exhibits the entire completion of the process of pith-absorption is the largest in which I have hitherto discovered structure, its

* In this respect the Calamites only exhibit the same phenomena as appear in *Dadoxylon*, *Dictyoxylon*, and other exogenous stems of the Coal-measures, in British specimens of which I have rarely seen concentric rings of growth that I could identify with periodic checks arising from secular variations of climate. Nevertheless such rings do occasionally occur.

medullary cavity alone having a diameter of 2 inches. I think this chain of facts justifies me in my conclusion that the common Calamitean medullary casts owe their form to a vital process carried on during the life of the plant, and not to an inorganic decay of the pith-cells occurring after its death*.

Of the common instances in which we find the well-defined Calamite composed of shale or sandstone, and covered by a carbonaceous layer, we have now no difficulty in discovering an explanation. The medullary cavities, and in the case of the *Calamopitus* the verticillate radial canals also, have been filled up with inorganic matter which formed an exact cast of each cavity and canal. This cast, having become indurated, retained permanently the grooves and ridges impressed upon it by the inner surface of the woody zone; in the case of the *Calamopitus* there also projected more or less prominently from the surface of the cast the inorganic contents of the verticillate infranodal canals, which in their uncompressed condition would stand out from the central cast like spokes from the nave of a wheel. So long as the woody zone retained its integrity these conditions remained unchanged; but pressure and chemical agencies gradually produced alterations. As the vegetable tissues of the plant became converted into coal, their structure disappeared along with much of the material composing them. And when the process was completed, what remained was deposited, as if by a process of electrotyping, in the form of a thin film of coal, moulded upon and taking the shape of the hardened central cast,

* That the common specimens of Calamites were inorganic casts of the interior of the woody cylinder, occupying the cavity left by the disappearance of the pith, was suggested by Mr. Dawes ten years ago ("Further Remarks upon the Calamites," Proceedings of the Geological Society of London, vol. vii. 1851); though I had overlooked the circumstance until after the present memoir was read before the Royal Society. I arrived at my conclusions from independent evidence. But Mr. Dawes differs from me in adopting the idea of decay to account for the disappearance of the pith, the reverse of my opinion that it was the result of a vital process of absorption. Transverse sections of sandstone Calamites exhibit a crenulated outline of geometric regularity; and this outline recurs throughout the entire length of specimens from 6 to 8 feet long. I can scarcely conceive of inorganic decay producing so sharply defined and uniform a result. Had the woody zone been a continuous cylinder, whose inner walls were unbroken, such conditions might have occurred; but this is not the case. As we have seen, it consists of a ring of detached wedges, separated from one another by radiating masses of cellular tissue, *which latter are continuous with, and prolongations of, the pith.* Now that the decay should not have extended along these prolongations (my primary medullary rays) but *invariably* have stopped at a crenulated line of which the tissues themselves afford no indication, is, to me, incredible. The uniformity of these results appears a sufficient proof of the correctness of my hypothesis. But there remains further evidence. In the case of *Calamopitus* we have to account for the infranodal canals. Whatever else these may have been, they were clearly outward prolongations of the central medullary cavity; and if decay was the agent producing the latter, it must equally have produced the former. In other words, this capricious agent, dependent upon a variety of outward conditions, though it respected the cellular primary medullary rays *as a whole*, yet attacked each one of them at certain circumscribed points, arranged in regular verticils which recurred with most unvarying uniformity, immediately below each node, from one end of the stem to the other. I must confess myself unable to accept such an explanation. These infranodal canals obviously existed in the living plant, in which they fulfilled some unknown function. At the same time they are but prolongations of a central cavity which must have coexisted with them. I therefore conclude that the entire structure resulted from the operation of that vital force which works out its designs with unbroken regularity, respecting boundary lines of which the eye, even though aided by the microscope, can frequently detect no trace.—October 3rd, 1871.

whilst the corresponding casts of the infranodal canals, unable to bear the pressure to which they were subjected, were forced down upon the medullary cast, on the surface of which they now appeared as very slightly elevated tubercles. The effect of these changes was the more or less complete reproduction, on the *exterior* of the carbonaceous layer, of forms which really belonged only to its *interior*, a reproduction which has occasioned much of the existing misapprehension respecting these fossils. Had the longitudinal ridges and furrows seen on the exterior surface of the carbonaceous film belonged to the corresponding portion of the living plant, they would have alternated with those of the medullary cast, as is shown by the restored diagram (fig. 1). But this is very rarely the case. Hence I cannot avoid the conclusion that external markings afford no absolute clue to the real nature of the external surface either of the bark or of the woody zone of the living Calamite. I have referred to the scars or tubercles so frequently seen on the medullary casts of Calamites; these are always arranged in verticils. The scar is sometimes circular, at others oblong, but always planted on the upper extremity of each vertical ridge, immediately below each node. The common aspect of the round variety is shown in Plate XXVII. fig. 27, and the oblong one in fig. 28. In both figures *l* represents the scars in question, *i* the node, *e* the inner angle of each woody wedge, and *c* the broad inner surface of each primary medullary ray. These scars are usually a little raised above the surface of the ridge upon which they are planted, but not invariably so. They, as I have already intimated, are the remains of the infranodal canals; and the degree of prominence which the scars exhibit has partly depended upon the extent to which inorganic matter has penetrated the interiors of the original canals, and partly on the chemical changes which the woody cylinder, through which these canals passed, has undergone. In the specimen of *Calamopitus* figured in the 4th edition of LYELL'S 'Manual of Elementary Geology' (fig. 478), as well as in my memoir on *Calamopitus* (fig. 1), we see that in the sandstone casts these scars are merely the bases or remnants of what that fine specimen exhibits so perfectly, viz. a verticil of radiating projections, each more than $\frac{1}{8}$ of an inch in length, and exactly resembling the spokes of a wheel, of which the central medullary cast is the nave. This specimen demonstrates that the common conditions seen in figs. 27 & 28 are very deceptive ones, which would inevitably mislead a student who had seen no other form; but read in the light afforded by the specimen above referred to, and of those represented in figs. 22, 23 & 25, their history becomes simple enough.

Before quitting the common forms of these internal casts I would call attention to a feature which I have noticed in several specimens, but of which I have as yet discovered no explanation. Plate XXVII. fig. 29 represents four internodes of a very long Calamite in the Cabinet of Mr. WILDE, of Glodwick Collieries, Oldham. The drawing is of the natural size. In the two lowermost internodes (k, k), and to a large extent in the uppermost one (k'), the longitudinal grooves are regularly parallel with each other and, like the internodes themselves, uniform in size; but in the shorter internode (k'') this is not the case. Plate XXVII. fig. 30 represents a few of the ridges and furrows of the latter enlarged about three diameters. Some of them are *much* thicker at one end than

at the other, whilst there are few in which the two sides are quite parallel*. But besides this peculiarity, the internode itself is unlike its neighbours, being only about half their length. Were this all, the internode might be regarded as an accidental anomaly; but when phenomena appear in regularly recurring series such an explanation is inapplicable. In Mr. WILDE's fine specimen, of which fig. 29 represents a very small portion, every *eighth* internode exhibits these peculiarities. Similar appearances are seen in another specimen in the same collection, but here they appear in every *fifth* internode. I have as yet failed to correlate these appearances of the medullary cast with any known external features of Calamites, but that they have some special significance cannot be doubted; they most probably indicate some specific features of the plant to which they belonged.

Amongst the abundant stems of Calamites exhibiting the peculiarities upon which I have already dwelt, are others of a very different aspect. Plate XXVII. fig. 31 represents a well-marked example of a not uncommon type belonging to Mr. NIELD, of Oldham. In the ordinary forms the casts are equilateral, being thickest towards the centre of each internode, and having their smallest diameter at the constricted node; but in the examples under consideration these proportions are reversed. The protuberance of the node is the most strongly marked on one side (fig. 31, *i*), whence a branch has been given off. The longitudinal surface-furrows are very strongly marked in the immediate vicinity of the node, but as they recede from it they become faint in outline and are at least doubled in number, indicating that, in the internodes (*k*), the number of the longitudinal woody wedges was multiplied, whilst the primary medullary rays became less conspicuous than at the node. On examining the prominent but concave phragma (Plate XXVII. fig. 32) or cicatrix left by the detached branch, we discover that there is a marked difference between the half of it above the nodal constriction and that below the line. In the former the surface-grooves and ridges bend uninterruptedly over the projecting margins of the cup-like cavity, and are prolonged to its centre; but in the latter, though they enter the cup, they are arrested very near its margin, the rest of the lower half of the cavity being occupied by fractured matrix. These differences indicate corresponding ones in the relations of the branch to each of the two internodes; it springs from, and is organically connected with, the one below the node, whilst it is merely in contact with the one above, which it indents by the pressure occasioned by the growth of its truncated base. Plate XXVII. fig. 33 represents a similar specimen to the last, only it retains the branch belonging to the middle node in its normal position, which is rarely the case. In these two and some other allied specimens the branches spring from the main stems at definite angles, indicating some regularity in the phyllotaxis of the plant. The amount of the angle of divergence depends upon whether the growth has taken place from left to right, or from right to left. If the former, the angle of divergence has been 140°, which, according to BRAUN's method of indication, would give the fractional symbol $\frac{7}{18}$; but if the latter, that angle would be 220°, giving the symbol $\frac{11}{18}$. I have met with a few stems in which these branches were given off in verticils, three being planted on each node; and I have

* A similar condition is represented on one out of four internodes in Tab. 6, fig. 7 of PETZHOLDT's "Ueber Calamiten und Steinkohlenbildung." The plant figured is *Calamites approximatus*.

observed that those of one node are intermediate, or alternating with, those of the contiguous one.

A question is at once suggested by these specimens. What are they? Are they a distinct species of Calamite, or are they merely some specialized portion of well-known forms? The plant which BRONGNIART figured and described under the name of *Calamites ramosus* exhibits some features in common with them (Histoire de Végétaux Fossiles, pls. 17. figs. 5 & 6), as also do some examples of his *Calamites arenaceus.* But after examining a large number of specimens, I have come to the conclusion that they have been *vertical* subterranean rhizomes. One specimen in the Cabinet of Mr. WILDE especially confirmed this conclusion. At its lower extremity it exhibits all the appearances of fig. 31, but at its upper end, after giving off a strong branch, it is prolonged in the form of an ordinary Calamite. A similar specimen in the Cabinet of Mr. NIELD does the same. That these rhizomes have been vertical is demonstrated by the direction of the branch in fig. 33, by the verticillate arrangement of the branches in the variety just referred to, and by the two significant specimens. When large branches are met with, detached from such parent rhizomes as fig. 31, the base, composed of several internodes, is seen to be truncated as shown in Plate XXVII. fig. 36; but very frequently the concentric shortened internodes are much more numerous than in that example. These internodes exhibit, even at the truncated base, the same longitudinal ridges and furrows as occur on the free stems, indicating that their woody wedges, to which these lines are due, radiate regularly from a central point representing the medulla. Of course such specimens as that figured are mere casts of the medullary cavity, and beyond indicating the size of the internodes and the arrangement of the innermost margins of the woody wedges, they throw no light upon the actual relations of the woody zone of the branch to that of the rhizome; but they do show that the medulla of the branch is only connected with that of the central rhizome by an exceedingly small cellular union, represented in fig. 36 by the dot forming its central point.

M. GRAND' EURY has published (Comptes Rendus, *loc. cit.*) some interesting statements respecting the Calamites of St. Etienne. He found long rhizomes running away from a central stem, giving off aërial shoots, and then continuing their subterranean course to repeat the process at successive and more distant intervals. But the condensed abstract of his memoir published in the Comptes Rendus does not enable me to identify his specimens of *horizontal* growths with those which I have just described, though the latter are the only ones I have met with in England which I can regard as rhizomes. Neither have I been able to satisfy myself as to the exact relationships between these rhizomes and such specimens as that represented in fig. 34, a common type of the subterranean base of an aërial stem, and a form which now demands a moment's consideration.

The conical bases of Calamites are not uncommon. Sometimes they are very obtuse, the internodes diminishing rapidly in size; at others they are drawn out in a more tapering manner. Occasionally they are quite straight, but much more frequently they are curved, as if they were lateral shoots from some other structure, which in the curved examples has doubtless been the case. The lowermost internodes of many such specimens

exhibit verticils of irregularly disposed protuberances (Plate XXVII. fig. 34, *p*) planted upon the inferior extremity of each internode. These I have always regarded as indicating the exact position of the roots of the Calamites in relation to each node, viz. immediately *above* it. The specimen represented in Plate XXVIII. fig. 35 settles this point. Mr. BINNEY had already published, in his Monograph, a drawing of a fine specimen (page 5), showing a large number of the lowermost articulations giving off roots; but it afforded no clear evidence whether those roots were planted above, upon, or below each node. Mr. WILDE'S specimen leaves no room for doubt*. Several of its nodes exhibit similar indications to that shown in fig. 35, which is a sandstone cast detached from the exterior of the stem, preserved by Mr. WILDE; *p* is a smooth surface or cast from which the actual root has been separated, but the latter remains *in situ* at *p'*. At its extremity it appears to have divided into several slightly diverging branches, or, what is possible, its branched portion may have been broken off and the part left split by pressure. Two points are clearly indicated by this instructive example:—1st, the cast of the root is perfectly smooth, exhibiting none of the ridges and furrows which are so strongly marked on the internodes (*k*) of the parent stem; 2nd, these ridges and furrows pursue their course right across the base of the root, almost entirely undisturbed by its close contact with them. These two facts seem to indicate that the root is an adventitious structure, and that if it received any vessels from the woody zone they were few in number, and did not exist as a cylindrical prolongation of the exogenous woody axis of the stem into the root, as would be the case in a recent phanerogamous Exogen. That these roots branch at their extremities into the plants known as Pinnulariæ appears to be established on the testimony of so many observers that no grounds exist for doubting the correctness of the conclusion, though I have had no opportunity of verifying it.

A much more difficult question to be determined is the nature and position of the aërial branches. When we obtain specimens in which subterranean rhizomes are directly prolonged into aërial stems, we find that the large phragmata or cicatriculæ, seen in Plate XXVII. figs. 31, 32 & 33, give place to others similarly located, but becoming very much smaller in size as soon as the stem emerged from the ground. The upper portions of the curved lateral aërial stems, appear entirely devoid of all but these smaller cicatriculæ. Hence we may conclude that in all instances the aërial branches were of small diameter. These inferences are sustained by what we know of their minute organization. In my memoir on *Calamopitus* I gave figures of several *transverse* sections of these branches as seen in *tangential* sections of the main stem (*loc. cit.* tab. 3, fig. 6; tab. 4, fig. 15), whilst in a diagram representing a longitudinal section (tab. 5, fig. 17) I showed how these branches spring directly from the innermost part of the woody zone exactly at the node; at the same time I pointed out that, in the sections in question, the diameter of each branch never exceeded the width of one of the small longitudinal ridges, seen on the surface of each Calamite. I have now further evidence of the correctness

* This position of the roots was long ago shown by LINDLEY and HUTTON in the 'Fossil Flora of Great Britain,' Tab. 78 A; but I have again dwelt upon it because the fact was disputed at the Liverpool Meeting of the British Association.

of these conclusions. Plate XXVIII. fig. 37 represents part of a longitudinal section of the specimen of *Calamopitus*, of which figs. 19 to 25 inclusive exhibit other aspects. Like figs. 23 & 24 it is a vertical section which has traversed one of the horizontal portions (37 *e*) into which each longitudinal canal, in that variety, divides at the node: we have the usual arched nodal arrangement of the wood-vessels; and at *m* is indicated the position of a branch of which the tissues have perished, such a branch being identical with those indicated in Plate XXVI. fig. 25 *m*, at the lower extremity of the primary medullary ray b' in fig. 22, in Plate XXIV. fig. 13, and in Plate XXIII. fig. 2 *m*. In all these examples, except 25, we find evidence not only that the branches originate in the innermost part of the woody zone, receiving additional vessels from the latter as they proceed outwards, but in several of them we have proof that a minute amount of cellular tissue in their centre forms a pith, continuous with that of the parent stem, as I have already shown to be the case with the larger subterranean branches. In the *Calamopitus* already so often referred to this is very marked. In the section delineated in plate 4. fig. 15 of the memoir quoted, and reproduced at Plate XXVIII. fig. 38 of this memoir, we see that this cellular tissue already appears as a well-defined pith. But since the above description was written, Mr. BUTTERWORTH has placed in my hands a transverse section of a stem which has been about an inch in diameter without its bark. The section has taken a slightly oblique direction, one half of it passing *through* the node, and the other a little on one side of it. But fortunately the nodal portion passes through the finest example of an aërial branch I have yet seen, and which accords with the remarks already made. Plate XXVIII. fig. 39 represents the branched part of the section referred to. The plant has belonged to the same type as fig. 16, not being furnished with longitudinal canals*. The base of the branch (*m*), and which is inserted into the woody zone, is wedge-shaped. It takes its rise from the innermost or medullary angle of one of the woody wedges, the two halves of which (f', f') are pushed asunder by it as it proceeds outwards, receiving additional vessels from each half as it does so, increasing its diameter. At its medullary extremity its vessels are distinctly seen to be intermingled with some large cells, reminding us somewhat of the arrangement seen in fig. 3; but the chief portion of the branch consists of parallel vessels which pass directly outwards. I conclude that the section has cut through the branch a little on one side of its medullary portion, which consequently is not seen. The part imbedded in the woody zone, as well as the remaining portion of what has been external to it, consists of one undivided joint or internode, no transverse nodal constrictions being visible in it. At its emergence from the woody zone the branch has a diameter of ·101 of an inch, which is about one half more than that of the exteriors of the woody wedges in its neighbourhood. The wedge from which it springs is thus dilated by it at that point to about double its usual dimensions. These measurements sustain what I have already said respecting the small size of the aërial branches. We thus have the longitudinal transverse and tangential sections of Calamites combining to fix with approximate certainty the position of these branches in relation to the central axis. These conclusions

* See note on p. 488.

respecting the small size of the aërial branches lead to some practical suggestions. If they prove to be of universal application, they leave us in no doubt as to which are aërial and which subterranean branches. Wherever a branch has had a diameter equal to that of several of the woody wedges combined, it appears to have been a subterranean one. The aërial ones, on the other hand, have rarely, if ever, had a diameter, apart from their yet unknown cortical investments, exceeding that of two woody wedges. This conclusion agrees with what we see on the indisputably aërial stems of ordinary Calamites. I have already observed that, in such, the concave depressions, indicating the points from which branches have fallen, are very small. The form of these concavities, allowing for the effect of surface-pressure, corresponds exactly with that of the base of the specimen now described. The exceeding rarity of fossil stems to which branches remain attached, at least in British strata, indicates that the connexion between the two has been very slight. The conical base of the twig imbedded in the woody zone has been united with the vessels of the latter throughout its entire surface; but the suddenness with which the vessels have been deflected from a vertical to a horizontal course may have been a source of weakness, and led to the almost habitual disarticulation of the branches. These appear to have been detached as easily as the leafy shoot is screwed out of the top of a pine-apple, leaving, in like manner, conical cavities behind them impressed upon the casts of the pith.

I do not propose to enter at length in the present memoir into the questions of the foliage and fruit of the Calamites. Of the foliage I have not seen sufficient to enable me to balance the discrepant testimony of the writers who have preceded me. My indefatigable friend, Dr. DAWSON, of Montreal, thinks he is able to distinguish the leaves of Calamites from those of the fern-like *Asterophyllites*. M. GRAND' EURY regards the *Asterophyllites* as belonging to stems that have but a vague and distant resemblance to those of Calamites. Mr. CARRUTHERS is inclined to believe that *Asterophyllites*, *Annularia*, and *Sphenophyllum* are but modified forms of one common genus, and that they collectively represent the foliage of Calamites. The structure of the stems and branches which I have described exhibit so marked a tendency towards verticillate arrangements, that we should naturally turn to fossil leaves similarly disposed in searching for the foliage with which to clothe them. At present, however, our information does not appear to me to be sufficiently definite to enable us to settle the disputed question. The three genera named are the only ones found in the Coal-measures possessing the needful verticillate arrangements, and I have no doubt that we must seek the required foliage amongst them, but under what limitations is yet to be ascertained. The remarks just made apply with almost equal force to the fructification of the Calamites. Several varieties of cones or strobili have been found in the shales of the Lancashire measures belonging, or allied to STERNBERG's genus *Volkmannia*; but these rarely retain their internal structure. Mr. BINNEY has figured, under the name of "Cone of *Calamodendron commune*," one from ironstone, in which the structure is preserved; whilst Mr. CARRUTHERS has described and figured the same cone* under the

* Journal of Botany, December 1867.

name of *Volkmannia Binneyi*, and referred it to *Calamites.* I have figured and described another very distinct cone, resembling a *Volkmannia**, of which the central axis affords the clearest proof of a Calamitean origin. Besides this, I have before me a third form of cone, discovered by Mr. BUTTERWORTH, in which the fruit-bearing organs are arranged in verticils, but of which the structure is very different from that of either of the other two†. Of the form described by Mr. BINNEY and Mr. CARRUTHERS I have made a number of preparations, and have examined a still larger number of specimens. Hence I have now in my cabinet three distinct types of verticillate spore-bearing Cryptogamic cones, in which the structure is exquisitely preserved; but of these three, that which I have described in the Manchester Transactions is the only one of which the central woody axis exhibits the same organization as the Calamitean stems now described. It does so in the most minute details; hence I have no doubt respecting its Calamitean character; but the other two differ so widely from it, from each other, and from the stems of all known Calamites, that I greatly doubt the propriety of uniting them. When stems, however varied in minor details, exhibit so remarkable a conformity to a definite type as I have shown to be the case with all the varieties of Calamites, I cannot conceive of the central axes of their fruit-bearing organs becoming so widely divergent from that type as must have been the case if the fruits described by Mr. BINNEY and Mr. CARRUTHERS belong to the same genus. In the Calamite the medullary axis is purely cellular, and its vascular zone is the more external one even in the youngest twigs; but in the fruit in question these conditions are reversed. The vascular tissues are all found in the central part of the axis, involving a metamorphosis to which I know no parallel amongst living plants. I would speak on this question with the reserve demanded by limited information, but I am at present disposed to believe that the only Calamitean fruit *of which the internal organization has hitherto been ascertained*, is that which I have described in the Manchester Transactions. The others, I suspect, belong to the *non*-Calamitean forms of Annularian plants, whichever they may be.

We have as yet failed altogether to correlate, with accuracy, the fruits of which we know the organization, and those just referred to as found in the Coal-shales, and of which we only know the external forms. Much work remains to be done ere we can succeed in this part of an inquiry of which we have but touched the threshold. Are we under these circumstances in a position to determine the position of Calamites in relation to living plants? Recognizing the necessity of proceeding with caution, I think we are. A further question also presents itself. Are we to recognize two genera of Calamitean plants, or are they all to be included in one genus? I fear my valued friend ADOLPHE BRONGNIART will scarcely agree with my reply to this second question, though I think that if he had the opportunity of carefully studying the sections in my cabinet

* "On a new form of Calamitean strobilus from the Lancashire Coal-measures," Memoirs of the Literary and Philosophical Society of Manchester, 3rd series, vol. iv. 1869–70.

† This cone is the subject of a memoir by the author, just published by the Manchester Society, "On the Organization of *Volkmannia Dawsoni*," in which the new fruit is referred to either *Asterophyllites* or *Sphenophyllum*.—October 3rd, 1871.

he would soon be induced to do so. It is clear to me that the separation of Calamites into two groups, the one Cryptogamic and the other Phanerogamic, the former represented by Calamites and the other by Calamodendron, must be wholly abandoned. Internal organization affords no solitary fact upon which such a division can rest. The specimen illustrated in the figures 19–25 is one of the *thin-walled* types which M. BRONGNIART, and those who hold his views, would regard as an unmistakable *Calamites*. Prior to his recent visit to England, Dr. DAWSON held the views of the French botanist as strongly as he himself does; and when he saw the specimen in question, he unhesitatingly identified it as one which was *not* a *Calamodendron*. It was this, and some allied examples, which led the Canadian geologist to declare to the Geological Society of London that "specimens in the collection of Professor WILLIAMSON show forms intermediate between *Calamites* and *Calamodendron*, so that possibly both may be included in one family"*,—language which, when we remember the strength of Dr. DAWSON'S previous convictions, bespeaks the true philosopher, to whom scientific truth is a sacred thing. The preceding descriptions and illustrations leave, I think, no reasonable room for disputing that the thin-walled and the thick-wooded plants, the latter being the *Calamodendra* of BRONGNIART, are but different species of one group, if they represent more, in some instances, than different ages of the same species.

The structure of the woody zone is unquestionably exogenous†. The arrangement of its vessels and medullary rays, its mode of growth, and that of its aërial branches all demonstrate the truth of my conclusions on this point. But the bark is the cellular covering of a Cryptogam. Whichever of the verticillate-leaved plants of the Coal-measures constituted its foliage, they are all equally cryptogamic. My strobilus, the Calamitean character of which it is impossible to doubt, is filled with round spores that are unmistakably Cryptogamic. The structure of the roots affords clear evidence that they were not the woody prolongations of the main axis seen in the roots of the Gymnospermous Conifers, but adventitious appendages of the Cryptogamic type. All these facts point to one conclusion, viz. that the Calamites were all Cryptogamic plants, but that they possessed a much higher organization than is seen in any of the Cryptogams living at the present day. Some writers affirm that the living Isoetaceæ exhibit an exogenous stem. Since I have had the opportunity of studying them for myself, I must confess I have failed to trace the evidence of the alleged exogenous growth; but the authority of HOFMEISTER and other botanists is sufficient to show that it exists; in the Calamites the proofs of such growth are incontrovertible.

There is no question that the only living plants with which Calamites can be compared are the Equisetaceæ, with which Mr. CARRUTHERS has unhesitatingly united them;

* Abstract of the Proceedings of the Geological Society of London, No. 217, May 1870.

† I have already pointed out the resemblance which a transverse section of a Calamite, made at an internode, bears to a similar section of a branch of the first year belonging to any exogenous plant. The Calamite may be regarded as exhibiting *permanently* a condition that is *temporary and transitional* in the living plants. This observation has especial reference to the non-multiplication of the woody wedges after their first appearance in the stem of Calamites.

but it seems to me undesirable to do so; though there are some points of resemblance between the two plants that sorely tempt a botanist to do so. But before attempting to determine the question, we must ascertain what are the several points of resemblance and of difference.

Different botanists have defined the Equisetaceæ in various ways; but most of these definitions of the family include two things: 1st, a sheath to each joint of the stem; and 2nd, hygrometric elaters attached to each spore. That the former of these has no existence in Calamites is universally admitted*. The plant figured in M. BRONGNIART'S great work under the name of *Calamites radiatus*†, exhibits what that distinguished botanist regarded as a true sheath. But I recently examined the specimen which is, or was, in the Strasburg Museum, and I satisfied myself that the supposed sheath is a mere verticil of leaves; a conclusion in which its eminent custodian, Professor SCHIMPER, fully agreed with me. Consequently the fact must be admitted, that one of the universally existent features of recent Equisetaceous plants does not appear in the fossil Calamites.

In his description of his *Volkmannia Binneyii* Mr. CARRUTHERS announces the discovery of spores with attached elaters, and on the strength of this supposed fact he further rests his conclusion that the Calamites are Equisetaceæ. But to this I entertain two objections. I have not seen Mr. CARRUTHERS'S specimen, but I have made sections of and examined a considerable number of strobili of the same species; and though I find spores in abundance, I have failed to detect a solitary example furnished with elaters. I think the evidence for a conclusion, so scientifically important as that to which the recognition of elaters would lead us, must be very clear and decisive before we accept it; and if we find one solitary example which may possibly be interpreted as bearing such organs and a large number of others, of the same species, in which no such organs can be found, we must be quite sure that the exceptional specimen is incapable of any interpretation that will bring it into harmony with all others of the same sort. Judging from his figures, I think it probable that, in Mr. CARRUTHERS'S specimen, the outer cell-walls of the spores have become accidentally ruptured, and the detached portions projecting from the spores have been mistaken for elaters. But even if, contrary to expectation, elaters should be shown to exist in these cones of Volkmanniæ, it still remains to be proven that the species in question belongs to Calamites; and until such connexion is established we must scarcely, on the strength of its possibility, alter our definition of a great and important natural family of plants.

I am not aware that the minute organization of the Equisetums has yet been illustrated in detail by any English writer; but it has been very effectively done by the late Dr. J. MILDE in his 'Monographia Equisetorum'‡. But even this admirable mono-

* If we regard the sheaths of Equisetums as verticils of coalesced leaves, the verticils of free leaflets supposed to exist in Calamites may be said to represent them. In the former case these verticils exist persistently on the deep subterranean rhizomes, as well as attached to the aërial stems. Nothing takes the place of these subterranean appendages in Calamites.

† Végétaux Fossiles, tab. 26, figs. 1, 2.

‡ Nov. Act. Academiæ Cæsariæ Leopoldino-Carolinæ Germanicæ Naturæ Curiosorum, Dresden, 1867.

graph leaves unnoticed some points of importance to my present inquiry. In Plate XXIX. fig. 41 I have represented a longitudinal section of a node in a cylinder of *Equisetum maximum*. Fig. 42 represents a segment of a transverse section of an internode of the same plant, whilst fig. 43 exhibits one of the canals (42, *e*) with its surrounding parts yet more highly magnified.

The internal organization of these plants presents some striking resemblances to that of the Calamites. Their stems are jointed and their internodes fistular, whilst the medulla stretches permanently across the stem at each node (41, *i*), forming a cellular diaphragm (41, *n*). There is a thick persistent medullary layer (41, *b*) separated from an outer or cortical layer (41, *h*) by a series of canals (42, *e*, 43, *e*), like those common in the Calamites, and which reach from node to node without penetrating either. At each of the outermost angles of these canals we find a few longitudinal spiral vessels (43, *g*), and in the mass of dense cellular tissue which lies externally to each canal, we have two other small clusters (43, *g'*) of similar vessels. Primarily cellular, these vessels become variously modified into annular, reticulated, barred, and scalariform types. The first glance at these sections tempts us to regard the canals (*e*) as the homologues of the longitudinal canals of the Calamite, and the vessels (*g*, *g'*) as the degraded representatives of the woody wedges, the copious vascular laminæ of which seem thus to have dwindled down to a few detached vessels. This resemblance is rendered yet more remarkable if we examine the corresponding vessels in some other species of *Equisetum*. Dr J. MILDE has shown that in *E. variegatum*, *robustum*, *brachyodon*, *hyemale*, and *ramosissimum*, these vessels are arranged in two laminæ, which radiate from the outer angles of the canal towards the cortex in two parallel lines. Let the intervening space between these lines be filled up with similar ones, and we should have the woody wedge of the Calamite; at the same time we have no evidence that this arrangement in the recent forms is the result of an exogenous growth*. When we turn to the longitudinal section (fig. 41), we again discover points of resemblance. The canals (42, *e*) are not very visible; but we can readily mark their course by tracing that of the vessels (*g*) which accompany them, whilst the outer series (*g'*) is also seen imbedded in cellular parenchyma. But the most striking features of this section appear immediately beneath the node. At this point we have a dense mass, convex externally, and consisting of very large reticulated vessels (41, *y*), in which the vessels (*g*, *g'*) coming up from below suddenly disappear. This cluster of enlarged reticulated vessels terminates very abruptly at the node, as if truncated, having previously given off a divergent cluster (41, *z*) to the base of an adjoining branch (41, *m*), and from which a few spiral vessels alone are prolonged into the branch, but between which and the cluster (*z*) whence they spring is placed an extension (*i'*) of the nodal septum which bounds the branch cluster (*z*) in the same way that the main node (*i*) does the cluster *y*. These dense additions to the vascular tissues in the neighbourhood of the node appear at first sight to represent the arched

vessels of the thickened woody zone in the same region of the Calamite; but when we compare the two more minutely, we discover them to be very different things. We have seen that, in the fossil genus, the woody zone is materially thickened at each node by the addition of a number of arched vessels, which differ only in their increased numbers from those of the internodes. There is no change in the character of the individual vessels, nor interruption to the continuity of their course. The woody zone of the inter node gradually thickens into a lenticular form as it ascends to the node, and as gradually diminishes again as it enters the internode above. There is no abrupt termination or change in the vascular mass, neither is there any nodal diaphragm crossing it and intersecting the course of the vessels; but it is otherwise in all these points with the Equisetums. Plate XXVIII. fig. 40 represents one of the coarsely spiral vessels (g) of fig. 41, taken from the point where it ascends from below, and enters the vascular mass ($41, y$). The ordinary spiral texture has already become yet more coarse and irregular (w), and this irregularity degenerates as the vessel increases in diameter into a large open network of lignine (x) deposited in the interior of the tube. Still higher the vessel is yet further enlarged, and now, forming part of the vascular mass ($41, z$), it is rendered angular by the pressure of other vessels like itself. Its angles (z) are here thickened by a continuous deposit of lignine, as in true scalariform vessels, and on its flat faces similar deposits occur in the form of minute and regular reticulations. We have nothing approaching to this condition in Calamites. We further discover in the node of the Equisetum that, in addition to the cellular diaphragm, or extension of the pith that stretches across the fistular cavity, a still more dense layer exists, not only within the diaphragm, but which, as shown in fig. 41, is continued in a direct line across both the vascular and cortical zones; at the exterior of the latter it merges in a second one ($41, i$) at right angles to the first, and which separates the base of the branch ($41, m$) from the main stem. In both instances, as already shown, this dense layer truncates the vascular masses y and z. The presence of this dense layer has perhaps something to do with one of the differences between the branches of Equisetums and Calamites, which in the former are very much more persistent than the deciduous ones of the latter appear to have been. I have shown that in the more matured stems of the Calamites the pith becomes wholly absorbed, which is not the case with the Equisetums. It may be urged that the reason for this difference lies in the fact that the stems of the former have been much more persistent than the annual growths of the latter. Such may be the case; but the distinction holds equally good in the case of the permanent subterranean rhizomes of the Equisetums, where also the pith remains intact, not even becoming fistular. The existence of these rhizomes in both instances presents another feature of resemblance; but as they occur equally in many other Cryptogams, *e. g.* the Marsileaceæ, they have little definite value in relation to the present argument. But this is not the case with the medullary rays that abound in the Calamites. Of course we should not expect to see the *secondary* ones represented in the Equisetums, because they do not possess the vascular laminæ between which these structures are located; but if the canals (e, e) of Plate XXIX. fig. 42 represent the similar ones in the Calamite, then the outward prolongation of the cellular pith (a)

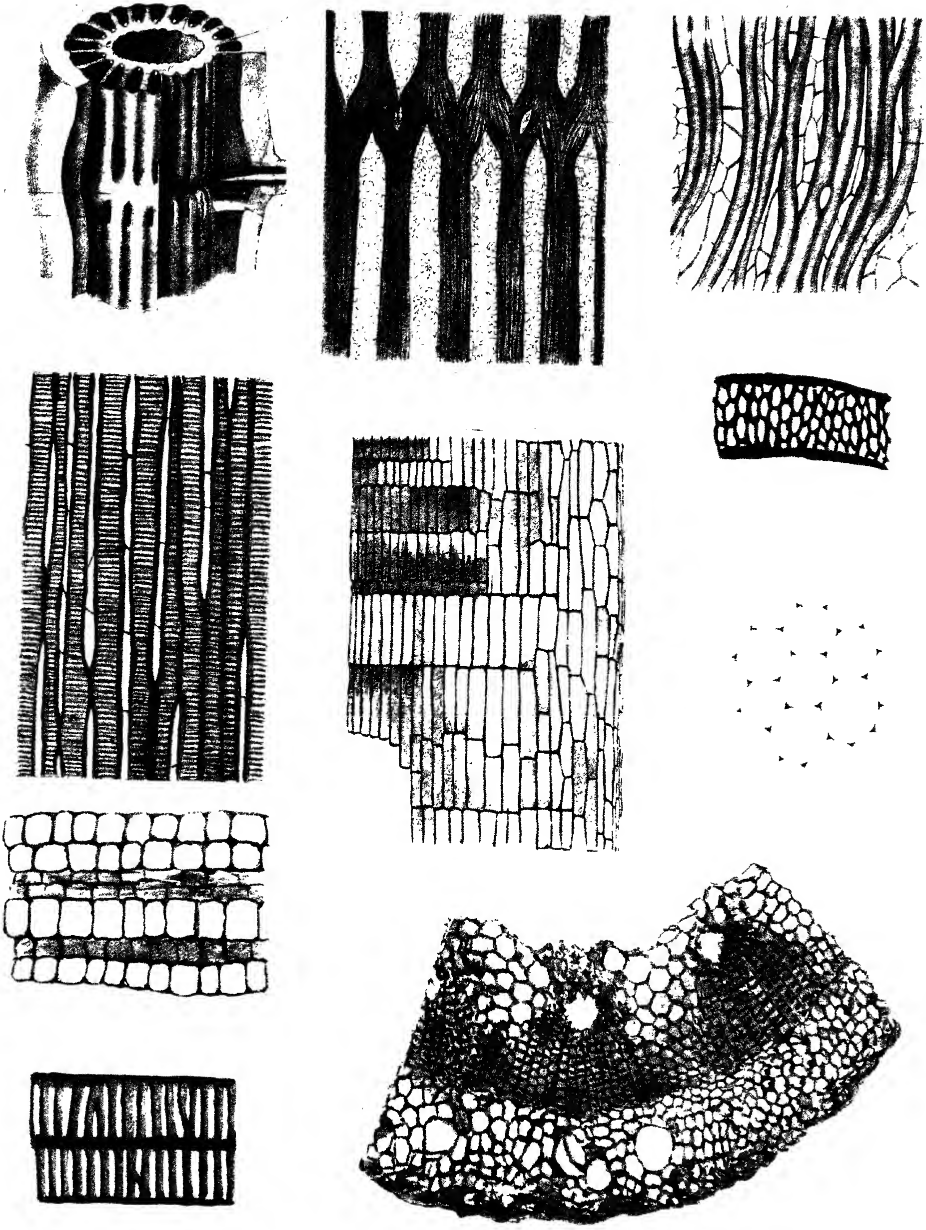
between the two canals, in the direction of *h*, corresponds with a primary medullary ray in a Calamite; yet we nowhere see, in the recent plants, the remarkable muriform arrangement of the cells common to all varieties of the latter objects. After fairly weighing the evidence for and against the admission of the Calamites amongst the true Equisetaceæ, as proposed by Mr. CARRUTHERS, it appears to me that the reasons against doing so preponderate over those which favour such a course; to disturb generally accepted definitions of a living family of plants for the sake of doing this seems to me unwise. I should therefore propose the recognition of a distinct family of *Calamitaceæ**, which from their complex organization must necessarily stand high up in the great Cryptogamic division of the vegetable kingdom, which division their exogenous stems would directly connect more closely than has yet been done with the true Gymnospermous Exogens—a connexion which I believe will be still further strengthened when some other plants of the Coal-measures, especially the Dictyoxylons, have received a more careful study than they have yet obtained†.

The only question that remains for consideration is that of generic and specific nomenclature. At present we have four generic names applied to Calamites,—*Calamites*, *Calamodendron*, *Calamopitus*, and *Astero-calamites*. The supposed distinction between the first and second of these I think I have proved to have no existence; and I doubt the sufficiency of the evidence, resting as it does upon a single specimen, for the recognition of Professor SCHIMPER'S genus *Astero-calamites* in this list‡. Nevertheless we have existing two very distinct types of Calamites, viz. those which have, and those which have not possessed the infranodal canals—a distinction that appears in every part of the fossil, whether it be the medullary cast or the carbonaceous investment; hence it is a distinction most easily recognized in nearly every specimen that falls into our hands. It appears to me desirable that we should distinguish these two types by different generic names. In order to avoid a needless multiplication of terms, I would gladly have given up my new genus *Calamopitus* and employed *Calamites* to represent one of the types and *Calamodendron* the other. But M. BRONGNIART retains his original views respecting the gymnospermous nature of his genus *Calamodendron*, and, doing so,

* Whilst deeming this course desirable I do not attach much importance to it, provided the accepted definitions of the natural order of Equisetaceæ be sufficiently extended to embrace these fossil forms. As those definitions now stand they exclude the Calamites.

† The greater portion of the Cryptogamous plants of the Coal-measures, with the exception of the Ferns, exhibit this exogenous growth of their woody zones, thus linking them with the Exogens rather than with the Endogens. The arborescent form which these carboniferous Cryptogams assume affords us a better opportunity of learning their true position in the vegetable scale than their dwarfed living representatives, none of which, with the exception of the Ferns, exhibit the arborescent condition. So far as the stem is concerned, we have in the fossil forms both the exogenous growth and the medullary rays of true Exogens; indicating close affinities with that highly developed class, and between which and the cellular Cryptogams they may be regarded as constituting a connecting link. On the other hand, may not the Ferns, with their detached bundles of vascular tissue, hemmed round and isolated by dense layers of woody fibre, connect in like manner the lower Cryptogams with the Endogens?—Jan. 26th, 1871.

‡ It appears to me not improbable that this curious specimen belongs to *Asterophyllites* or to *Annularia* rather than to the true Calamites.



he naturally objects to my employing his name in a new sense. Under these circumstances I propose designating all the plants which have no infranodal canals, indicated by the absence of verticils of round or oblong scars, *Calamites*, whilst to those which have possessed such canals, I would assign the name of *Calamopitus*, already applied to them in my previous memoir on the subject.

I am disposed to regard all existing specific names and definitions as worthless. They separate things that I believe to be identical, and confound others that are obviously distinct. The medullary casts, with their altered carbonaceous coverings, appear to afford very imperfect data for the determination of species. Practical purposes may require the provisional employment of a few such names, but they have little if any scientific value. One remarkable plant has been included amongst Calamites which I believe has no affinity with them,—I refer to the *Calamites verticillatus*.

I have to acknowledge the kind assistance which I have received during this inquiry from several gentlemen. Mr. Wild, of the Glodwick Collieries at Oldham, has allowed me the free use of his fine collection of specimens, and Mr. Nield, of the same town, has been indefatigable in giving me facilities for obtaining others. I have also received help from Mr. Whittaker, of Oldham, and from the Rev. W. Higgins, of Huyton. But I am mainly indebted to Mr. Butterworth, of Shaw, without whose invaluable cooperation this investigation could scarcely have been carried on.

Description of the Plates.

To facilitate comparative references, each letter of the alphabet up to *p* is employed throughout the following figures to indicate homologous structures.

- *a.* Fistular medullary cavities.
- *b.* Pith-cells.
- *c.* Primary medullary rays.
- *d.* Secondary medullary rays.
- *e.* Longitudinal internodal canals.
- *f.* Woody wedges.
- *g.* Vessels of woody wedges.
- *h.* Cortical layers.
- *i.* Nodes.
- *k.* Internodes.
- *l.* Infranodal canals.
- *m.* Branches.
- *n.* Nodal diaphragm.
- *o.* Divergent woody wedges.
- *p.* Roots.

PLATE XXIII.

Fig. 1. Diagrammatic restoration of part of the stem of a *Calamopitus*, from the left portion of which the bark has been removed exposing the exterior surface of the woody zone, and from the right both bark and wood are removed, revealing the outer surface of the pith. The branch at *m* exhibits its medulla (b'), surrounded by its woody zone, f'.

Fig. 2. Tangential section of a *Calamites* crossing the node.

Fig. 3. Portion from fig. 2, *a*, more highly magnified.

Fig. 4. Reticulated vessel of Calamites.

Fig. 5. Tangential section of part of a woody wedge with barred vessels and medullary rays

Fig. 6. Transverse section of part of fig. 5.

Fig. 7. Cells from the centre of the nodal medullary diaphragm of a *Calamites*.

Fig. 8. Longitudinal section of a primary medullary ray with medullary cells at *b*.

Fig. 9. Part of a transverse section of a Calamite with the cortex *in situ*.

PLATE XXIV.

Fig. 10. Longitudinal section of the same. *x*, spores accidentally introduced into the fistular cavity.

Fig. 11. Longitudinal section of a woody wedge, showing the pith, longitudinal canal, and secondary medullary rays.

PLATE XXIII.

Fig. 12. Barred vessels of Calamites.

PLATE XXIV.

Fig. 13. Tangential section of a Calamite near the pith, showing the transverse section of the commencement of a branch.

Fig. 14. Two woody wedges of a *Calamites* with large vessels.

Fig. 15. Part of the transverse section of a thin-walled *Calamites* with a smooth exterior to its woody zone.

PLATE XXV.

Fig. 16. Inner part of the transverse section of a woody wedge of a *Calamites* (?) unprovided with longitudinal canals.

Fig. 17. Transverse section of a portion of a woody wedge and part of a primary medullary ray of a *Calamites* in which vascular laminæ encroach upon the primary medullary rays.

Fig. 18. Tangential section of part of a primary medullary ray of the last specimen, further enlarged. *g*, vessels of the vascular laminæ; *c*, cells of the primary medullary ray.

Fig. 19. Transverse section (natural size) of a thin-walled *Calamopitus*, imbedded in its dark matrix.

Fig. 20. Small segment of fig. 19, more highly magnified.

PLATE XXVI.

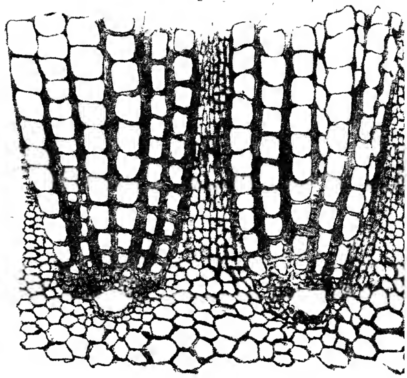
Fig. 21. Small portion from opposite the star in fig. 19, viewed diagonally, and exhibiting part of the outer surface of the specimen.

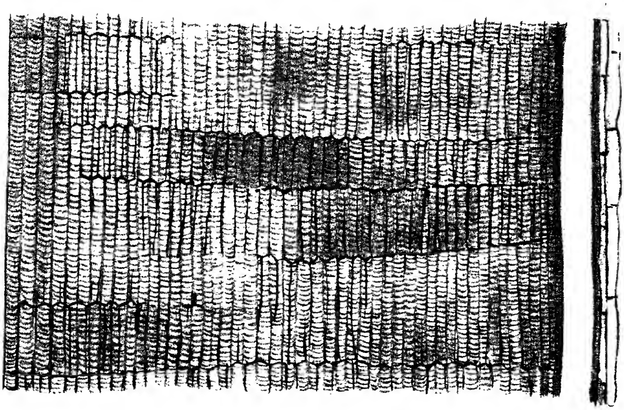
Fig. 22. Tangential section of the same *Calamopitus* crossing the node in the plane of the internodal longitudinal canals, or close to the surface of the medulla.

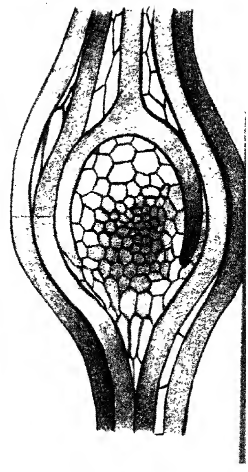
Fig. 23. Vertical section of part of the same crossing one of the infranodal canals, *l*.

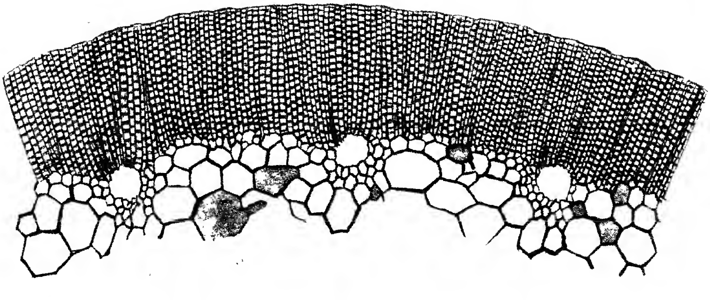
Fig. 24. Similar section across the node in the plane of the two woody wedges above and below it.

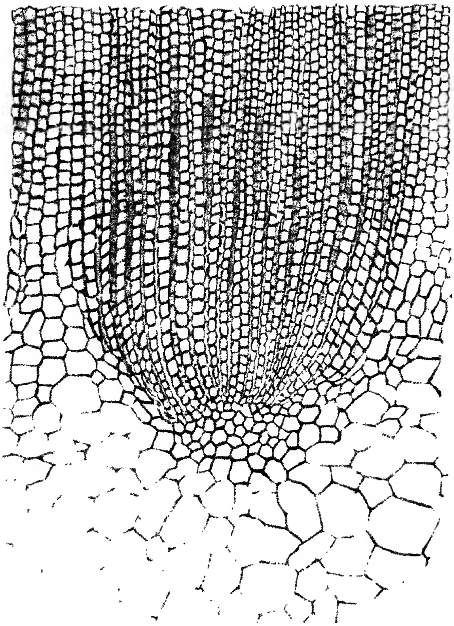
Fig. 25. Tangential section of the woody zone of the same near its external surface, and viewed as an opaque object.

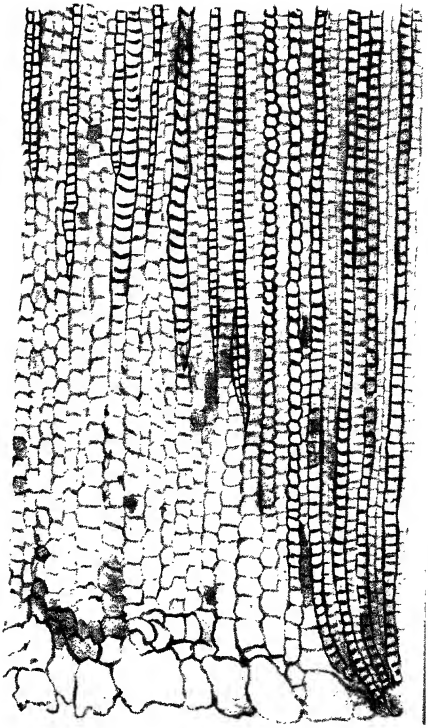


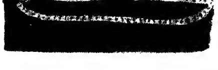


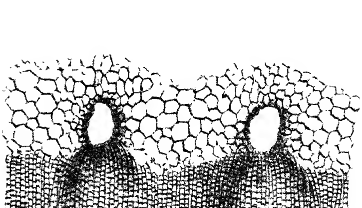


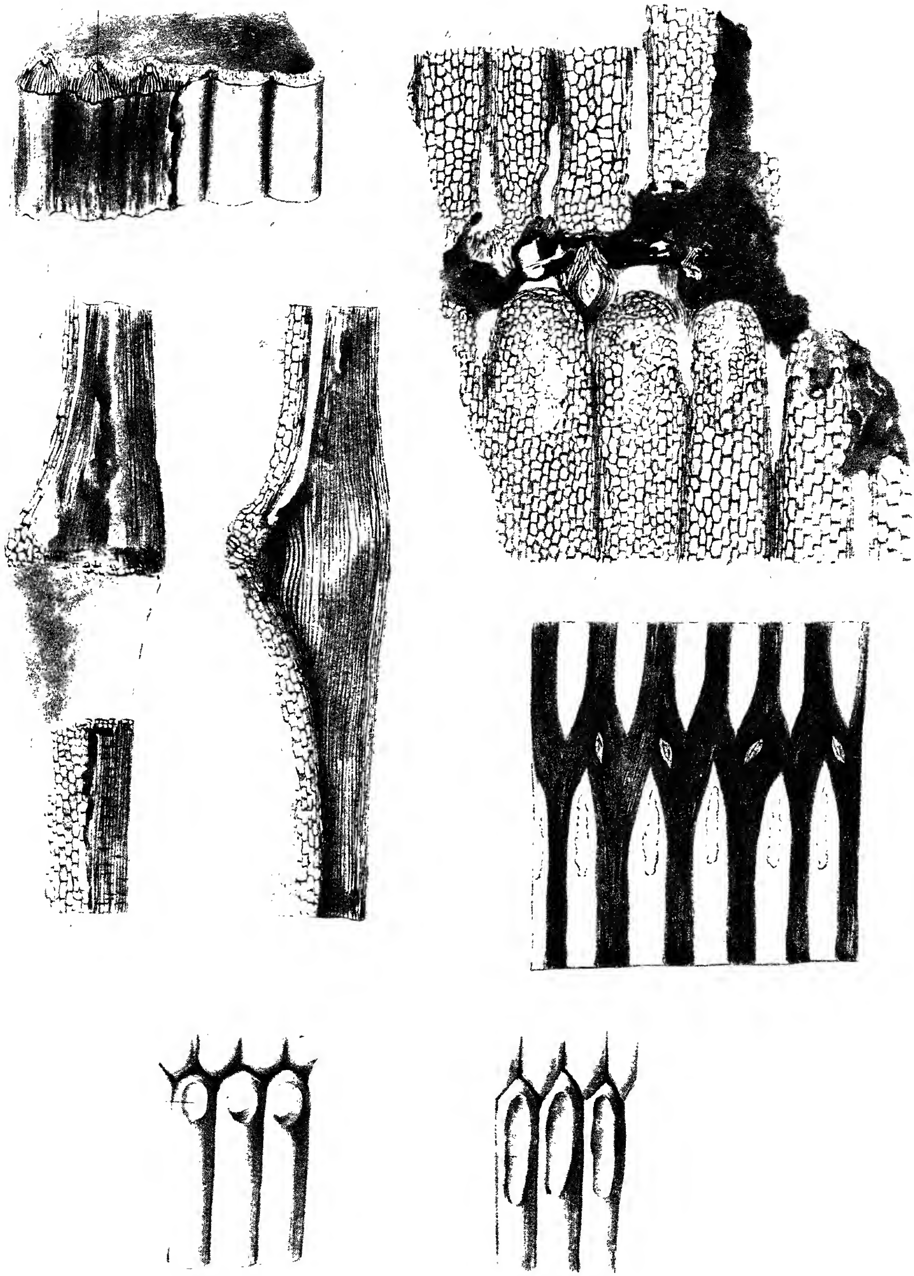


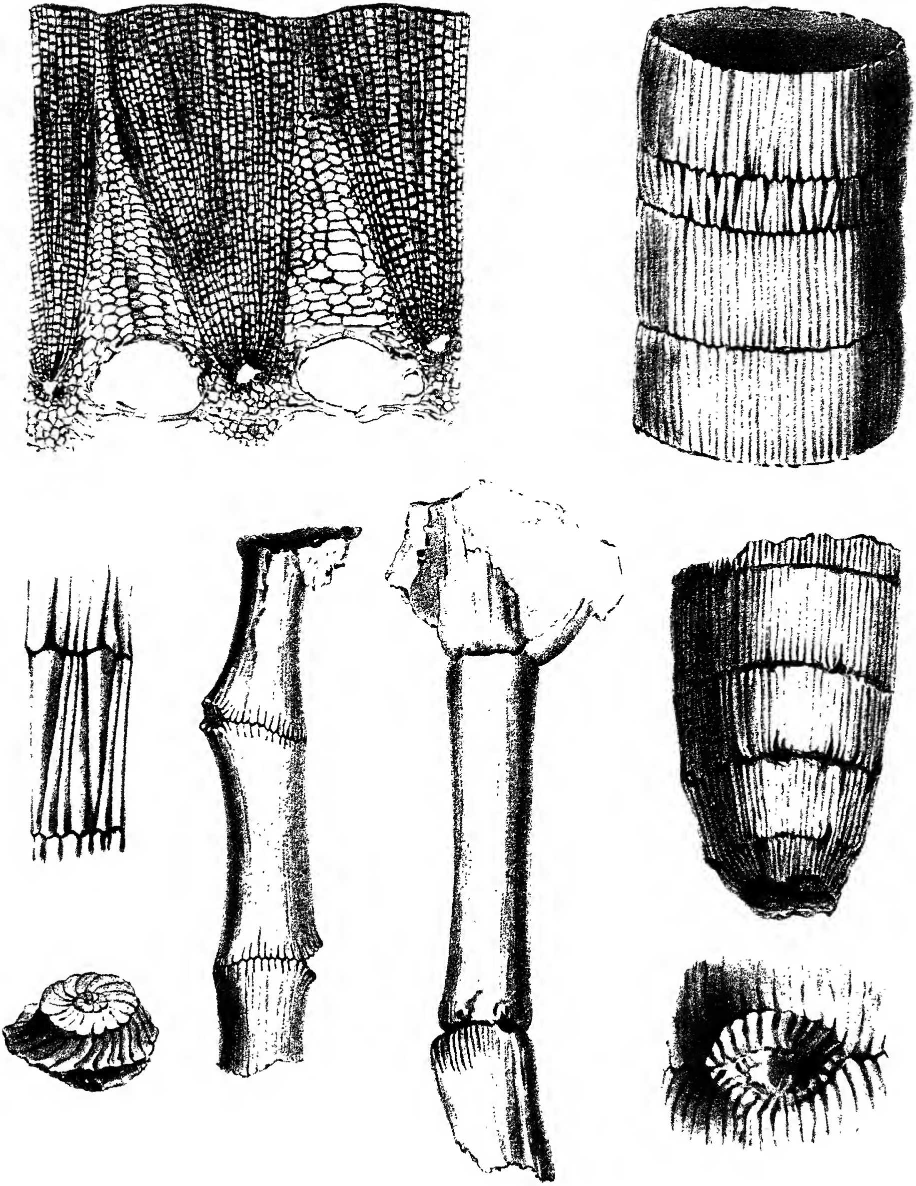


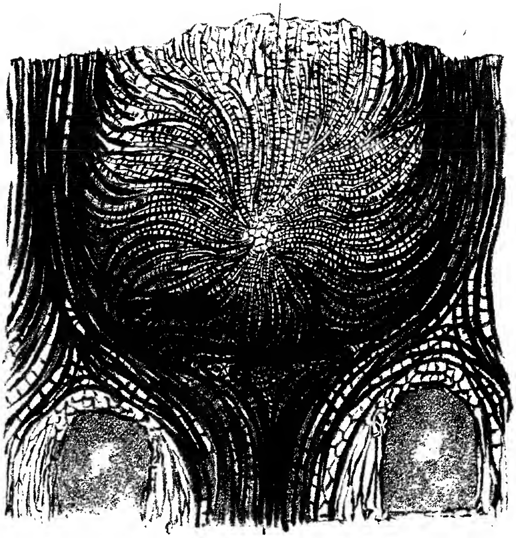
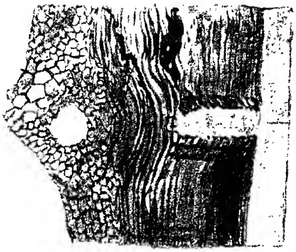


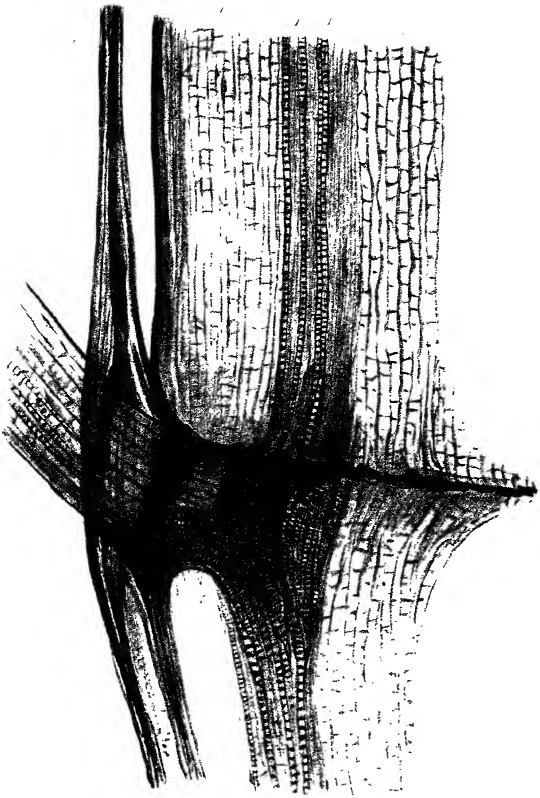


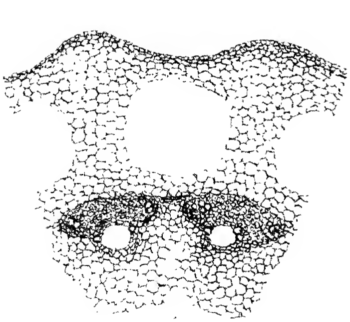










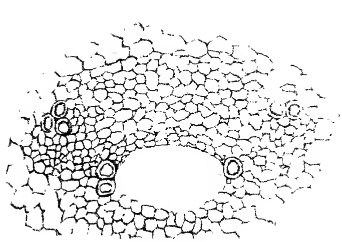


PLATE XXVII.

Fig. 26. Segment of the transverse section of a *Calamites* in which the pith was being absorbed as far as line of arrested absorption, *c'*.

PLATE XXVI.

Fig. 27. Round scars left by the infranodal canals of a *Calamopitus*.

Fig. 28. Oblong scars of another variety of *Calamopitus*.

Fig. 29. Internodes of an arenaceous medullary cast of a *Calamites*.

Fig. 30. Portion of the internode (*k''*) of fig. 29, enlarged.

Fig. 31. Portion of a subterranean rhizome of a *Calamites* with the phragmata or scars left by the detached branches.

Fig. 32. One of the phragmata of fig. 31.

Fig. 33. Part of a rhizome with a branch in situ.

Fig. 34. Subterranean base of the aërial stem of a *Calamites*, exhibiting the points whence the roots were given off.

PLATE XXVIII.

Fig. 35. Base of the root of a *Calamites in situ.*

PLATE XXVII.

Fig. 36. Lower extremity of the medullary cast of a subterranean branch.

PLATE XXVIII.

Fig. 37. Vertical section of the *Calamopitus* fig. 19, indicating the position of a branch, *m*.

Fig. 38. Tangential section of part of a large *Calamopitus*, exhibiting a transverse section of an aërial branch.

Fig. 39. Transverse section of a *Calamites*, exhibiting a longitudinal section of the base of an aërial branch.

Fig. 40. Single vessel from near the node of an *Equisetum maximum*.

PLATE XXIX.

Fig. 41. Longitudinal section of the stem-wall of an *Equisetum maximum*. *m*, part of the sheath.

Fig. 42. Segment of a transverse section of *Equisetum maximum*.

Fig. 43. One of the vascular canals (*e*) of the last figure, with its surrounding tissues, more highly magnified.

RELATIVE DIMENSIONS OF THE VARIOUS PARTS OF INDIVIDUAL CALAMITES.

Number of specimen.	Diameter of the entire specimen.	Thickness of wall of entire cylinder.	Thickness of woody cylinder independent of the pith.	Approximate proportion of the thickness of ligneous zone to that of the entire cylinder.	Distance between contiguous canals.	Approximate proportions of the distance between the canals to the thickness of the woody zone.	Average diameter of the longitudinal canal.	Average diameter of the largest vessels.	Number of woody wedges in the cylinder.	General remarks.	References to figures in the memoir.
1.	1·16	25 ·12	·25 ·12	$\frac{1}{4}$ to $\frac{1}{9}$	·10 ·12	1 to $\frac{1}{2}$	·02 ·005	·003 ·004	....	Pith nearly absorbed.................	Figs. 5, 8, 9 & 11.
2.	·66	·08	·06	$\frac{1}{11}$	·08 ·1	$1\frac{1}{3}$ to $1\frac{2}{3}$	·013	·0025 ·0016	about 80	Fistular cavity large	Figs. 19–25.
3.	·31	·1 ·12	·07 ·1	$\frac{1}{3}$ or $\frac{1}{4}$	·04 ·06	$\frac{1}{4}$ to $\frac{1}{8}$	....	·0025	21		
4.	·2	·035	·05	$\frac{1}{7}$	·03 ·04	$\frac{1}{5}$ to $\frac{1}{7}$	·015	·0025	13	Bark present. Thickness at internode 025	Figs. 9, 10.
5.	·62	....	·2 ·25	$\frac{1}{3}$	·07 ·11	$\frac{1}{6}$ to $\frac{1}{7}$	....	·005	24	No long canals	Fig. 16.
6.	·21	·09	·07	$\frac{1}{3}$	·045 ·05	$\frac{1}{5}$	·005 ·0075	·01	22	Vessels very large	Fig. 14.
7.	....	....	·81	....	....	....	....	·005	......	Section apparently made at a node. 354 vessels in linear series.	
8.	·25	·075	·045	$\frac{1}{5}$	·075	$\frac{1}{3}$	·01	·0025	15	....	Fig. 15.
9.	....	....	·27	....	·123 ·143	$\frac{1}{2}$	·01 ·013	·005	....	Pith-cells being absorbed	Fig. 26.
10.	·18	·133	·07	$\frac{1}{2}$	·033	$\frac{1}{6}$	·0066 ·01	·0025	21	Pith thick.	
11.	·5	·29	·24	$\frac{1}{2}$	·05 ·073	$\frac{1}{10}$ to $\frac{1}{7}$	·01	·005	27	One intercalated wedge without any canal.	
12.	·31	·123	·083	$\frac{1}{4}$	·05 ·06	$\frac{1}{6}$	·01	·0033	25		
13.	....	....	·23	....	·24	....	....	·005	....	Vessels in primary medullary rays...	Figs. 17, 18.
14.	·37	·12 ·2	·11 ·13	$\frac{1}{3}$	....	....	·01	·005 ·0064	21		
15.	2	....	....	....	....	....	....	·0017	above 60	Innermost part of woody zone alone preserved. No canals present.	Fig. 38.

In the above Table the distance between the longitudinal canals indicates the distance between the centre of one woody wedge and another The actual thickness of the ligneous zone will be subject to great variation according to whether the transverse section is made at the node or at the internode. In most of the above examples it has been made at the latter point.

XIX. *Description of* Ceratodus, *a genus of Ganoid Fishes, recently discovered in Rivers of Queensland, Australia. By* Albert Günther, *M.A., Ph.D., M.D., F.R.S.*

Received February 7,—Read March 16, 1871.

Contents.

Introductory Remarks.

At the beginning of last year my valued correspondent, Mr. Gerard Krefft, Curator of the Australian Museum, Sydney, informed me of the discovery in Queensland of a large *Lepidosiren*-like animal, of which he enclosed a photograph*, and which he assigned to *Ceratodus*, a genus of fishes known from fossil teeth only, naming the species after its discoverer, the Hon. William Forster, *Ceratodus forsteri*.

Before my reply had time to reach Mr. Krefft, a short communication of his on the subject was read before the Zoological Society of London (April 28, 1870); it is entitled "Description of a gigantic Amphibian allied to the genus *Lepidosiren*, from the Wide-Bay district, Queensland" (Proc. Zool. Soc. 1870, p. 221). The author describes in this note the principal points of the external appearance and dentition, and mentions also that "the skeleton is partly ossified, partly cartilaginous, the vertebræ being pure cartilage, and the ribs hollow tubes filled with a cartilaginous substance. The palate and upper part of the skull are bone, and the head is covered with two enormous scales."

* From this photograph the woodcut in the Proc. Zool. Soc. 1870, p. 222, is taken.

Mr. KREFFT'S specimen was evidently only roughly preserved, and the internal organs had been removed.

Another specimen, acquired by the Sydney Museum some months after, was forwarded to me, through the great liberality of the Trustees of that Institution, for the purpose of a more detailed examination. It reached me towards the end of August last, and proved to be a fine example, 38 inches in length, unfortunately without any of the soft internal organs, but with the external and skeletal parts nearly perfect. From this specimen, which I have deposited in the British Museum, the subsequent descriptions of the skeleton, scales, and fins are taken.

Professor OWEN received a third (male) specimen from Professor A. M. THOMSON, of Sydney, in the month of November; he very kindly handed it over to me; and it proved to be of the greatest value, as it was not only but little inferior in size to the one previously received (32 inches), but also had the soft organs in a good state of preservation. Finally, by the same mail, and apparently from the same source, the Secretary of the Zoological Society received a fourth example, smaller than the preceding (26 inches), but of great interest, because by it I was enabled to ascertain the structure of the female organs in an immature condition. As it is destined for the collection of the Royal College of Surgeons, I availed myself of the permission to dissect it only so far as to elucidate some points which could not be satisfactorily made out from the preceding specimens*.

Mr. KREFFT was certainly most fortunate in assigning, from the beginning, to this fish its proper place in the system, by describing it as "allied to *Lepidosiren*," and referring it to *Ceratodus*. Indeed the principal reason which appears to have induced him to state in so definite a manner an affinity to *Lepidosiren*, was his having been informed that this fish was in the habit of living temporarily on land; but the affinity extends much further, and consists in a nearly perfect identity of the skeleton, in the coexistence of a lung with gills, in a great resemblance of the intestinal tract, and also of the dentition—a resemblance recognized by Professor OWEN† at a time when *Ceratodus* was still considered to be a Shark, but denied by Mr. KREFFT‡.

The genus *Ceratodus* was established by Professor AGASSIZ§ for teeth which are found in strata of Jurassic and Triassic formations in various parts of Europe. Professor OLDHAM, F.R.S., has described teeth which were found in India, at Maledi, south of Nagpur‖, and differ scarcely from Muschelkalk examples; the stratum from which the

* Whilst this paper was passing through the press, the Trustees of the British Museum received from those of the Sydney Museum three other examples in a perfect state of preservation. One proved to be a female with fully developed sexual organs, a description of which will be added hereafter.

† Trans. Linn. Soc. 1839, vol. xviii. p. 331. "These teeth, in their paucity, relative size, and mode of fixation to the maxillæ, resemble those of the *Chimæra*, and some of the extinct cartilaginous fishes, as *Cochliodus* and *Ceratodus*."

‡ "This newly found amphibian has a dentition different from that of *Lepidosiren*."—Proc. Zool. Soc. 1870, p. 221.

§ Recherch. Poiss. Foss. vol. iii. p. 129.

‖ Geological Survey of India, p. 295.

Indian specimens have been extracted is not yet determined with certainty. These teeth have always been found isolated, sometimes with a portion of the bony base attached to them; no other part of the fishes to which they belong has hitherto been found associated with them; and with our present knowledge of the organization of the living representative of these extinct species, we can hold out but little hope that other parts may have been preserved which can be recognized as the remains of *Ceratodus*.

These fossil teeth (Plate XXXI. fig. 10), of which there is a great variety with regard to general shape and size, are much longer than broad, sometimes 2 inches long, depressed, with a flat or slightly undulated, always *punctated* crown, with one margin convex, and with from three to seven prongs projecting on the opposite margin. Professor AGASSIZ pointed out, from their shape, that there must have been only two of them in the upper jaw and the same number in the lower, that the convex margin was directed inwards and the prongs outwards—a view also held by PANDER*, who made considerable additions towards an accurate knowledge of the structure of these fossils.

We shall see hereafter that all the characters mentioned are found in the teeth of our living fish, that the teeth of some of the fossil species (for instance *Ceratodus runcinatus*) are surprisingly similar to those of the living (see Plate XXXI. figs. 9 & 10), and that their position and their number is exactly as shown by AGASSIZ and PANDER (to whom, however, the vomerine teeth were unknown). Therefore Mr. KREFFT was quite right in referring the recent fish to this genus†.

Geographical Distribution and Habits.

Before I proceed to the description of the fish, I may notice the little that is known of its geographical distribution and habits. Hitherto it has been found in Queensland

* Ctenodipt. Devon. Syst. p. 33.

† At a time when nothing was known in this country of *Ceratodus forsteri*, except the description and photographs which were afterwards published by Mr. KREFFT in Proc. Zool. Soc., doubts were expressed as regards the propriety of associating a recent fish with a genus living in the Jurassic and Triassic epochs. It may be said in reply that fishes very closely allied to, although generically distinct from, *Ceratodus* and *Lepidosiren*, are known from a much earlier epoch, viz. the Devonian (*Dipterus, Cheirodus, Conchodus, Phaneropleuron*), and that the scanty representation and wide distribution of this ichthyic type in the present epoch (one in Africa, one in South America, and one in Australia) is a sure proof of its extreme antiquity. Further, there is not the slightest evidence that the recent and fossil Ceratodonts differed from each other. It is true we have only the teeth for our guidance; but these are so well marked by peculiar characters, and the recent teeth so similar to those of certain extinct species, that we should be better justified in making generic distinctions among the fossil forms, than in separating the living from the extinct. Naturalists can be guided only by the evidence before them, and not by such vague hypothetical notions as that distance in space or in time has *necessarily* effected generic or other differences. I expressed these views at the meeting of the Zoological Society when Mr. KREFFT's paper was read. However, an anonymous reporter in a semipopular journal could not withstand the temptation of proposing a new generic name by translating the term "Wonder of the Rivers" into Greek, a term more expressive of the amount of knowledge of the author than of the peculiarity of the fish. If *Ceratodus forsteri* had proved to be the type of a distinct genus, the honour of naming it would, of course, have been claimed by that writer!

only. Mr. E. S. HILL writes*, "The fish *Ceratodus forsteri* is found in most of the rivers north of, and abundantly at Wide Bay; the northern limit, so far as is yet known, is the Burdekin, and the southern the Mary river; these fishes do not go higher than the brackish water, and at night leave the streams and go out among the reeds and rushes on the flats subject to tidal influence: this has been particularly observed on the banks of the Mary, where of a still night they may be distinctly heard. The aborigines catch many. Locally the Europeans call them 'Flat-head.' In the Fitzroy river, above Yamba and the Falls, there is a fish known to the aborigines as 'Barramundi' of excellent quality, and attaining the size of twenty pounds; it does not come down to the brackish or salt water" (June 30, 1870). The specimen sent to Professor OWEN was accompanied by the following notes†:—"The fish was captured at Gootchy, on a tributary of the Mary river, in freshwater about thirty miles inland. The common name 'Barramundy' given to the fish is applied to different fishes in different places." Specimens 6 feet in length have been mentioned. The intestinal tract of two specimens was found by me crammed full of more or less masticated leaves of various plants, which my colleague, W. CARRUTHERS, Esq., F.R.S., determined as fragments of various *Myrtaceæ* and *Gramineæ*. In both specimens they had lost the green colour entirely, being of a uniformly deep black, as if they had lain in water for some time, and were eaten when in a decomposing condition. The quantity of these vegetables contained in the intestine is enormous; and there is no doubt that they constitute the principal food of the fish. Some fragments of small shells, which, Mr. KREFFT informs me, have been found in the stomach, appear to have been swallowed accidentally with leaves. Whether a fish with such a diet "will at certain seasons rise to a fly" is a point requiring further observation. Some remarks regarding the alleged amphibious habits of this fish are, perhaps, better reserved for the chapter on the organs of breathing (p. 541). Nothing is known about its propagation or development.

External Parts.—Ceratodus forsteri and *Ceratodus miolepis.*

With regard to the *general habitus* of the body (Plate XXX.), *Ceratodus* much resembles *Lepidosiren*; but it is less elongate, and the large scales of the body, joined with fin-like paddles and distinctly rayed vertical fins, give to it an appearance which approches more nearly to the ordinary fish-type than that of *Lepidosiren*.

The *head* is broad and depressed, with the upper surface slightly convex and gently sloping towards the sides; the snout rather short, spatulate, becoming narrower towards its extremity, which is truncate. The total length of the head (to the gill-opening) exceeds considerably its greatest width; and its depth is less than one half of its length. The upper surface, the jaws, and cheeks are covered with thick skin perforated by small pores, which, however, are not regularly arranged. The gill-cover and the throat behind the mandibles are covered with large scales, scarcely different in size from those of the

* Note kindly communicated by Sir D. COOPER, Bart.

† Letter of Professor ALEX. M. THOMSON to Professor OWEN, dated Sydney, Sept. 6, 1870.

trunk. The eye is small, lateral, much nearer to the extremity of the snout than to the gill-opening; orbit with a free edge. The mouth narrow, its corner being at some distance in front of the eyes; lips rather thick and soft, simple, except on the side of the mandible, where the skin forms a simple, pendent fold.

At the angle of the mouth, and hidden below a duplicature of the skin, there is an opening wide enough to admit an ordinary quill (Plate XXX. fig. 2, *a*); it leads into a spacious cavity (*b*), irregular in shape, clothed with a mucous membrane, and containing coagulated mucus in which an immense number of mucous corpuscles are deposited. This cavity is separated from the cavity of the mouth by the membrana mucosa only, and there is no direct communication between them; a branch cavity runs forward into the interior of the upper lip.

The gill-opening begins on the side of the occipital region, and descends to below the pectoral paddle; its entrance is covered, not by the bony gill-cover, but by a broad cutaneous fringe of the operculum, the scales of this part of the cutis being very small.

The foremost portion of the trunk is depressed, like the head, but it soon passes into the compressed remaining portion—the boundary between trunk and tail being externally indicated by the vent only, which is *in the median line of the abdomen.* The tail evidently varies in length; it is sometimes shortened, as I have also observed in *Heterotis* and other diphycercal fishes; and it appears that injuries of this part, particularly when inflicted in early youth, are readily repaired. The depth of the trunk decreases slightly behind; and the tail diminishes rapidly in vertical dimension, till it ends in a thin point which is externally scarcely distinguishable, being enveloped by the vertical fin.

The dorsal part of this *fin* commences as a low membrane on the back behind the middle of the trunk, and the anal part at a short distance behind the vent. It is supported or formed by innumerable simple rays, becomes gradually deeper in the same proportion as the depth of the fleshy portion of the tail decreases, and finally tapers rapidly into a point.

The *limbs* consist of two pairs of paddles, similar in appearance to the termination of the tail; viz. a longitudinal axis, formed by the endoskeleton and muscles and covered with scales, is surrounded by a broad rayed fringe. These paddles are structurally identical with the fins of *Lepidosiren*; only the axis and also the fringe are much dilated. The pectoral and ventral paddles taper to a fine point, the former being longer than the head, the latter rather shorter. The ventral paddles are inserted at a short distance in front of the vent.

The entire body is covered with very large *scales*, presenting on the exposed portion a smooth surface with several faint concentric lines of growth; the margin is smooth and membranaceous. Towards the extremity of the tail the scales rapidly diminish in size, passing into the small scales with which a great part of the vertical fin is covered. The axial portion of the paddles is also covered with small scales; whilst the gill-cover is protected by small scales nearest to the branchial cleft, some large scales overlapping the small ones. The lateral line is clearly marked, its scales being perforated at the

base of the exposed portion. From the head to opposite the vent there are 22–23 scales in the lateral line; thence the size of the scales diminishes, and about seventeen more scales may be counted in the continuation of the lateral line. So far our description applies to all the specimens known at present; but with regard to the number of longitudinal series of scales, a remarkable difference exists between the examples first known and the two others obtained at a later period from the Mary river.

In the former the middle of the trunk is surrounded by eighteen series of scales, five of which are above, and eleven below the lateral lines. This form has been obtained from the Burnett river; and the type of *Ceratodus forsteri* belongs to it.

In the second the scales are conspicuously smaller, and their external sculpture is less intricate than in the former. The middle of the trunk is surrounded by twenty-one series of scales, six of which are above, and thirteen below the lateral lines. This form may be named *Ceratodus miolepis*.

Structure of the Scales (Plate XXXI.).

In an example of *C. forsteri* 38 inches long, a scale taken from the middle of the side is $2\frac{3}{8}$ inches long and $1\frac{6}{8}$ inch broad (figs. 1 & 2). Anteriorly its outline is rounded, the lateral margins being almost parallel, and meeting the hind margin at nearly a right angle. The exposed part of the scale is not more than about a fourth of the entire surface of the scale, and is covered with a brownish-black membrane. The outer surface, especially of the imbedded portion, is rough, the inner perfectly smooth. Four areas can be distinguished on the outer surface:—1, the exposed portion, covered with a thin brownish-black membrane; 2 and 3, the lateral areas of the imbedded portion, divided into oblique rows of trapezes arranged like scales of a Ganoid fish; and, 4, a median area, triangular in shape, extending from the centre of the scale to its posterior angles, and irregularly longitudinally striped. The inner surface (fig. 2) is quite smooth, but furrowed by lines indicating the course of blood-vessels, with which the membrane of the pouch of the scale is provided; they form an irregular network on the exposed and median areas, densely distributed on the former and sparingly on the latter, and follow the sutures between the trapezes of the lateral areas. The inner surface of the exposed portion shows also numerous pores by which vessels enter the interior of the scale.

The microscopic examination of the scale reveals the following structure*. In a vertical section (figs. 3 & 4) it is shown that each scale is composed of an inner thicker, and an outer thinner and harder stratum. The former consists of numerous layers of fibrous cartilage ("membranous laminæ" of WILLIAMSON), more numerous in the centre of the scale (I count about forty-two), the number of layers decreasing towards the margins. The fibres of one layer run parallel to one another, but generally cross those of the next layer at an angle of either 90 or 45 degrees (figs. 4 & 5). I have not found any of the isolated lenticular calcareous bodies which WILLIAMSON saw imbedded in the

* For comparison with the structure of other cycloid scales I refer to WILLIAMSON's original researches, in Phil. Trans. 1851, p. 647.

membranous laminæ. The outer stratum (fig. 4, *d*) is calcified, the inorganic matter consisting chiefly of phosphate of lime, and in much less quantity of carbonate of lime. The thinnest vertical and horizontal sections give but an indistinct image of the arrangement of its constituent tissues; the greater portion is composed of a great number of transparent calcareous globules, the form of which is less regular towards the surface, which is studded with conical prominences; and these prominences are constantly reproduced from the underlying globules, of which I have seen several assuming a conical form (fig. 4, *d*). When we destroy the organic matter by burning, the scale breaks up into the smaller divisions indicated by the superficial sutures. In the central parts the surface of the calcified matter (fig. 6) is then found to be raised into numerous conical spines, the base of each spine being surrounded by small foramina*. In the marginal and basal portions of the scale (fig. 8) the spines are placed on ridges running parallel with the longitudinal axis of the scale, and separated from one another by valleys. The bottom of these valleys is perforated like a sieve. The spines are more densely placed on the basal part of the scale than on the sides, and are entirely absent on its exposed portion (fig. 7), where the ridges run at a right angle to the free border of the scale, and are frequently connected with each other by transverse ridges.

In the scales of *Protopterus annectens* the outer stratum contains but a small quantity of salts of lime, which form a very thin sieve-like lamina.

In no part of the scales of *Ceratodus* have I discovered bone-corpuscles, which is very singular, inasmuch as Kölliker has not only found them in *Lepidosiren*, but also in those fishes the bones of which are provided with those corpuscles.

Nasal Cavity.

As in *Lepidosiren*, there are two nasal openings on each side (Plate XXXIV. fig. 3, *n*), both being situated within the cavity of the mouth, the anterior somewhat in front of the vomerine teeth, and the posterior outside of the front part of the molar. The nasal cavity is coated with the transversely folded pituitary membrane, the transverse folds being divided by one longitudinal fold; there are about twenty-three transverse folds.

Eye.

The eyeball has a transverse diameter of 13, and a longitudinal one of 10 millims. The optic nerve perforates it somewhat out of its axis. The pupil is circular; the lens spherical, and not steadied by a falciform process. Also the choroid gland is absent, as in *Polypterus* and *Lepidosteus*. The uvea is of a deep black colour, but leaves a great portion round the entrance of the optic nerve uncovered. The sclerotic capsule is strengthened by a single cartilaginous spheroid, which, in a horizontal section made through the greatest periphery of the eyeball, is three fourths of a millimetre thick.

The *Organ* of *Hearing*, being enclosed in the cranial cavity, will be described in connexion with the skull (p. 526).

* Resembling the sculpture of a shield of *Asterolepis*.

Dentition.

The *dentition* is essentially that of *Lepidosiren*, and resembles that of certain extinct genera (*Dipterus*, *Cheirodus*, *Conchodus*, *Psammodus*, &c.), as we shall see hereafter. It consists of a pair of vomerine teeth, and of a pair of maxillary and mandibulary dental plates; but whilst the dentition of *Lepidosiren* is chiefly adapted for piercing and cutting, that of *Ceratodus* is modified for the functions of cutting and crushing. The *vomerine* teeth (Plate XXXIV. fig. 3, *v*, and Plate XXXV. fig. 1, *v'*) are broad and rather low laminæ with a convex and trenchant margin, the outer or posterior part of which is slightly serrated. Each lamina is 13 millims. long, and in the middle 5 millims. deep. They are inserted in an oblique direction to the longitudinal axis of the vomer, and meet in the middle at a right angle; being implanted in cartilage, they are slightly moveable. Each maxillary dental plate (Plate XXXIV. fig. 3, and Plate XXXV. figs. 1 & 2) is an oblong piece with a grinding-surface, a convex inner side, and with the outer side divided into six prominent trenchant ridges or prongs by five notches, of which the foremost is the deepest, the others becoming shallower posteriorly. The foremost ridge passes to the inner border of the tooth, which is likewise somewhat raised. The grinding-surface has a great number of minute depressions (punctuations). The total length of a maxillary tooth is 32 millims. ($=1\frac{1}{4}$ inch), and its greatest width 13 millims. ($=\frac{1}{2}$ inch). In form and size the mandibulary teeth (Plate XXXV. figs. 1–3) are very similar to the maxillary; only the grinding-surface is less uneven. These teeth are anchylosed to the bone, and inserted in an oblique direction: the upper teeth nearly meet each other in the median line; but there is rather a wide interspace between the lower. The double kind of action which these teeth have to perform may be easily understood when the mouth is closed. The flat surfaces of the upper and lower molars are opposed to each other, and serve for crushing or grinding food, whilst the sharp lateral ridges of one tooth fit into the notches of the opposite tooth (Plate XXXV. fig. 1) like the shells of a *Cardium*, this part of the arrangement being adapted for cutting. The foremost ridges of the upper molars are received in the wide space between the lower ones, the vomerine teeth being opposed to the concave dilatation of the symphysial part of the lower jaw.

In a vertical section of one of the grinders (Plate XXXII. fig. 1) it is seen that the real depth of the tooth (that is, of that portion which is formed by dentine) is much less than it appears from a merely outward inspection. It rests, in fact, on an elevated plateau of the dentary bone (fig. 1, *c*), which has exactly the same outlines as the tooth itself, and the substance of which passes so gradually into that of the tooth that it is only by the difference in the shade of colour that the boundary between osseous base and dentinal crown is indicated. This anchylosis, however, is limited to the circumference of the base of the tooth; for its central parts are separated from the bone by the extensive but shallow pulp-cavity (fig. 1, *b*). We must remember that our specimens of living *Ceratodus* are by no means aged individuals, certainly much smaller and younger than those gigantic individuals of extinct species must have been, of which teeth 2 and more inches long are preserved. In such fossil teeth no pulp-cavity is

visible, but the dentine passes into the bone across the whole base of the tooth. It is not at all improbable that the pulp-cavity disappears altogether with age.

In our specimens the structure of the bony base of the tooth differs in nothing from that of the remainder of the dentary bone (Plate XXXV. fig. 5): there is the same spongious structure, the same proportion of bone-corpuscles, &c.

Microscopical Structure of the Teeth (Plates XXXII. & XXXIII.).—The teeth of *Ceratodus* present that modification of the tubular structure which is known from *Cestracion*, *Ptychodus*, *Psammodus*, and other fossil genera*; the resemblance in this respect to the structure of a *Psammodus* tooth is particularly striking. In a vertical longitudinal section (Plate XXXII. fig. 2) the substance of the mandibulary tooth is seen to be traversed in the direction from the root towards its upper surface by about fifty-five or fifty-seven medullary canals, following a slightly undulated course, and running nearly parallel to, and at nearly equal distances from, one another. Some of them dichotomize; but no anastomosis can be observed between them. Their terminations clearly correspond to the punctate impressions on the surface of the crown; and those canals which do not actually terminate on the surface of the crown, have the ends surrounded by a great number of dentinal tubes ramifying in every direction. The boundaries between their respective systems are indicated by an intermediate space, into which the dentinal tubes do not penetrate, or in which only their minute terminations can be traced.

In a horizontal section (Plate XXXIII. fig. 2) made near the crown of the tooth the lumina of the medullary canals appear opaque, of an irregularly ovate shape, surrounded by an opaque ring. The opaque centre and opaque ring are separated by a clear interspace traversed by the wavy dentinal tubes. These tubes penetrate through the dark ring, branching off into a great number of finer and extremely minute tubules, so that the outer periphery of the ring appears to be surrounded by a crown of tubules like a fungoid growth. The width of the clear interspace is about equal to that of the dark ring. Sometimes the lumina of two or three and even more medullary canals are surrounded by the same dark ring; they represent the branches of an originally single medullary canal which has been split up by bifurcation into two, three, or more branches. Each of these branches retains its own system of dentinal tubes, not anastomosing with that of the next.

All these remarks refer to the principal vertical medullary canals in the body of the tooth; but towards the surface of the processes (prongs) of the tooth (in the same horizontal section) smaller medullary canals may be observed which have a horizontal position, running obliquely towards the surface, and ramifying in an irregular manner into dentinal tubes which penetrate near to the enamel-like surface of the tooth. These canals are entirely similar to those figured by PETERS in *Protopterus* (MÜLL. Arch. 1845, Taf. 3. fig. 4). There are no bone-corpuscles in any part of this section.

In a second horizontal section, made near the base of the tooth (Plate XXXIII. fig. 3), the lumina of the medullary canals are rather wider than in the first; they are open,

* See OWEN, 'Odontography,' p. 11.

not filled with opaque substance, and the opaque ring is narrower and very close to the periphery of the canal, with a very narrow clear interspace. On the level of this section fewer of the medullary canals are bifurcate. Bone ascends from the base of the tooth to the level of this section, especially at the ends of the tooth; and some bone-corpuscles are observed also in the notches between the processes near the surface.

In vertical (Plate XXXIII. figs. 4 & 5) and horizontal (Plate XXXIII. fig. 6) sections of a large tooth (2 inches long) of the fossil *Ceratodus runcinatus* I find exactly the same structure, making due allowance for the much greater age of the individual, which must have been very large, and for changes due to fossilization. We find the medullary canals arranged in the same manner, of nearly the same width, surrounded by a dark ring from which densely set (fungoid) dentinal tubes proceed. In consequence of greater attrition of the tooth's surface, all the stems of the medullary canals penetrate to the surface, and the corresponding punctations lead directly into the canals.

On the other hand, the microscopical structure of the tooth of *Psammodus* (Plate XXXII. figs. 6 & 7) does not sufficiently agree with that of *Ceratodus* to justify us in assuming, from this part alone, a close affinity between these two fishes. The medullary canals are conspicuously narrower, the dentinal tubes are much less numerous and more simple; there is scarcely a trace of a dark ring; and no tufts of surrounding minute tubules can be seen.

The dentition of *Lepidosiren* differs still more, as far as the minute structure is concerned. That of the upper tooth has been examined and figured by OWEN ('Odontography,' p. 168, pl. 59. fig. 4). The medullary canals form there a network; the main branches run "nearly parallel with the plane of the upper surface of the tooth." In the mandibulary tooth (which I have examined) (Plate XXXII. figs. 3, 4 & 5) the medullary canals are few in number, chiefly running from the base towards the upper parts of the tooth, emitting coarse processes which abruptly break up into very fine spreading ramifications. Near the anterior and posterior surfaces of the tooth the medullary canals are narrower, running in an oblique or vertical direction, parallel to the outer surface of the tooth, emitting dendritically branched processes. In a horizontal section of the tooth the lumina are not surrounded by a dark ring; their arrangement and size, as well as the distribution of dentinal tubes, is exceedingly irregular.

Before we proceed to the examination of the skeleton, it must be mentioned that the general arrangement of the *muscles*, especially of the trunk, is identical with that of *Lepidosiren*; the limit between the dorsal and ventral series is indicated by the course of a continuous mucous canal corresponding to the lateral line.

The Skeleton.

The greater part of the skeleton is cartilaginous; where ossification appears, it is in the form of a covering replacing the perichondrium and more or less completely enveloping the cartilaginous substratum, but never taking its origin in the interior and by

transmutation of the cartilage. A layer of white connective tissue sometimes binds the bone and cartilage together; but generally the bone rests immediately upon the cartilage. (See, for instance, the vertical transverse section through the mandible, Plate XXXV. fig. 5, or through the vertebral segments, Plate XXXVIII. figs. 3–8.) Most of the bones are thin, sometimes flexible; and those which are thicker, like the ethmoid, mandibularies, and pterygo-palatine, have a spongy texture with numerous small medullary cavities (Plate XXXV. fig. 5). Bone-corpuscles are found in nearly all the ossified portions of the skeleton; they have but few and very short processes. Their shape is subject to great variations; for instance, they are short in the sclero-parietal (Plate XXXIV. figs. 5 & 6) and other parts of the skull, and very elongate in the articulary bone of the mandible (Plate XXXV. fig. 6).

General Configuration of the Skull.—The Relations of its Cartilaginous and Osseous Parts (Plates XXXIV. & XXXV.).

The skull consists of a completely closed *inner cartilaginous capsule* (Plate XXXIV. fig. 2, and Plate XXXV. fig. 2) and an *outer incomplete osseous case*, to which, again, some outer cartilaginous elements are appended. In the former the confluence of cartilage is so complete that no distinct divisions are traceable by sutures; its parts can be designated only by reference to the locally corresponding bones of the teleosteous skull. The bones of the outer case have their outlines more or less distinctly marked; but, although they are few in number, their determination is very difficult. The vertebrate skull which approaches most closely the type of *Ceratodus* is that of *Lepidosiren*; and nearly every part of this latter has been differently named by its describers.

In order to facilitate the description of this skull, we may distinguish four regions of the *cartilaginous capsule*, viz.:—(1) the *central* region, much depressed, extending from the occiput to the orbit, enclosing the brain and auditory apparatus, raised in the middle into a slight longitudinal crest; (2 & 3) a *lateral* region on each side of the former, in which the cartilage is expanded into a broad thin roof (Plate XXXIV. *p*) covering the gill-cavity, and which, anteriorly, is formed into the suspensory pedicle (*s*); (4) the *facial* region, a continuation of the central, but much more narrowed, tapering in front, and armed (in its vomerine portion) with two incisor teeth.

The bones which form the *outer osseous case*, and will be described in detail hereafter, are attached to the cartilaginous case in the following manner. The upper side of the *facial* region is covered by a triangular bone (*ethmoid*) (Plate XXXIV. *b*, and Plate XXXV. *b*); it is intimately united with the underlying cartilage (which is perforated by the olfactory nerves), and forms the roof over the nasal cavity. At its hinder angle this bone forms a serrated suture with a second bone (*os frontale*) (Plate XXXIV. *c*, and Plate XXXV. *c*), which emits a process downwards and in front of the orbit, to form a firm connexion with the palatine portion of the *os pterygo-palatinum*. This latter bone (Plate XXXIV. *l*, and Plate XXXV. *l*) occupies the edge of the lower side of the facial region, and bears the large upper pectinated tooth. It is narrow, but very

strong, and forms, in front, a short and firm suture with its fellow, both bones diverging behind, the space between them being entirely filled by the anterior part of the basal bone (*o*).

The *central* region is covered above by a median and two lateral flat thin bones, the latter being the continuations of the frontals: the former might be designated as *sclero-parietal* (Plate XXXIV. *a*, and Plate XXXV. *a*). These bones are united with one another by squamous sutures, are more solid in front where they are connected with the ethmoid, and become thin and almost membranaceous near the occiput; they form a continuous roof over the skull, but are separated from the cartilaginous capsule by the enormously developed and dilated *musculus temporalis* (see Plate XXXV. fig. 2, *mt*). This muscle is expanded as in a carnivorous mammal, and covers entirely the upper surface of the cartilaginous brain-capsule, taking its origin from the longitudinal median crest, and from some protuberances above the occiput. The lower side of this region is nearly entirely covered by the *basal* (*sphenoid*) bone (*o*).

The *lateral* region has only one ossification; it is a covering of the outer edge of the suspensory pedicle; superiorly it is dilated into a lamina (Plate XXXIV. *d*, and Plate XXXV. *d*), which assists in forming the supramuscular roof of the skull. To the edge of the cartilage of this region is attached the *gill-cover* apparatus, which consists of an *operculum* (*h*) and *suboperculum* (Plate XXXV. fig. 1, *h'*); a small cartilage behind, and hidden by the opercle, may be regarded as a rudimentary *præoperculum* (Plate XXXIV. *k*).

A further insight into the relations between the cartilaginous and osseous portions of the skull is obtained by a vertical section along its longitudinal axis (Plate XXXV. fig. 2. To this figure we shall have to refer subsequently, and in the explanation of the figures at the end of this memoir).

Detailed Description of the Parts of the Skull.

We proceed now to describe those details of the definable bones, cartilages, and other parts of the skull which are not included in the preceding general account.

Distinct maxillary or intermaxillary elements are not developed; they are replaced by *facial cartilages*, which can only be separated artificially from the external cutaneous integuments, and pass into the front part of the cartilaginous skull, and into the suborbital ring.

These superficial facial cartilages (Plate XXX. fig. 2, and Plate XXXIV. fig. 1) may be divided into three groups:—1. The upper labial (*f'*), surrounding the front part of the upper jaw, extending over the nasal region, and attached to the side of the ethmoid and, at the upper anterior angle of the orbit, to the frontal; it is nearly continuous with (2) the infraorbital ring (*f*), in which three ossifications (*e*) are imbedded, viz. two on its origin behind the orbit, and one just below the eyeball. The infraorbital ring passes by a broad bridge into the lower lip. There is a large cavity (Plate XXX. fig. 2, *c*) with several divisions in the interior of this bridge; it is quite distinct from the labial cavity (p. 515), and appears to be entirely closed, without external openings; but

in one (badly preserved) example a wide foramen at the inside and near the angle of the mouth leads into this pouch; during life it was probably closed by the mucous membrane of the mouth. 3. The lower labial cartilage (see also Plate XXXV. fig. 3, *la*), forming a broad trenchant fringe round the extremity of the mandible.

The part of the cartilage which bears the two trenchant front teeth I consider to be the *vomer* (Plate XXXIV. fig. 3, *v*). It is very similar in form to the vomer of many Teleosteous fishes, and has two free lateral edges, whilst behind it is suturally connected with the pterygo-palatines; it is entirely cartilaginous, and passes into the sphenoid cartilage above the symphysis of the pterygo-palatines. The corresponding part in *Lepidosiren* has been regarded by Professor OWEN* as the intermaxillary, and by Professor BISCHOFF, PETERS, and others as belonging to the ethmoid or nasal bone, Professor BISCHOFF and PETERS considering the cartilaginous space *behind* the palatal arch to be the vomer. From the fact that the vomer occupies nearly always the same position in the skull of fishes, and from the frequency of its being armed with teeth, I conclude that this part of the *Ceratodus*-skull is the vomer. A comparison with the skull of *Menopoma* will also much assist in determining it as such.

Towards the upper surface of the skull the vomerine cartilage passes into the *ethmoid* or *nasal* portion†, which is easily distinguished by its double perforation by the olfactory nerves (Plate XXXV. fig. 1, *b'*, and fig. 2, *ol'*); its upper surface has a thick osseous layer (Plate XXXIV. fig. 1, *b*, and Plate XXXV. *b*), which appears as a distinct triangular bone without median suture, the apex being directed forwards, whilst its base forms a serrated suture with the frontals and sclero-parietal; anteriorly it passes into the rostral cartilage.

The *frontal* bones (Plate XXXIV. fig. 1, *c*, and Plate XXXV. fig. 1, *c*) are entirely separate from each other; their anterior portion, which forms part of the border of the nasal cavity, and also the upper edge of the orbit, is very solid and closely connected with the underlying cartilage; it emits, in front of the orbit, a strong, broad, concave process (*c'*) for union with, and support of, the upper molar. The posterior part is an oblong, thin lamina, extending to the hindmost edge of the skull, to which it is fastened by fibrous tissue; this part covers the temporal muscle, and is attached by squamous sutures to the tympanic lamina and sclero-parietal. It is the same bone which appears as a free process on each side of the head of *Lepidosiren*; Professor OWEN determined it as postfrontal (Trans. Linn. Soc. xviii. p. 334, pl. 23. fig. 5, *m*), adding that it represents the postorbital and supratemporal bones in Ganocephala (Anat. Vertebr. i. p. 85). HYRTL does not much differ by designating this bone as superciliary, which, although a distinct bone in most fishes, is in reality nothing but a detached part of the frontal.

* In referring to Professor OWEN's account of the osteology of *Protopterus*, not only his first memoir in the Linn. Trans. vol. xviii. is to be consulted, but also the figure (No. 41) in Anat. of Vertebr. vol. i., where some inaccuracies of the former account are corrected.

† So designated in *Lepidosiren* by PETERS and MÜLLER; BISCHOFF considers it to be the intermaxillary; and OWEN also does not distinguish it from the "tooth-bearing cartilage."

Bischoff's opinion, that it is homologous with the jugal bone, does not appear to have been adopted by other anatomists.

The space between the two frontals, which in *Lepidosiren* is filled by membrane, is ossified in *Ceratodus* (Plate XXXIV. fig. 1, *a*). Morphologically it is entirely distinct from the parietal of other fishes, corresponding to this bone in situation only. Its distinctness from the parietal of *Lepidosiren* is evident, as this is a partial ossification of the skull-cartilage, and covered by the musculus temporalis, whilst it forms a case over this muscle in *Ceratodus* (Plate XXXV. fig. 2). It is evidently a conspicuous example of ossification of a fibrous membrane, and may be designated as *sclero-parietal*. It is a single bone, without median suture, subquadrangular, and of the same thickness as the adjoining part of the frontals; it passes behind into an occipital fascia, serving for the attachment of superficial muscles of the neck.

To the description of the *pterygo-palatine** bone given above, I have nothing to add (Plate XXXIV. *l*, and Plate XXXV, *l*). The two pieces which are found distinct in Fishes and Reptiles, and indicated by the compound name chosen by me for this bone, are here evidently merged into one; a bony arch extending from the tympanic suspensorium to the vomer necessarily includes both those pieces. Anteriorly the bones are united by a distinct suture (Plate XXXV. fig. 2, *l'*), which seems to disappear entirely in *Lepidosiren*.

The *tympanic pedicle* (Plate XXXIV. fig. 3, *s*, and Plate XXXV. fig. 1, *q*), the substance of which is cartilaginous, is strongly compressed in the direction from the front backwards; it is strengthened on its inner surface by the apposition of the end of the osseous pterygo-palatine, on the outer by a curiously shaped bone (Plate XXXIV. fig. 3, *q*, and Plate XXXV. fig. 1, *q*) which, as it is at least part of the *os quadratum*, may bear this name. This bone forms a not very thick covering of the outer edge of the tympanic pedicle. At its upper end it is dilated into a thin triangular lamella (Plate XXXIV. *d*, and Plate XXXV. fig. 1, *d*), the body of which is detached from the skull, entering into the supracranial roof formed by the frontals and sclero-parietal, and being of the same structure as those parts. Each suspensorium terminates in two condyles, the inner of which is rather narrower and much longer than the outer; they are separated by a deep and rather wide groove; these two condyles correspond to two articular surfaces of each ramus of the lower jaw. On the hinder side of the tympanic pedicle, near its base, there is a small round tubercle for the suspension and articulation of the hyoid arch† (Plate XXXIV. fig. 3. *r*).

The body of the *mandible* (Plate XXXV.) is persistent cartilage; but its entire outer and inner surfaces are covered by bone, forming an articular and a dentary piece. In figure 1 the mandible is shown from the outer (articulary) side, in figure 2 from the inner (dentary). The relative position of the two osseous pieces is represented in figure 3. By making a vertical section across the middle of the mandible, as is repre-

* In *Lepidosiren paradoxa* the "os palatinum" of Bischoff.

† The agreement of the suspensory pedicle of our fish with that of the other *Dipnoi* is obvious.

sented in figure 4 of the natural size, and magnified in figure 5, it is readily understood why the fossil teeth are always found isolated, without, or with scarcely any trace of, their osseous support. The bones are of a spongy consistency, with a dense network of wide medullary canals and cavities. The articulary especially has in its interior large cavities filled with marrow. The tooth can easily be broken off, its inner basal margin being anchylosed to the dentary by a narrow strip of bone only, and the outer resting on a layer of connective tissue which intervenes between the tooth and the articulary. The cells of the cartilage (fig. 5) are arranged in concentric rows,—the outer strata containing a greater number of cells, of an elongate ovate shape (fig. 7), whilst the cells of the central portion are fewer in number and more rounded. The articulary and dentary bones meet near the top of a low but strong coronoid process, and again at the symphysis, which is formed by fibrous tissue, and may easily be severed by the knife. The two articular concavities are separated by a narrow groove; and this part consists entirely of cartilage. A wide oval foramen penetrates the cartilage in a vertical direction, opposite to and outwards of the second notch of the tooth (fig. 3, *ar'*). In front of the jaw the cartilage is expanded into a slightly concave lamella (lower labial cartilage).

The *gill-cover apparatus* (Plate XXXV. fig. 1) consists of the same bones as in *Lepidosiren*; but being more developed, it assists in determining the two slender bones in that genus. The *operculum** (*h*) is a flat subrhombic bone, fairly protecting the gill-cavity; its upper edge is inserted in a long groove of the tympanic cartilage behind the base of the suspensorium. To its lower edge is attached, by fibrous tissue, the long styliform *suboperculum*† (*h'*) terminating at a considerable distance from the mandibulary joint. A small moveable piece of cartilage is found inside of the articulary groove of the opercle (Plate XXXIV. figs. 2 & 3, *k*); it is a rudiment of a *præoperculum*. The gill-cover is adpressed to the head by a singular broad *muscle*, which takes its origin from the lateral edge of the occipital cartilage, attaches itself to the upper margin of the opercle, and penetrates into the soft portions of the gill-cover, descending as far as the subopercle. It is divided into eight or nine fascicles by fibrous sheaths which run parallel with the muscular fibres.

The *basal*‡ bone (Plate XXXIV. fig. 3, *o*, and Plate XXXV. fig. 2, *o*) covers the greater part of the lower surface of the brain-capsule; it is lance-head-shaped, broadest between the tympanic pedicles, tapering in front, and still more behind, filling out the entire space between the pterygo-palatines, and extending backwards far beyond the commencement of the vertebral canal, to the level of the third neural spine. It is a thin bone, except in the middle of its length, where large medullary cavities are imbedded in its substance.

* In *Protopterus*, "Præoperculum" of OWEN; "Kiemendeckelstück" of PETERS.

† In *Protopterus*, "Branchiostegal" of OWEN (Anat. Vert.); "Kiemendeckelstück, *o'*" of PETERS. There are also two opercular pieces in *Lepidosiren paradoxa*; but their shape and place of attachment to the skull are, according to BISCHOFF, widely different from the same in *Ceratodus*.

‡ Basi-occipito-spheroidal bone of OWEN in *Lepidosiren*.

Before leaving the description of the skull, I have to add some notes on the *cerebral cavity*, with which the acoustic cavity is in direct communication. These parts may be seen in the vertical section of the skull (Plate XXXV. fig. 2). The brain itself was, unfortunately, found to be destroyed; but we suppose that, as in *Lepidosiren*, it is much smaller than, and does not fill, the cavity; and it appears to be very similar in form to that of the other Dipnoi. We distinguish the broad canals for the diverging lobi olfactorii (*ol'*), a considerable excavation for the hemispheres, a small groove (the lowest portion of the cavity) for a well-developed pituitary gland (*pt*) which is still preserved, and the space for the corpus quadrigeminum and cerebellum, gradually passing into the medullary canal.

Opposite to the origin of the nervus trigeminus, on each side of the brain-cavity, there is a large irregular opening leading into the *cavity of the organ of hearing*; it is closed by membrane (*ac*), which is perforated by the acoustic nerve, and on which are distributed its branches. After the removal of the membrane, three irregular subdivisions of the cavity may be distinguished—one considerably larger than the others, and containing otolithic masses of a chalky appearance and easily dissolved in water. By microscopical examination they are found to be composed of the same prismatic crystals as in *Lepidosiren* (see Hyrtl, *l. c.* tab. 1. fig. 6). The three semicircular canals are rather wide; the planes of the two inner ones are vertical and partly visible through the semitransparent upper surface of the cranial cartilage (see Plate XXXIV. fig 2.). The organ is entirely enclosed within the cartilage, without any other opening beside the communication with the cerebral cavity.

The *hyoid* arch (Plate XXXV. fig. 1) is more complex than in *Lepidosiren*, and approaches more the Teleostean type. It consists of a pair of ceratohyals, a basihyal*, and glossohyal*. As mentioned above, the ceratohyal (*ch*) is suspended from a tubercle at the base of the tympanic pedicle; it is a long, subcylindrical bone, externally well ossified, dilated at its proximal extremity into a subtriangular lamella. The basihyal (*bh*) is short, thick, cartilaginous, interposed between the ends of the ceratohyals and the acutely conical glossohyal (*gh*).

The *skeleton of the branchial apparatus* does not differ from that of Teleosteous Fishes, but is entirely cartilaginous. There are five branchial arches, the last rudimentary and attached to the base of the fourth. There is no peculiar modification of any part of this apparatus; and the middle pieces have the usual groove for the reception of the vessels and nerves.

Vertebral Column (Plate XXX. fig. 2).

Ceratodus agrees perfectly with *Lepidosiren* in the structure of the vertebral column. Its axis consists of a simple cartilaginous chorda dorsalis enclosed in a thick fibrous sheath, and with a gelatinous cylindrical thread along its centre. No transverse divisions in this notochord are visible; and it passes uninterruptedly into the cartilaginous capsule

* Absent in *Lepidosiren*.

of the skull (Plate XXXV. fig. 2); yet in a vertical section of the skull the course and termination of its tapering extremity can be distinctly traced, as the cartilage composing it is of a somewhat lighter colour than that of the skull. The boundary line between the cartilage of the skull and the notochord is still more distinct under the microscope (Plate XXXVI. fig. 1). The cranial cartilage (*c*) is distinguished by a considerable number of cells, many of which are spindle-shaped, having each end produced into a narrow process. These cells are entirely absent in the central parts of the notochord (*n*), the substance of which appears in the form of bundles of undulated fibres running in the direction from the central cylinder towards the periphery.

The central gelatinous body (Plate XXXV. fig. 2, *cd'*) is continued nearly to the extremity of the notochord, which lies opposite to the entrance into the acoustic cavity. The caudal portion of the notochord is tapering behind; and its extremity terminates in a thread which is gradually lost between the upper and lower series of neural and hæmal elements, which, in one specimen, coalesce into two tapering bands, and are persistent further backwards than the notochord.

Wherever an organ is reduced to a rudimentary condition, individual variation occurs. Thus scarcely two specimens of *Ceratodus* will be found in which the caudal termination of the vertebral column is exactly alike. One of the most remarkable variations is figured on Plate XXX. fig. 3. The notochord, with its whitish fibrous sheath, terminates here abruptly at a considerable distance from the end of the tail, its termination (*n*) being rather obtuse. The neural and hæmal arches are continued beyond the end of the notochord, but, being no more separated by it, are now confluent, form one tapering band which extends to within half an inch of the tail, and *which shows a distinct vertical segmentation*. This is all the more worthy of notice, as a similar segmentation of the posterior extremity of the vertebral column has been observed in some other fishes with notochordal skeleton. In *Ceratodus* such a segmentation is evidently within the limits of individual variation; it is confined to a continuation of the neural and hæmal elements, and does not extend to the notochord.

The notochord forms the base for about sixty-eight sets or rings of neural and hæmal elements, the hindmost being quite rudimentary and so indistinct that it is impossible to give the exact number. Twenty-seven of these rings bear *ribs*; and the first (caudal) ring in which the hæmal apophyses coalesce into a spine is the twenty-eighth. The bases of the apophyses are so deeply imbedded in the fibrous sheath of the notochord as to be in immediate contact with the cartilage of the notochord. The boundary line between them may again be discerned by the colour, the cartilage of the notochord being yellowish and slightly iridescent, whilst the cartilage of the apophyses is bluish. Under the microscope the latter appears as cartilage without fibrous basal substance, but with a great number of regularly ovate cells; whilst the cartilage of the notochord shows a distinctly fibrous structure with similar cells sparingly distributed near its periphery, the cells disappearing entirely towards the central parts. We have seen, above, that no cells whatever could be found in the attenuated foremost portion of the noto

chord; its attenuation, therefore, appears to be due to the gradual loss of the outer cell-bearing strata.

The neurapophysis of each segment is perfectly distinct from the preceding and following; but all are most intimately bound together by intermediate fibrous ligaments, in which frequently true hyaline cartilage is deposited. Each neurapophysis (see also the sections, Plate XXXVIII. figs. 3–9) consists of a basal cartilaginous portion forming an arch over the myelon, and of a superadded second portion, which is separated from the former by a distinct line of demarcation, and the two branches of which are more styliform, cartilaginous at the ends and in the centre, but with an osseous sheath, and coalesced at the top, forming a gable over the *Ligamentum longitudinale superius*, which is enveloped in a layer of fat (Plate XXX. fig. 2, *l*). To the top of this gable is joined a single long cylindrical neural spine, again with the central cartilaginous centre enclosed in an osseous sheath, and with an upper swollen cartilaginous extremity. This is the arrangement of the first ten segments; but from the eleventh the cartilaginous swelling at the top begins to lengthen into a distinct interneural spine of the same structure as the neural. Further on, from the fifteenth segment, another interneural is developed in the same manner; so that in this part of the vertebral column we have the following series of neural pieces:—

a. Cartilaginous arch of neurapophysis (*c*) for the formation of the medullary canal (*d*).
b. Semiossified gable portion of neurapophysis (*e*) over the Ligamentum longitudi nale (*e'*).
c. Neural spine (*f*).
d. Lower interneural (*g*).
e. Upper interneural (*h*), to which the dermo-neurals (*i*) are attached.

Whilst the increase of the number of neural pieces is the consequence of a lengthening of the distal part of the segment, it commences to decrease from the base. We observe that all at once, from the thirty-third segment, the neural spine coalesces with the gable portion of the neurapophysis; further behind, this portion is gradually shortened and finally disappears entirely, so that only the two interneurals remain. Towards the end of the tail the neural elements are reduced to a low and narrow cartilaginous lamella (neurapophysis) and a single short interneural spine; and this latter piece disappears in the last two inches of the vertebral column.

In the first three segments the neur- and hæmapophyses of each set are confluent, but without forming complete rings round the notochord, as the hæmapophyses of a segment do not coalesce below (see Plate XXXVIII. fig. 3).

The *hæmapophyses* (Plate XXX. fig. 2, and Plate XXXVIII. figs. 3–9) are, in form, size, and structure, very similar to the neurapophyses: those belonging to the same segment coalesce below the notochord into a narrow transverse band, at least in the anterior portion of the trunk (from the fourth to the twentieth segment). Those of the first twenty-seven segments bear well-developed *ribs*, about 2 inches long, bent outwards and downwards in the fore part and middle of the trunk, and backwards and downwards

in its hind part. Like the neural and hæmal spines, and other parts of the skeleton, the basal and distal portions of the rib remain cartilaginous whilst the central part is enclosed in bone. The terminal cartilages penetrate for some distance into this bone, but without being continuous, the centre of the bone being filled with cellular tissue and marrow. There is no real joint between the rib and hæmapophysis; its cartilaginous head fits into a shallow concavity of the latter, both surfaces being united by short connective tissue, allowing no free motion to the rib. The *first rib* (Plate XXX. fig. 2, Plate XXXIV. fig. 3, x, and Plate XXXVIII. fig. 3) deserves to be particularly noticed, not only because it somewhat differs in shape from the others, but because the corresponding bone in *Lepidosiren* has received various interpretations. It is a long bone, considerably thicker and more cylindrical than the other ribs, horizontally directed outwards and slightly backwards, forming the posterior lower limit of the gill-cavity. Its insertion is opposite to the first neurapophysis, being joined to the first hæmapophysis, and not to the basal bone, which at this place is slightly contracted. It crosses the suprascapula, from which a short ligament passes to a small cartilaginous protuberance (x') on the front edge of the rib. This tubercle, which is placed exactly at the spot where those two bones cross each other, has evidently only a functional, and not a homological significance. PETERS (Müll. Arch. 1845, p. 12, pl. 2. no. p) is the first who distinctly notices this bone in *Protopterus*, as "a peculiar bone which is to be compared to a similar bone in *Batrachus*." BISCHOFF (Ann. Sc. Nat. 1840, vol. xiii. pp. 123, 126) represents the neurapophyses of the first vertebral segment in *Lepidosiren paradoxa* as occipitalia lateralia, and the neural spine of the same segment as a cartilage "qui remplace, en quelque sorte, l'écaille de l'os occipital;" consequently he describes the first rib as inserted in the occipital lateral and in the body of the sphenoid, declaring it to be the "os suspenseur de la ceinture pectorale des poissons." According to this view, suprascapulary elements would be absent in *Lepidosiren paradoxa*; future investigations must show whether this is really the case. HYRTL and OWEN (Anat. Vert. i. p. 83, fig. 41. no. 51) have adopted BISCHOFF'S view; and the bone is figured and described by the latter as scapula, in direct connexion with the pectoral arch. Finally, W. K. PARKER ('Monograph of the Shoulder-girdle,' 1868, p. 21) not only determines the homology of this bone, but also of the small cartilage attached to the middle of its length; the former is stated to represent the "large first pharyngo-branchial," the latter the "small unossified second pharyngo-branchial."

A comparison of *Lepidosiren* with *Ceratodus* shows that a positively defined boundary between the vertebral column and skull does not exist, that parts which in one genus appear to belong to the skull, are distinctly portions of the vertebral column in the other, and that, with regard to the particular bone which has been so differently interpreted by the authors named, the opinion first expressed was nearest to the truth. The peculiar bone in *Batrachus*, to which PETERS directed attention, is in fact nothing but the first rib (see GÜNTHER, 'Fishes,' vol. iii. pp. 167, 172), extending and fixed to the upper end of the humerus. Although its insertion into the top of the neurapophysis is a most

singular fact, the gradual change of insertion of the following ribs, first to the bottom of the neurapophysis, then to the centre of the axis, and finally to the parapophysis, shows clearly that this bone is a rib. Nor is the application of parapophysial elements of the first vertebra, for support of the scapular arch, of very rare occurrence in fishes; it is observed in *Bagrus* and other Siluroids.

The *hæmal* pieces of the tail (Plate XXX. fig. 2, and Plate XXXVIII. figs. 7–9) are built up similarly as the corresponding dorsal parts. We have again

a. A cartilaginous portion of the hæmapophysis (c^2), supplemented by

b. A hæmal spine (*l*), with its proximal end osseous and forked to form a canal for blood-vessels (*k*), but without a trace of a ligamentum longitudinale inferius.

c. Interhæmal first (*m*).

d. Interhæmal second (*n*).

These pieces remain distinct very nearly to the end of the tail; but all become gradually weaker and shorter, and also ossification ceases in some of the posterior ones. Finally the number is reduced by the coalescence of the hæmal spine with the hæmapophysis, and the pieces near to the extremity of the tail are quite rudimentary and scarcely distinguishable.

Small and short *dermo-neurals* (Plate XXX. fig. 2, and Plate XXXVIII. figs. 4 & 5 . . .) may be distinguished from the interneural of the seventeenth segment; they gradually increase in length towards the middle of the tail, where they are very long; they are all obliquely directed backwards, assuming a more horizontal direction the nearer they are to the end of the tail. They are exceedingly numerous, four or five or more corresponding to a single vertebral segment—and form a double series, one series on each side of the fin. This peculiarity, which *Ceratodus* has in common with *Lepidosiren*, reminds us of those fin-rays of Teleosteous fishes which can be more or less completely split into a right and a left half. The dermo-neurals of *Ceratodus* are not articulated to the extremities of the interneurals, but overlap them for a considerable distance of their length. The shape and arrangement of the dermo-hæmals is exactly the same as that of the dermo-neurals. No ossification takes place in either of them; they consist entirely of cartilage, in which numerous spindle-shaped cells are imbedded, many of these cells being produced at both ends into a very long process (Plate XXXVI. fig. 7).

The Scapular Arch (Plate XXX. fig. 2, Plate XXXV. fig. 1, Plate XXXVI. figs. 2 & 3).

The scapular arch of *Ceratodus* is, with regard to the persistence of the primary cartilaginous condition and to the development of superficial bones, extremely similar to that of *Lepidosiren**. The primordial cartilaginous arch consists of three pieces, viz. a single

* Professor Owen was the first who, from the examination of a small example (far from being "half-grown"), pointed out the primordial condition of the scapular arch in *Protopterus*, whilst Professor Peters was enabled to supplement the account of his predecessor from the examination of fresh specimens. Recently the shoulder-girdle of this fish has been made the subject of special research by Professor Gegenbaur and Mr. Parker; and it is singular that these two anatomists prefer to criticise the almost unavoidably imperfect first account, instead of availing themselves of the researches of the second of their predecessors.

median transverse strip (Plate XXXVI. figs. 2 & 3, *a*) and a large irregularly shaped piece (Plate XXXVI. figs. 2 & 3, *b*) with the articular condyle for the pectoral limb (Plate XXXVI. figs. 2 & 3, *c*, and Plate XXXV. fig. 1, *hc*). The latter cartilage forms the base of a large concave bone (coracoid of OWEN, clavicle of most other authors) (Plate XXXVI. figs. 2 & 3, *d*, and Plate XXXV. fig. 1, *co*); and the whole arch is suspended from the skull by means of a broad suprascapula (Plate XXXVI. fig. 2, *e*, and Plate XXXV. fig. 1, *ss*). We proceed now to a detailed description of these parts from the skull outwards.

The *suprascapula** is a thin, broad lamella of an obliquely ovate shape, entirely ossified; it forms the posterior wall of the capacious gill-cavity, fitting into the hinder angle of the lateral cartilaginous region of the skull, to the edge of which two of its sides are fixed by a short continuous ligament. Its distal portion is connected with the coracoid by a broad ligament, which allows the ends of the two bones to slide over each other.

The *coracoid* is a long, curved, rather thin bone, strengthened by several longitudinal ridges, and extending downwards nearly to the median line of the arch. A suture running right across its middle (Plate XXXV. fig. 1, and Plate XXXVI. fig. 2), nearly on a level with the pectoral condyle, divides it into two subequal portions†.

The *humeral cartilage* (Plate XXXVI. figs. 2 & 3, *b*) has an oblong form, spreading out into a shorter upper lamella covering the hinder side of the lower half of the upper coracoid bone, and into a longer lower lamella covering a great part, and projecting beyond the margin, of the lower coracoid; it does not quite extend to the median cartilage. The humeral cartilage is swollen behind into a thick condyle for the pectoral joint.

The *median cartilage* (Plate XXXVI. figs. 2 & 3, *a*) is a single band connecting the two coracoids, into the interior of which it extends for some distance. More of it is covered by bone on the posterior side of the scapular arch than on the anterior.

The Pectoral Limb.

The parts of the pectoral limb are entirely cartilaginous, without a trace of ossification. The cartilage is distinguished by cells of rather short ovate shape, a great number of which have a double nucleus, or are even perfectly divided into two (Plate XXXVI. fig. 8).

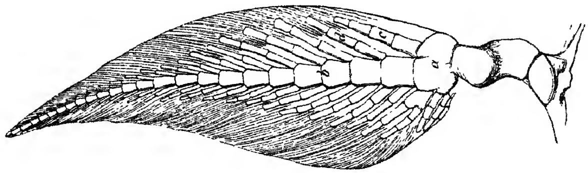
The paddle (Plate XXX. fig. 2) is joined to the scapular arch by an elongate, flattish,

* In *Lepisodiren* "suprascapulare" of PETERS.

† I cannot attach much value to this division; the upper piece is certainly not homologous with the scapula of Teleosteous fishes, which is far removed from the region of the pectoral condyle. The division is also present in *Lepidosiren*; only the upper piece is much the smaller, and Mr. PARKER, by naming it superclavicle, has regarded it as the homologue of the scapula. A division of this bone has also been observed in *Polyodon folium*, where the lower piece is the smaller (see GEGENBAUR, Schultergürtel, Taf. 6. fig. 3).

slightly curved cartilage (*m*); its proximal end has a glenoid cavity, fitting into the humeral condyle; the joint is simple, free, allowing of a considerable amount of motion, its parts being held together by a ligament fastened round its circumference. This is the only true joint in the limb, all the other parts being fixed to one another by connective tissue. I consider this cartilage to be the forearm; a horizontal section along its longitudinal axis does not show any primary division. The next following cartilage (marked *a* in the accompanying woodcut) forms the base of the paddle; although externally it appears as a single flat, broad, short piece, unevennesses of its surface indicate that several primary pieces are coalesced in it. I am confirmed in this view by a horizontal section, in which the lines of the former divisions are preserved in the shape of tracts of a white connective tissue. Three such divisions may be distinguished, corresponding to the three carpals of most Plagiostomes*. If this determination is correct, then the antibrachial cartilage just described is not represented in that order.

The remaining framework of the paddle shows an arrangement unique among the Vertebrata. From the middle of the basal cartilage a series of about twenty-six subquadrangular pieces takes its origin, forming a longitudinal axis (*b*) along the middle



of the paddle to its extremity; the pieces become gradually smaller, and are scarcely distinguishable towards the end of the paddle. On the two posterior corners of each piece a branch (*d*) is inserted, running obliquely backwards towards the margin of the fin; the branches of the first eight or twelve pieces are three-jointed, the remainder two-jointed, the last having no branch at all. Slight irregularities, such as the origin of two branches from one side of a central piece, occur, as also several four-jointed branches (*c*) being inserted immediately on the basal cartilage. As in the vertical fins, so also in the pectoral, the fin-rays are very fine, imbedded in the skin, and arranged in two layers, between which the ends of the cartilaginous branches are received. The analogy of this framework to that of the caudal portion of the vertebral column is striking. *Ceratodus* is not only truly diphycercal as far as the termination of the body is concerned, but this term may be also applied with regard to the extremity of its paired fins. The many-jointed pectoral axis may be compared to the series of neural and hæmal apophyses, both forming the base to a system of superadded processes (here two- and three-jointed branches, there neural and interneural, hæmal and interhæmal spines), which are

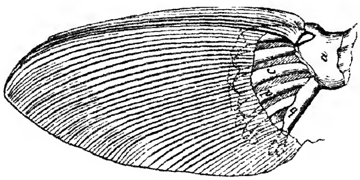
* Pro-, meso-, and metapterygium of GEGENBAUR.

destined to serve as a framework to the surrounding soft parts, and as a support to the rays of the fold surrounding the fin.

I may add here that the muscle moving this fin imitates the form of the cartilaginous framework; a flat, broad band covers and runs along the central axis, sending off towards the right and left small fascicles, one to each cartilaginous branch.

When I designated the arrangement of the parts of this pectoral skeleton unique, I did not mean to convey the idea that no homological relation could be pointed out between the parts of the pectoral skeleton of *Ceratodus* and that of other fishes. It is quite evident that we have here a further development of the simple pectoral axis of *Lepidosiren* in the direction towards the Plagiostomes. The pectoral skeleton of *Lepidosiren paradoxa* consists merely of the central series of cartilages of *Ceratodus*; there is no fin-like expansion of the skin of the pectoral limb, which is a simple tapering filament. In *Lepidosiren annectens* this pectoral filament is bordered by an expansion of the skin along the lower edge; and even minute fin-rays are imbedded in each lamina of the fold; in order to support this low one-sided rayed fringe, very small, single-jointed cartilages are added to the axis*. The fin is still more developed in *Ceratodus*: it has become a broad, scythe-shaped paddle, dilated by a fold of the skin, with two layers of fin-rays surrounding it in its entire circumference; therefore supporting cartilaginous branches are added on both sides of the axis; and most of the branches are composed of several joints, in order to reach the more distant parts which require the support.

The arrangement of the limb-skeleton of *Ceratodus* is foreshadowed in the pectoral fin of *Acipenser*. In the left fin of an *Acipenser sturio*, var. *oxyrhynchus*, four cartilaginous rods are attached to the basal cartilage (*a*); the innermost (*b*) consists of three joints, and represents the jointed axis of the *Ceratodus*-limb; it is provided with three two-jointed branches (*d*), but on one side only, namely the outer, the branches being intercalated between their axial base and the next cartilaginous rod. Having compared the arrangement in *Ceratodus* to a diphycercal tail, we may designate the plan observed in *Acipenser*



Acipenser.

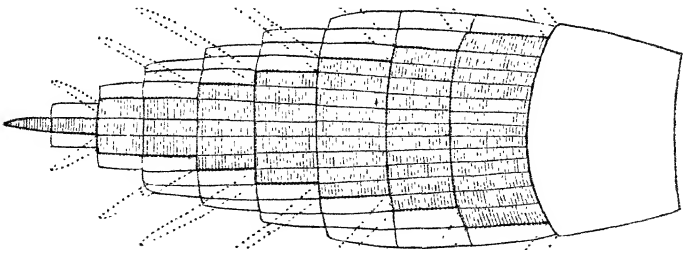
as offering analogy with heterocercy. The three outer rods (*c*) are two-jointed, the posterior joint being much the smaller, and obliquely bent inwards. The outermost rod

* Four or five of these ray-bearers are obliquely attached to each joint of the axis. PETERS, Müll. Arch. 1845, Taf. 2. fig. 2.

forms the base to the attachment of the pectoral spine, which is deeply furrowed, and undoubtedly formed by coalescent and stiffened soft fin-rays. As in *Lepidosiren* and *Ceratodus*, the fin-rays are not joined to the supporting cartilages, but, being disposed in two layers, their ends overlap the cartilages. The attachment of the pectoral spine to its basal cartilage is not effected by a joint, but by apposition.

The arrangement in *Ceratodus* evidently resembles more that in *Acipenser* than that in *Polypterus*.

On further inquiry into the more distant relations of the *Ceratodus*-limb, we may, perhaps, be justified in recognizing in it a modification of the typical form of the Selachian pectoral fin. Leaving aside the usual treble division of the carpal cartilage (which, indeed, is sometimes simple), we find that this shovel-like carpal forms the base for a great number of phalanges, which are arranged in more or less regular transverse rows (zones) and in longitudinal rows (series). The number of phalanges of the zones and series varies according to the species and the form of the fin; in *Cestracion philippi* the greater number of phalanges is found in the proximal zones and middle series, all the phalanges decreasing in size from the base of the fin towards the margins. In a Selachian with a long, pointed, scythe-shaped pectoral fin, like that of *Ceratodus*, we may, from analogy, presume that the arrangement of the cartilages might be somewhat like that shown in the accompanying diagram, which I have divided into nine zones and



fifteen series. When we now detach the outermost phalanx from each side of the first horizontal zone, and with it the other phalanges of the same series—when we allow the

remaining phalanges of this zone to coalesce into one piece (as, in nature, we find coalesced the carpals of *Ceratodus*, and many phalanges in Selachian fins), and when we repeat this same process with the following zones and outer series, we arrive at an arrangement identical with what we actually find in *Ceratodus*.

The Pelvis and Pelvic Limb (Plate XXX. fig. 2, and Plate XXXVI. figs. 4–6).

In the example figured the right ventral paddle is conspicuously narrower and generally smaller than the left; this is merely an individual peculiarity. The pelvis is essentially the same as in *Lepidosiren* and *Protopterus* (fig. 6); it is a single flattened subquadrangular cartilage with two paired processes and a single process. The posterior pair of processes are separated from each other by a deep semicircular notch, each terminating in a condyle to which the basal cartilage of the ventral paddle is joined. The anterior pair of processes are directed outwards and upwards, offering a point of attachment to the neighbouring lateral muscles of the trunk as well as to others stretching to the basal cartilage of the paddle. In our example the process on the side of the larger paddle is considerably broader and stronger than that of the opposite side. Thirdly, the single process arises from the middle of the front margin of the pelvic cartilage; it is very long, cylindrical, tapering, and enclosed in a thick, shining, fibrous sheath (fig. 4, *a*), which is the continuation of the perichondrium of the pelvic cartilage. As is evident from this description, this process points forwards, towards the head, *not backwards*. So it is also in *Lepidosiren* and *Protopterus*. It lies imbedded between the abdominal muscles, in the median line of the abdomen in *Protopterus*, but bent somewhat towards the left side in our example of *Ceratodus forsteri*. If a vertical section is made through the pelvic cartilage (fig. 5), a capacious cavity (*b*) is found to exist in its interior; but the state of preservation of our example did not allow of forming an opinion of the nature of its contents.

The paddle is attached to the pelvis by means of a separate, rather short, subcylindrical cartilage which has a rounded process on its lower outer side; it articulates with the pelvic condyle by a true joint, having a glenoid cavity in front. The endoskeleton of the paddle itself is almost identical with that of the fore paddle; but the segments of the axial series of the right paddle (Plate XXXVI. fig. 4) are more liable to confluence; the insertions of the branches are consequently more confused, and the phalanges are shorter, more feeble, and somewhat less in number. The skeleton of the left paddle (Plate XXX. fig. 2) is much more symmetrical and regular.

The Organs of Circulation (Plate XXXVII.).

The *heart* lies far forwards, its basal half being protected by the middle of the scapulary arch. It is enclosed in a pericardial sac with thick, tendinous walls; the base of this sac is confluent with the diaphragm, which, like the proper tunic of the abdomen, is a strong glistening fibrous membrane. This is the only direct connexion between pericardium and diaphragm; and a separate additional lamina extending from the dia-

phragm to the middle of the pericardium, as it has been described by HYRTL in *Lepidosiren paradoxa*, and as I have found it also in *Protopterus*, is not developed in *Ceratodus*.

After the pericardium is opened (fig. 1) the ventricle (v) is observed on the right side, and the large single atrium (a) on the left. From the former arises a conus arteriosus* (b), forming a short spiral; that part of it which is visible without further preparation, runs transversely from the right towards the left. Its muscular layer is much more developed along the lower side of the spiral than along the upper. The grooves between the divisions of the heart are covered with a thick layer of a dark semifluid fat. At the upper end of the cônus the insertion of the pericardial sheath causes a slight swelling round this portion. The heart hangs freely in the pericardium, except at the two opposite poles, viz. where the sinus communicates with the auricle, and where the conus arteriosus passes into the aorta. A single very short and thin filament fixes the point of the heart to the pericardium.

We begin with opening the ventricle on the side facing the observer (figs. 2 & 3). Its parietes are thick, with well-developed trabeculæ carneæ on its inner surface. Its cavity is spacious, and incompletely divided into two by a papillary muscle (m) rising from the apical portion of the ventricle, and passing into a cartilage (m') which is considerably broader and thicker than the muscle; a number of chordæ tendineæ fix the end of this body to the walls of the atrium, into the cavity of which the cartilage extends. The cartilage, with its muscle, lies in the plane of the atrio-ventricular opening, which is partially closed by it during the systole. Opposite to the base of this papillary muscle (which is also present in *Lepidosiren* and *Protopterus*, and has justly been described by HYRTL as an incomplete septum ventriculi) is a broad valve (c) with semilunar margin; it likewise assists in preventing the blood of the ventricle from regurgitating into the atrium during the systole.

The cavity of the atrium (fig. 3) is about twice as spacious as that of the ventricle; it does not extend beyond the left half of the heart; and there is no long, produced auricular process; its walls are membranous, with a great number of thin muscular fasciculi decussating and forming an open network. Those of these little muscles which are connected with the cartilage are strongest, and terminate in short chordæ tendineæ; but although a careful preparation under water gave a clear view of their arrangement, I could not convince myself that they indicated a division of the atrium, such as HYRTL found in *Lepidosiren paradoxa*.

The vessels carrying the blood to the atrium open into a *sinus venosus* which lies within the pericardium. The sinus communicates with the atrium by a simple *ostium*

* Before I became acquainted with a paper by GEGENBAUR in Jena. Zeitschr. f. Med. und Naturwiss. ii. pp. 365–375, I distinguished the dilatation of the ventricle in Ganoids and Selachians under the name of *Bulbus arteriosus*, from the swelling of the aorta of Teleosteans, for which I retained the term *Bulbus aortæ*. But finding that Professor GEGENBAUR had already proposed the term "conus arteriosus" for the former division of the heart, I shall, of course, adopt the prior nomenclature.

venosum without valve; this ostium is on the dorsal surface of the atrium, a little towards the right of its middle. A right and left *vena cava superior*, beside the principal *vena cava*, terminate in the sinus venosus, the former perforating the pericardium at nearly equal distances on the left and right from the mouth of the latter, which empties its contents a little towards the right of the median line of the end of the pericardium. The single *vena pulmonalis* does not communicate with the sinus, but, passing along its dorsal wall, enters the atrium by a separate opening more towards the left than the sinus; its mouth is provided with a valve. The diameter of the vena pulmonalis is about equal to the sum of those of the two venæ cavæ superiores, and about one half of the vena cava.

The *conus arteriosus* differs from that of *Lepidosiren* less externally than internally. Its anterior wall is thin, though provided with a muscular stratum, which is thickest along the posterior rim of its spiral course. Its beginning is indicated by the absence of trabeculæ carneæ, the inner surface being smooth. The valvular arrangement is entirely different from that which was considered to be the characteristic of the subclass *Dipnoi*. No valve exists on the boundary line between conus and ventricle. Before the conus turns to the left, its interior is rather spacious; but this compartment is separated from the transverse portion of the conus by a cartilaginous valve, which from a certain view appears as a merely papillary prominence (*d*), but when viewed from various sides proves to be a spiral performing a half turn, to the lower end of which a muscle (*d'*) is attached, which reaches down into the ventricle. This valve closes the lumen of the conus most effectually during the diastole of the heart. In the systole, the muscle attached to it contracts, and draws the valve from its position downwards, thus opening free egress to the blood out of the heart. When the muscle relaxes during the systole, the valve resumes its position through its own elasticity, shutting up the communication between the heart and arterial system.

Beyond this valve the conus turns towards the left, and then for a very short distance forwards. Quite at the end of it, and immediately before it bifurcates, there are two pairs of (ganoid) valves (*e*), narrow, and rather long, with stiff non-collapsing walls, thicker along the middle than at the sides, and without tendinous chordæ, in a single transverse series. Their tunics are continued in four narrow raised strips behind their bases* (*f*). So far I have found the arrangement of this part of the heart nearly identical in two examples. On examining the first example, a pair of small papillary prominences (fig. 7, *g'*) were found in a line between the series of stripes and the spiral valve, immediately in front of the latter. These stripes and papillæ appeared to me to represent rudiments of a second and third series of valves, analogous to the plurality of series in other ganoid fishes. Remembering, at the same time, the fact that individual variations in the development and number of valves are not of uncommon occurrence in these fishes, I examined the heart of a second (smaller, female) specimen, and had the

* Such longitudinal pads have been also found in *Acanthias* by GEGENBAUR, Jena. Zeitschr. f. Med. und Naturwiss. ii. p. 366.

satisfaction of finding my supposition confirmed. In this specimen there are, in the series corresponding to the pair of small papillæ, four valves (fig. 6, *g*) corresponding in position to, but much smaller than, the permanent large valves which I have described above.

The conus arteriosus branches off into four *arcus aortæ* (figs. 1 & 5, *h*, *i*, *k*, *l*) on each side, the origins of the two anterior being more on the ventral side, and entirely distinct from each other, those of the two posterior being confluent for a very short distance, and situated more on the dorsal side. The valves are placed close to, and nearly opposite, the origins of the arcus aortæ, so that one valve corresponds to the common origin of the two posterior arches (*k* & *l*), and one to the origin of the first arcus (*h*), whilst the second arcus (*i*) has no separate valve opposed to its root.

These arches diverge, severally entering the ventral end of each of the four gills (fig. 8), and, reappearing at the dorsal end (fig. 9), converge again to form the aorta. They are united in such a manner that the two anterior are confluent into a common stem, and the two posterior into a second. The first arcus, immediately after having left the gill, emits a considerable branch as *Carotis* (*n*), and a second much smaller recurrent branch (*o*), which also enters the brain-capsule on the margin of the basal bone (*ba*).

The common stem of the aorta is formed opposite the beginning of the narrow posterior part of the basal bone. Its walls are at its commencement easily compressible, and collapse when empty; however, very soon they are strengthened by firm, tendinous bands, which at the origin of the arteria cœliaca are so much developed as to form a rigid half-canal covering the ventral side of the aorta.

The strongest lateral branch of the aorta is the *arteria cœliaca* (*p*), which leaves the aorta at a right angle about an inch below its origin: it is present on the right side only; and, as mentioned in the description of the liver, it winds like a collar round the narrow neck of that organ, distributing its contents through the intestinal tract.

At the same place where the arteria cœliaca branches off, a right and left *arteria subclavia* (*q*, *q'*) take their origin. The right is much narrower than the left, and carries blood merely to the fore limb of its side. The left is about twice as strong, sends off a small muscular branch, and then divides into two arteries. One of the latter runs straight towards the left paddle, emitting two small branches, which enter the foremost extremity of the testicle. The second artery forms a very peculiar anastomosis (*r*) with the left *vena cava superior* (*s'*). This anastomosis is also indicated on the right side; but its arrangement is somewhat different; namely, the right *vena cava superior* (*s*) emits a branch (*t*) inwards, towards the aorta, running parallel with the arteria cœliaca. The branch is widely open at its origin, and colouring fluid can easily be injected into its cavity from the *vena cava*; but the canal is gradually obliterated, and closed entirely before it reaches the aorta. Probably there exists an open communication between the vein named and the aorta at an earlier period of life.

The number of intercostal arteries, which are of inconsiderable calibre, has not been ascertained.

The venous trunks, by which the blood is carried to the atrium of the heart, have been mentioned above. The two anterior *venæ cavæ* collect the blood from the head and fore limbs, and offer no further peculiarity. The course of the vena pulmonalis is noticed with the organ to which it belongs. The single vena cava posterior (Plate XL. *x*, and Plate XLI. *e*) is the largest vessel in the whole system, having a diameter of 3 lines, and collecting the blood from the trunk, tail, and the abdominal organs, except the lung and intestine. Immediately below the diaphragm it enters the liver, collecting the blood from this organ, and descends through the substance of the right lobe of the liver (Plate XXXIX. *x*) to and along the inner margin of the testicle (or ovary) of the same side, being rather superficial whilst it accompanies this organ. It receives in succession from the front towards behind the following branches:—

a. A number of very small branches from the right testicle (or ovary) which join the main trunk at right angles.

b. A stronger branch from the hindmost lobe of the right testicle and upper part of right kidney (Plate XLI. *g*).

c. A very strong vein from the left testicle (Plate XLI. *f*), which corresponds in situation and function to the main trunk, and might be called a left vena cava posterior; but the currents of blood in the two run in opposite directions, that of the right (main) trunk running towards the head, that of the left towards the tail.

d. The *vena caudalis* (Plate XLI. *h*) is the strongest tributary; it enters the main trunk on its dorsal surface, immediately after having penetrated through the muscles and fascia which cover at this place the vertebral column.

e and *f.* A pair of venæ renales revehentes (Plate XLI. fig. 1, *i i*).

The Gills (Plate XXXVII.).

The gill-apparatus differs considerably from that of *Lepidosiren*, and approaches in structure that of the common Teleosteous type. Of the five branchial arches, four bear complete gills (figs. 8 & 9). Each gill is a broad membranous fold, to which a series of numerous simple branchial laminæ is attached in front and behind; most of the laminæ extend somewhat beyond the connecting membrane, having their extremities projecting free beyond its edge. The outer point of attachment of the gills is not on the branchial arches, but beyond them, on the wall of the gill-cavity (*u*, *u*)—the three anterior being fixed to the lower surface of the lateral expansion of the skull, and the fourth to the inner surface of the suprascapula. The fifth pair of arches (*w*) (commonly called "lower pharyngeals") retain in our fish their primitive function, and, although not bearing a perfect gill, are, at least, provided with gill-rakers. Two series of short, closely set gill-rakers, closely interlocking with those of the neighbouring arch, run along the concave side of each of the four principal arches.

Further, to this gill-apparatus is added a very distinct *pseudobranchia* (*x*). This pseudobranchia is the gill originally belonging to that branchial arch which is modified into the hyoid. It is also accompanied by two series of gill-rakers, one of which (*x'*)

runs in an oblique direction from the front end of the pseudobranchia inwards; the other (x') remained attached to the cerato-hyal ($c\,h$). The pseudobranchia of *Ceratodus* corresponds evidently to the opercular gill of *Lepidosiren* and *Protopterus*; but it cannot have the same function, inasmuch as in these two genera a separate portion of the first arcus aortæ branches off to carry (venous) blood to this gill, thus proving its function to be respiratory (Kiemendeckel-Kieme of Müller), whilst in *Ceratodus* the first arcus aortæ has no branch for the pseudobranchia before it enters the foremost real gill.

Spiracles are absent.

Thus we can add another combination with regard to the coexistence of opercular gill, pseudobranchia, and spiracle in Ganoids, to those pointed out by Müller (Ganoid. p. 135):—

1. Opercular gill, pseudobranchia, and spiracle: *Acipenser*.
2. Opercular gill, pseudobranchia, no spiracle: *Lepidosteus*.
3. Opercular gill, no pseudobranchia, no spiracle: *Scaphirhynchus*, *Lepidosiren*, *Protopterus*.
4. No opercular gill, but pseudobranchia and spiracle: *Planirostra*.
5. No opercular gill, no pseudobranchia, but spiracle: *Polypterus*.
6. No opercular gill, but pseudobranchia, and no spiracle: *Ceratodus*.

The Lung (Plate XXXVIII.).

The *pneumatic apparatus* of *Ceratodus* may be described either as a single lung with symmetrical arrangement of its interior, or as two lungs confluent into a single sac without any trace of a septum. The sac is wide, and extends from one end of the abdominal cavity to the other; it occupies the middle of the dorsal region, being firmly attached to and along each side of the aorta. Its external surface shows numerous small rounded prominences, corresponding to the minor cells of its cavity, and surrounded by the network of the pulmonal vein. These prominences are absent in a stripe (a) running along the middle of its ventral surface, and bordered on each side by the branches of an arterial vessel (f, f'). The minute ramifications of the blood-vessels extend only sparingly into this smooth stripe, having probably only a nutritive, and not a respiratory function. The attachment between intestine and lung is also most intimate along this stripe, which is opposed to the aorta, and indicative of a division of the lung into two lateral halves. Such a division is also indicated by a slight prominence in front (b), and behind (c). The right half is contracted at its anterior extremity, slightly bent towards the right side, and opens, by a very short duct terminating in a *glottis* (gl), *into the ventral side of the œsophagus, somewhat to the right of the median line*. The *glottis* is a slit about one eighth of an inch long, and provided on one side with a duplicature of the membrane, which acts as a valve.

The interior of the lung is best seen by cutting it open along the smooth stripe in its median line (fig. 2). Both halves are then seen to have an identical and nearly symmetrical structure, being divided into a number of compartments formed by strong trans-

verse septa; there are twenty-eight on the left side, and thirty-three on the right, the latter half extending more forwards than the former. The foremost and hindmost compartments are much shallower than the others, which have a cubic shape, being about half an inch wide, and one third of an inch long and deep. The septa separating the compartments are imperforate membranes, so that each compartment can be separately filled with fluid, only the side towards the median line of the lung being open. The bottom of these compartments is again divided into a number of larger and smaller cells by reticulated septa, which are very irregularly disposed. A very dense network of the branches of the pulmonal arteries and vein extends all over the surface of the lung, and its principal and secondary septa; the terminal branches of both arteries and vein are rather wide, and can be injected with great facility.

The principal arterial vessels of the lung are on the dorsal side of the organ, running along, and very close to, each side of the aorta; that on the right side is much stronger than the left, and can be injected from the arteria cœliaca. There is no direct arterial communication between the lung and the arcus aortæ, as in *Lepidosiren*. The two arteries mentioned distribute blood through the entire length of the lung, to its posterior end; their ramifications are confined to the inner surface of the organ, and not visible externally.

On the median line of the ventral side of the lung, opposite to the dorsal arteries, are two vessels (f and f'), of the arterial nature of which I could not satisfactorily convince myself. Their ramifications are also spread over the inner surface and collected into the two stems running along each side of the smooth band in the median line of the lung, the left stem being wider than the right. After their union near the anterior end of the lung, the single vessel turns off towards the right side of the œsophagus, which it then crosses from the right towards the left, at a short distance above the glottis. Arrived on the left side, it descends again, as far as the basal portion of the first rib, and splits up (at g) into three very small branches, which I could not follow further. Great care was taken in ascertaining the very singular termination of this vessel, which I expected to take its origin from the left arteria subclavia; but no such connexion could be discovered.

The ramifications of the venæ pulmonales (e and e') are more visible on the external surface of the lung than on the internal; they are collected into a common branch, running near to the outer margin of the ventral surface of each half of the lung; the left branch is stronger than the right; both unite near the anterior end, and on the right side of the lung; its passage through the diaphragm and separate entrance into the atrium have been noticed above in the description of the heart.

At the beginning of this paper (p. 514) statements have been referred to, from which it would appear that *Ceratodus* is in the habit of going on land, or at least on mud-flats; and this assertion seems to be borne out by the fact that it is provided with a true lung. On the other hand, we must recollect that a similar belief was entertained with regard

to *Lepidosiren*, of which now numerous examples have been kept in captivity, but none have shown a tendency to leave the water. I think it much more probable that this animal rises now and then to the surface of the water in order to fill its lung with air, and then descends again until the air is so much deoxygenized as to render a renewal of it necessary. The fish is said to make a grunting noise, which may be heard at night for some distance. This noise may be produced by the passage of the air through the œsophagus, when it is expelled for the purpose of renewal. From the perfect development of the gills we can hardly doubt that, when the fish is in water of normal composition, and sufficiently pure to yield the necessary supply of oxygen, these organs are sufficient for the purpose of breathing, that the respiratory function rests with them alone, and that the lung receives arterial blood, returning venous blood, like all the other organs of the body. But when the fish is compelled to sojourn in thick muddy water charged with gases which are the product of decomposing organic matter (and this must be the case very frequently during the droughts which annually exhaust the creeks of tropical Australia), it commences to breathe air with its lung in the way indicated above. Under this condition the pulmonary vein carries purely arterial blood to the heart, where it is mixed with venous blood and distributed to the various organs of the body. If the medium in which the fish happens to be is perfectly unfit for breathing, the gills cease to have any function; if only in a less degree, the gills may still continue to assist in respiration. *Ceratodus*, in fact, can breathe by either gills or lungs alone, or by both simultaneously.

It is not probable that this fish lives freely out of the water, the limbs being much too flexible for supporting the heavy and unwieldy body, and too feeble generally to be of much use in locomotion on land. However, it is quite possible that it is occasionally compelled to leave the water, although I do not believe that it can exist without it in a lively condition for any length of time.

Relative Situation of the Abdominal Viscera (Plates XXXIX. and XL.).

The part which first attracts attention on opening the abdminal cavity is the extremely large and wide intestinal sac (*a*), which fills the cavity nearly entirely. It is perfectly straight from its entrance below the diaphragm to the vent, without any circumvolutions, and of a nearly black colour, whilst the inside of the walls of the abdomen is of a silvery white. In the specimens examined it was throughout distended with food, apparently provided with thin walls, except below the stomachic region, where the walls were thicker, offering greater resistance to pressure. In the middle of its course it is crossed by six lines placed at regular intervals and indicating the insertion of the internal spiral valve. Below the last line the intestine is gradually contracted to its end.

This large intestinal sac is fixed by a ligament (*b*) to the ventral surface of the cavity; this very peculiar ligament commences from the first turn of the spiral valve, and is continued to the end of the intestine, fixing it, not exactly along the median line of the

abdomen, but somewhat to the right hand of it. It is a very strong ligament, and, behind, firm like a tendon; there is a slit in it (b'), on the level of the pelvis, allowing of communication between the two sides of the abdominal cavity. The ventral portion of the upper part of the intestine is without mesenteric ligament.

On its dorsal side the intestine is fixed by its attachments to various organs—thus, along the median line, to the smooth band of the lung; more towards the side a portion of the testicles or ovaries adheres so firmly to the intestine that it is difficult to separate them. Along each side of the intestine lie the generative organs (g), the lung (l) occupying the place below the vertebral column, without the peritoneal sac. A greater portion of the lung is visible on the left side than on the right; and in order to expose it to view more fully, we have (at least in the male) to penetrate an extremely singular cellular tissue (h) filling up the interspaces between the peritoneal sac and the walls of the abdomen*. HYRTL appears to have observed a similar tissue in *Lepidosiren paradoxa.* Its cells are very wide, and the meshes so strong that, at the first glance, it may be taken for a distinct lung-like organ.

The kidneys (k) are entirely hidden from view, lying without the peritoneal sac, and enveloped in an adipose tissue. They occupy the side of the posterior part of the abdominal cavity, forming, with regard to position, merely a continuation of the generative organs, to which they are intimately attached.

The liver (c) lies immediately below the diaphragm (d), to which it is attached only in the neighbourhood of the large vessels penetrating the diaphragm; its upper lobe is thin and short, covering the uppermost part of the intestine, and subdivided in the middle by the very large and pear-shaped gall-bladder (e), which thus occupies the median line of the abdomen. This upper lobe is connected by a narrow bridge with a lateral triangular lobe (c') lying on the right side of the intestine; it is also thin, and its tapering posterior end is firmly attached to the extremity of the testicle or ovary (g) of the same side. The liver has no other attachment to the intestine, except at the place where the ductus choledochus enters the wall of the latter.

Intestinal Tract (Plate XXXIX.).

The buccal cavity is clothed with a soft and rather thin membrane of brown colour, the surface of which is uniformly covered with small papillæ. A small and short prominence, a fold of the mucous membrane between the front prongs of the lower molars, represents a rudimentary *tongue* (Plate XXXV. fig. 3, *t o*). The *œsophagus* (Plate XXXIX. fig. 2, *oe*) is rather narrow at its commencement, but widens rapidly behind the glottis, passing into the stomach (f) without a distinct boundary line. The membranes of the œsophagus do not show either folds or striæ; but there is on each side a longitudinal flat pad (i) of an orange colour; it is a layer of fat deposited below the mucous membrane; the right pad commences at a short distance behind the glottis, the

* I have found this tissue much more developed in the male specimen than in the female, in which it would interfere with the free passage of the ova through the abdomial cavity.

left somewhat more forward. The part which must be regarded as the *stomach* (*f*) is short, in its posterior portion much wider than the œsophagus, and nearly as wide as the succeeding part of the intestine. It has very thin walls, without folds or crypts. Its end is indicated by a double circular fold (*p*) of the mucous membrane, each fold being about three lines broad (*pylorus*). On the right side of the stomach, below the extremely thin mucous membrane and the pad of fat mentioned above, there is an extensive rather thin layer of a very soft substance of brownish-black colour (*m*). Water or any other substance coming into contact with it is coloured brown. This organ descends below the pylorus, and is continued for some distance along the axis of the spiral valve. I am inclined to regard it as a *spleen*; it has no communication with the inside of the intestine*.

The intestine is traversed throughout by a spiral valve, which performs nine gyrations. The extent of the intestine traversed by the first turn is the greatest, measuring about 4 inches; the second is much shorter, and the following are of nearly the same length as the second; the last two or three are again lengthened. The valve retains its spiral course nearly to its end, which is close to the vent.

The part below the pylorus receives the contents of the gall-bladder—the mouth of the ductus choledochus (*e'*) being on the right side of the ventral surface of the intestine, at a short distance below the pyloric valve. The mucous membrane of this portion of the intestine (*l*) is finely wrinkled, the folds having an obliquely transverse direction parallel to the pyloric valve. The ventral wall is much thicker than the dorsal; and numerous flat glands (*q*, *q'*) are imbedded between its membranes. These glands are either simple follicles without opening, or much larger, composed of a homogeneous firm substance, and with a small opening which leads into a short simple or bifurcate duct. The mucosa of the remainder of the intestine is smooth; but glands are scattered over all parts, disappearing only within the last two or three gyrations. In their most simple form they are flat, circular or ovate bodies of from 1 to 3 millims. diameter; many have an evenly convex surface; in others the membrane over the centre is sunk in, as if this part of the follicle had been filled with a fluid which has now disappeared; in a third kind the membrane in the centre is actually perforated by a more or less wide opening. Some of these follicles are isolated, in other places two or more are aggregated and more or less confluent. Beside these glands other much larger and thicker ones are placed along or near to the axis of the spiral valve, the largest being within the third and fourth gyrations, where some are more than an inch long, half an inch broad, and about two lines thick. Each of these large glands has several depressed points or openings on its surface, leading into two or three short ducts. These glands are much thicker than the spiral valve in which they are imbedded; consequently some of them project over both the anterior and posterior surfaces of the valve, so that one and the same gland discharges its contents towards two surfaces, or, in other words, into two adjoining compartments of the intestinal spire.

* HYRTL, who appears to have found a similar organ in *Lepidosiren* and the *Sterlet*, has come to the conclusion that it is a rete mirabile.

I have to add a few words on the termination of the intestinal tract, and on the openings of the neighbouring excretory organs (Plate XL. figs. 1 & 2). The *vent* (*v*) lies in the median line of the abdomen, and leads into a very short and rather narrow cloaca, which is subdivided into two by a projecting fold of the dorsal wall of the rectum (*r*). This fold answers also the purposes of a valve, preventing the contents of either of the cloacal divisions from entering into the other. The abdominal portion of the cloaca passes uninterruptedly into the cavity of the rectum, the dorsal portion (*u*) being a small receptacle for the urine and generative products.

Immediately behind the vent are the two peritoneal openings (*w*), leading directly into the peritoneal cavity.

Liver (Plate XXXIX.).

A general description of the liver and its situation in the abdominal cavity has been given above. Its texture is spongious, not dense, in consequence of the great width of all the venous and biliferous cavities and ducts in its interior; certain portions may be inflated like the lung of a mammal. The gall-bladder (*e*) is very large, pear-shaped, and continued into the ductus choledochus, which is rather wide in its commencement, and enters by a small opening (*e'*) below the pyloric valve on the right side of the ventral wall of the abdomen. In order to reach this spot, the ductus choledochus has to traverse a rather long course below the mucous membrane of the stomach. Of the ductus hepatici, one, coming from the lateral lobe of the liver, and running nearly along the entire length near to the inner surface of the lobe, is particularly conspicuous. These ducts are collected into one trunk, which enters the ductus choledochus in the upper half of its course, before it has reached the wall of the stomach. The common opening of the hepatic ducts is much wider than that of the ductus choledochus.

The *vena cava*, which ascends along the line of attachment of the peritoneum to the right testicle (Plate XL. *x*), enters the hindmost extremity of the lateral lobe of the liver; it becomes much wider within this lobe, and penetrates through its substance, through the bridge connecting the two lobes, and through the upper lobe, where it reappears to enter the sinus venosus communis. Its inner walls are perforated by the openings of numerous small branches; and the venous system of the liver can be filled with matter of injection from either end of the vena cava.

The *arteria cœliaca*, which takes its origin on the right side of the aorta, runs round the bridge, connecting the lateral lobe of the liver with the upper one, and divides into several branches on entering the intestinal canal at the end of the axis of the spire. One of its branches is destined for the liver itself, another for the dorsal portion of the lung. At the same spot several venous trunks leave the intestine, and, entering the liver, where they branch off into smaller stems, represent the portal system.

Uropoëtic Organs (Plate XLI.).

The kidneys (*c*, *c'*) lie in the posterior part of the abdominal cavity, being only about 3 inches long. They are paired, each being firmly attached to the testicle (*b*, *b'*)

or ovary of its side, and enveloped in fat. Each kidney is composed of from eight to ten lobes (see fig. 3). The two or three anterior lobes lie on the side of the air-bladder (*a*), by which they are entirely separated from those of the other side; but behind the air-bladder and the vena caudalis (*h*) the lobes of both sides are contiguous and connected by larger blood-vessels, though the renal substance itself is not coalesced. The posterior lobes become smaller and thinner, and, being apposed to the ureters, are again entirely separate from each other.

The ureters (*d*, *d'*) are provided with thick walls, and become superficial in the second anterior fourth of the kidney; they are perforated by only a few primary uriniferous tubes; about half an inch before their termination they externally unite into a single trunk, although they remain separate internally, the right entering the left close to the single opening in the urinal cloaca (see Plate XL. fig. 2, *k''*); this opening is round, dorsad to the genital opening (*g''*), from which it is separated by a slightly raised wall of the mucous membrane. The urinal cloaca (Plate XL. fig. 2, *u*) is small, incompletely separated from the termination of the rectum; in fact it may be described as a dorsal diverticle of the latter. The urine, consequently, passes by the vent.

The kidneys are but sparingly supplied with blood from the arterial system, by a small vessel (*n*) which descends on the dorsal surface of the intestine, and therefore is one of the extreme branches of the *arteria cœliaca*. On the other hand, a great quantity of venous blood is carried to them by *venæ renales advehentes* (*k*, *k'*), which form a portal system of the kidneys, such as has been described by Hyrtl in various fishes. I describe here the arrangement of the right side, remarking that that of the left differs in several unimportant points of detail, as also probably many individual variations occur. The posterior *vena renalis advehens* comes from the lower parts of the root of the tail, and communicates with its fellow of the other side by a short anastomosis lying on the back of the cloaca; it emits (or receives?) a slender branch to (or from?) the ureter, and distributes a part of its contents by three stronger branches over the posterior renal lobes. Then it enters into communication with several intercostal veins (*l*, *l'*, *m*), which make their appearance through slits in the fascia of the wall of the abdomen, traverse the peculiar cellular tissue of that region, and merge into the posterior *vena renalis advehens*. The hindmost of these intercostal veins (*l*, *l'*) is much stronger than the anterior (*m*, *m*). From these veins the middle lobes receive their blood. The anterior lobes are provided partly by the remaining portion of the posterior vein, partly by intercostal veins which enter the lobes by themselves. There is one pair of *venæ renales revehentes* (*i*, *i*) running along the base of the posterior and middle lobes and emptying their blood into the *vena cava*. The blood of the anterior lobes is probably collected in one vein, together with that of the posterior lobe of the testicle.

Organs of Reproduction.

The generative organs are paired. Their products pass outwards by a paired oviduct, or vas deferens. These ducts are entirely separate from the ovaries or testicles, each

having a distinct abdominal orifice immediately below the diaphragm. They accompany the ureters in their posterior course, but are nowhere confluent with them, and terminate in a common opening into the cloacal dilatation, immediately in front of the uretral orifice. A pair of wide slits *behind* the vent lead immediately into the peritoneal cavity.

Female Organs (Plate XLII.).

The sexual organs of the two sides differ from each other in form. The right ovary is considerably broader than the left ($2\frac{1}{4}$ inches across its middle), but extends forwards only to the extremity of the lateral lobe of the liver, to which it adheres by a fold of the peritoneum enclosing the vena cava. The left ovary extends from one end of the abdominal cavity to the other, and is only $1\frac{1}{4}$ inch broad across the middle; its terminal portions are tapering.

The ovaries (fig. 1) are elongate bands, with an inner and an outer surface, fixed along the dorsal edge to each side of the notochord, otherwise free, with a sharp ventral margin; they are separated from each other by the lung and intestine, and nowhere confluent. They are not closed sacs, being covered by the peritoneum on their inner (intestinal) side only. The outer side is in contact with the walls of the abdomen, and crossed by a great number of transverse lamellæ, the bearers of the stroma in which the ova are developed. These ova are in immense number, and, when ready for exclusion, have a diameter of $2\frac{1}{2}$ millims. The stroma of the ovary with mature eggs is of a very dark colour, irregularly and finely mottled with black and dark brown. The eggs themselves are now brown, and black on that portion of their convexity which is turned towards the surface of the ovary. Remains of a former state of development are visible on both sides, in the form of a linear tract (c, c') commencing at the posterior extremities of the ovaries, and running for some distance along the inner side of the oviduct. The anterior extremity of the left ovary terminates in a similar appendage (b'), but it is broader, band-like, and about 2 inches long; it extends to the abdominal aperture of the oviduct, surrounding a part of its circumference, and contains considerable venous vessels enveloped by an adipose tissue.

It is evident that the ova drop into the cavity of the peritoneum; at first they are received into a shallow gutter (d, d') at the base of the ovary, to commence a very circuitous journey before their final expulsion.

There is, namely, attached to, and along the base of each ovary a thick-walled oviduct (o, o'), 8 or 9 millims. thick, and still thicker in its posterior portion, but nowhere expanded into a sac with thinner walls. It is much convoluted, the convolutions forming a thick knotted rope along the whole length of the abdominal cavity; only posteriorly the convolutions are more simple, and finally open into a serpentine course. These oviducts lie outside the peritoneal cavity, and therefore are entirely separated from the ovaries. Their abdominal aperture (r, r') is a gaping slit, 3 or 4 millims. wide, in the foremost portion of the abdominal cavity, immediately below the diaphragm, at a short distance from the notochord. Consequently on the right side the liver intervenes

between the ovary and the orifice of its oviduct. The ducts follow the base of the ovary and the course of the ureters, and coalesce immediately before their termination in the urinal cloaca. In the mature female, during the breeding season, I found their internal structure to be the following. The mucous membrane clothing the duct in the posterior 3 inches of its course (fig. 1, *p*) is raised into numerous transverse folds from 1 to 2 millims. deep, with a sharp margin, closely set, parallel to one another, and more oblique anteriorly than towards the end of the oviduct. So far its structure is extremely similar to that of *Menopoma*. Further above, the duct changes its appearance. Its wall continues to be highly turgescent, and to have a thickness of about 3 millims.; but its substance, instead of being a firm and resistant membrane, like the posterior portion, is a gelatinous mass enclosed in the thin fibrous outer membrane of the duct. This gelatinous mass (which is the mucous membrane) is deeply longitudinally fissured; and when the duct is cut through transversely (fig. 3), the fissures appear to the naked eye as numerous striæ, radiating from the central canal towards the periphery. On a closer examination we find that the transverse folds of the hindmost portion of the duct gradually assume, towards the middle of its length, a more oblique, and finally a longitudinal direction; they become very thin, semitransparent lamellæ, are easily ruptured, very closely packed, and therefore much more numerous. In a section made transversely through the lamellæ, but in the longitudinal axis of the duct (fig. 5), they appear like undulated bands (sometimes with short branches), with a linear tract of fibrous tissue (*b*) along their centre, into which blood-vessels extend, and which is the basis for a thick stratum of epithelial cells (*c*). In a second section, made right across the duct (fig. 4), the lamellæ appear as slightly attenuated cones, again with the epithelial stratum (*c*) and the dark tract (*b*) in the centre, the latter being a continuation of, or proceeding from, the outer fibrous coat (*a*) of the oviduct.

It was of interest to know whether the similarity in structure observed in the lower part of the oviducts of *Ceratodus* and *Menopoma* extended into the middle and upper portions. This is not the case. In a female Menopome, sexually mature, but killed out of the breeding season, the mucous membrane of those parts of the duct is not lamellated, but perfectly smooth, and in a section (fig. 6) through the wall in the longitudinal axis the lumina of the cavities of numerous tubular glands are seen.

The question naturally arises whether the condition of the oviduct described above, especially the gelatinous consistency of its wall, is not partly due to imperfect preservation of the specimen. I can hardly believe that this is the case, because the neighbouring parts have not suffered from decomposition, and even the epithelium has been preserved in its natural position and continuity. But when we find the mucous membrane turgescent with gelatinous matter, we may reasonably suppose that its secretion is of a similar nature, and destined to form a protecting case for the ova during their passage through the duct, as in Batrachians*.

* Whilst my examination was limited to a female 26 inches long, the undeveloped sexual organs of which will be described on the following page, I expressed the belief that the ova might be expelled through the

This is the condition of the sexual organs in a fish 34 inches long, fully mature, and evidently caught during the spawning season. I am enabled to add a description of the organs in a less developed state (fig. 2) from an example 26 inches long. The extent of the ovaries is the same as in the mature example, but a much smaller portion of their substance is ovigenous; at both ends they terminate in strips of a soft substance (b, b'; c, c') containing a considerable quantity of fat, the posterior strips being narrow and tapering in a fine point. The anterior strip of the left ovary (b') is $2\frac{1}{2}$ inches long (e), and extends, besides, backwards to some distance along the edges of the ovary. The anterior strip of the right side is very short, and replaced by the lateral lobe of the liver (l). The ovigenous part is from two thirds of an inch to an inch broad, and crossed by transverse lamellæ containing an immense number of very small ova, which are not visible to the naked eye. No pigment is as yet deposited in the stroma.

The oviducts (o, o') are in an equally undeveloped state; they are of the thickness of a pigeon's quill, and although tortuous in their course, show only a few complete convolutions. The duct of the right side opens by a minute slit (r) at the same place as in the mature example; but the left duct terminates (r') blind in the substance of the non-ovigenous strip of the ovary of that side. The specimen was in a good state of preservation, and neither air nor fluid could be driven through the end of that duct.

Male Organs (Plates XL. and XLI.).

The following observations have been made on two examples, one of which is the male, 32 inches long, mentioned above (p. 512). The other example*, although larger (36 inches), had the testicles considerably less developed, as if they were shrunk; but I was enabled, by injection of mercury, to trace the vas deferens in its entire course, whilst I had only partially succeeded in this respect by the use of coloured fluid in the former example.

The testicles are entirely separate from each other; they are long flat bodies lying on each side of the lung, between the intestine and the walls of the abdomen. The left extends from the diaphragm to the middle of the kidney in one example, whilst in the other the anterior 2 inches of its length are replaced by an adipose band, as in the adult female described above (p. 547). The right does not extend so far forwards, being arrested by the lateral lobe of the liver, with the extremity of which it is intimately con-

postanal orifices, and that the oviducts had no function, representing merely the remains of the ducts of the Wolffian bodies. However, now it it is clear enough that these ducts resemble a Urodelous oviduct in every way. But have the peritoneal canals no function whatever? They might be useful for discharging semen or ova, which, having lost their way to the abdominal aperture of the oviduct, might injuriously act if retained in the abdominal cavity. The presence of a pair of peritoneal openings in *Acipenser*, one on each side of the vent, is mentioned by several authors. I do not find them in a young example of *Acipenser sturio*, var. *oxyrhynchus*, 24 inches long. In a specimen of *Acipenser maculosus*, 20 inches long, there is one large aperture on the left side, but none on the right. *Lepidosteus* has two *pori abdominales*, lateral to the vent.

* Received whilst this paper was passing through the press.

nected by the *vena cava* and cellular tissue (Plate XXXIX. fig. 1, *g*). The outer margin of the testicles is sharp, slightly undulated or indented. In one example a deep notch almost entirely cuts off a posterior lobe from the right testicle; therefore it is evident that this organ varies considerably with regard to the details of its form, as in other Fishes and Batrachians. The greater (outer) portion of the testicle is free, covered by the peritoneum, which passes thence over to the intestine. The inner third of the ventral surface of the testicle has no peritoneal covering, and is firmly attached to the intestine. Its innermost margin is fixed to the side of the lung.

The structure of the testicle was found not to be identical on both sides. In the *left* testicle three strata can be distinguished on external inspection and by transverse sections made about the middle of its length, viz.:—a light liver-coloured substance, forming by far the greater portion of the organ; then a much thinner and narrower stratum of whitish colour, lying on the liver-coloured substance along the line of attachment to the intestine; finally a still more superficial and still narrower layer of a dark yellow fatty blastema which accompanies the *vena testicularis*. A duct traverses the whitish substance from one end of the testicle to the other (Plate XL. fig. 3 & 4, *a*); it is widest in the middle (scarcely one sixteenth of an inch), and tapers towards its extremities, without penetrating to the surface of the testicle; its walls are perforated by innumerable pore-like openings, leading immediately into the *canaliculi seminiferi* (*d*). Coloured fluid injected into the duct was equally distributed throughout the substance of the testicle, through the whitish portion as well as the liver-coloured; but in the former the canaliculi seminiferi were more distinct, visible to the naked eye, densely packed, parallel to one another, arranged in obliquely decussating rows. The course and arrangement of the canaliculi in the liver-coloured substance of this (left) side could not be clearly made out, as it had too much suffered by decomposition; but on the right side they could easily be filled with fluid, at least those nearest to the surface: they run parallel to one another, across the testicle, at a right angle to its longitudinal axis; they have a slightly wavy course, do not subdivide, and appear to be equally wide throughout their length. The longitudinal duct (*a*) is present, as on the left side but the whitish stratum, if present at all, must be extremely thin, whilst the adipose substance is spread over the inner third of this testicle, surrounding the *vena cava*. In the second example with shrunk testicles I was unable to find again the longitudinal duct.

The *vasa deferentia* are, with regard to their course and orifices, entirely analogous to the oviducts. Their abdominal orifices occupy exactly the same spot as those of the oviducts. They run (Plate XL. fig. 1, *g'*), separated from the testicles by the peritoneum, in a slightly undulated course, towards the posterior end of the testis, are intimately attached to the ventral margin of the ureters (Plate XLI. *o*, *o'*), their canal remaining perfectly distinct from that of the latter, and finally they terminate in a common opening in the dorsal wall of the cloaca (ibid. *p*, or Plate XL. fig. 2, *g''*). This opening is separated from that of the ureters by a low fold of mucous membrane. In their anterior third they are very thin, but their lumen is nearly 2 millims. wide.

They gradually become narrower, until in the middle of their length the diameter of the duct is scarcely more than half a millim. Posteriorly they widen again, and the wall becomes thicker, the mucous membrane being raised into longitudinal folds; but even towards the end they are scarcely half as wide as the ureter.

The manner in which the semen is expelled is not quite understood at present. There is no connexion between the ureters and the testicles. Mercury injected through the ureter penetrated into all parts of the kidneys, and in one case (evidently where the substance had been ruptured) into the vascular system, but never into any part of the substance of the testicle; and as there is no direct communication between the testicle and vas deferens, it is probable that the semen flows into the cavity of the peritoneum, and thence passes through the two abdominal openings of the deferent ducts, taking the same course as the ova in the female. But how does it get from the testicle into the peritoneal cavity? To answer this question, it will be necessary to examine specimens obtained during the process of spawning. Of those examined by me, one was, as I suppose, only approaching such a condition, whilst the other was far removed from it. Perhaps the following observation may assist in solving the question. I found in the left testicle, between the longitudinal seminal duct and the inner margin, a number of cavities, the form or connexion of which could not be determined, as the substance was very soft and evidently somewhat advanced in decomposition. In the corresponding portion of the right testicle I found only two of those cavities, behind the middle of the length of the organ; both were of the size of a pea, close together, and communicating with each other. During the injection of the posterior half of the seminal duct, air had been driven by the injected fluid into those cavities, thus proving a connexion between them and the duct; a narrow branch of the duct led *towards* them; and their interiors were clothed by a distinct membrane. Therefore it is possible that the semen is conducted by the common longitudinal canal to those cavities, where it accumulates. These cavities have very thin walls, and lie immediately behind the peritoneum.

The Affinities of Ceratodus *to other Recent Fishes.*

In the preceding description of the external and internal structure of *Ceratodus*, reference has so frequently been made to *Lepidosiren* (including *Protopterus*) that only a short recapitulation of the more important features of the organization of these fishes will be required to establish their close relationship. We have seen that, with regard to the form of the body and its integuments, the resemblance is striking even to the superficial observer. In both forms the dentition consists of two pairs of molars, with the addition of a pair of vomerine teeth; it is the same type, modified in the one for a carnivorous, in the other for an herbivorous diet. Two pairs of nasal openings within the mouth. The cartilaginous skeleton, with its tegumentary ossifications, not only exhibits the same embryonic condition, but in certain portions (as, for instance, in the central part of the scapular arch, in the first rib, in the pelvis, in the apophyses and dermal appendages) the one is almost a repetition of the other. The limbs are paddles supported by an axial skeleton.

In both the air-bladder has attained to the function of a lung, the blood being brought into contact with air by gills as well as by this lung. The separation of the atrium from the ventricle is effected by the same intricate contrivance, the principal vessels by which the atrium is filled being the same in number and having the same arrangement. The difference of food is not accompanied by a corresponding amount of modification of the intestinal tract; there is the same want of stomachic dilatation, the boundary of this region being indicated merely by a pyloric fold; the same short and straight intestine, with its complete spiral valve and scattered voluminous glands.

The greater part of the characters enumerated are peculiar to *Lepidosiren* and *Ceratodus*; and there is no other recent fish known at present which approaches them in having a similar combination of peculiarities of structure. *Therefore, in a natural system, these fishes must be more closely approximated to each other than to any third living form.* Yet, on the other hand, unexpected and extraordinary differences have been pointed out between them. Instead of finding in the conus arteriosus the longitudinal valves of the "Dipnoous" heart, transverse series of "Ganoid" valves were discovered. In *Lepidosiren* a paired lung is developed, at the expense of the water-breathing apparatus, a considerable portion of the branchial arches being destitute of gill-laminæ. In *Ceratodus* the case is reversed: the gills are in the most perfect state of development, whilst the air-breathing apparatus is confluent into a single cavity, receiving a scantier supply of blood from secondary branches of the aorta descendens. In *Lepidosiren* the gill-cover supports an opercular gill with respiratory function, in *Ceratodus* this rudimentary gill is reduced to a pseudobranchia. Finally, the ovaries of *Ceratodus*, instead of being closed sacs connected with the oviducts as in *Lepidosiren*, are lamellated, discharging their products into the peritoneal cavity, the orifices of the oviducts being separated and even remote from the ovaries. The peritoneal *porus* of *Lepidosiren* is narrow, single, and in front of the vent, in *Ceratodus* this way of communication is paired, wide, and behind the vent.

Thus, Lepidosiren *and* Ceratodus *are well-marked modifications of the same type, the former diverging more towards the Amphibians than the latter.*

With regard to the systematic value of some of the differential characters mentioned, we may be guided by analogous relations in Teleosteous fishes. Thus there will be not much doubt that the modifications of the respiratory organs should be regarded as of generic degree, whilst the difference in the structure of the ovaries may be used as a family character here as well as for Salmonoids or Characinoids. The singular arrangement of the valves in the conus arteriosus is a point of much deeper interest. There cannot be a question that *Ceratodus* should be referred to MÜLLER'S subclass of Ganoids, and excluded from that of *Dipnoi*, according to the chief characteristics by which he has defined these divisions.

But I have pointed out above that *Ceratodus* and *Lepidosiren* are most closely allied to each other, and that, even if we regard them as representatives of two distinct *families*, they certainly cannot be referred to two separate *subclasses*. Consequently we

shall be obliged either to abandon the subclass *Dipnoi* altogether, or, if we maintain it, to materially alter its definition, so as to comprise *Ceratodus* as well as *Lepidosiren.* It appears to me that of these two courses the former ought to be adopted, and that the *Dipnoi* should be united with the Ganoids, among which they may form a well-marked subdivision, like the *Holocephala* among the Chondropterygians.

In order to justify this view, I will compare the chief characters of the *Dipnoi* with those of Ganoid and Chondropterygian fishes. The presence of a pulsating conus arteriosus is common to all three divisions; a notochordal skeleton is found in the Acipensers* and Chimæras; a cartilaginous skeleton of the paired fins is fully developed in the Chondropterygians, being only foreshadowed by the single row of cartilaginous rods in the fins of *Polypterus*; the cellular air-bladder of many Ganoids, which in *Polypterus* communicates with the ventral side of the œsophagus, is by all anatomists admitted to be a very close advance towards the Dipnoous lung; the gills of *Ceratodus*, which extend and are fixed to the walls of the branchial cavity, represent an arrangement indicating the first step towards the fixed gills of the Sharks and Rays; accessory external gills have hitherto been found in certain examples of *Protopterus* and *Polypterus*†; the presence of a spiral valve in the intestine, again, is common to Ganoids and Chondropterygians; finally, the convoluted oviducts with the addition of peritoneal openings, and the numerous ova of small size, remind us unmistakably of similar conditions among Ganoids, for instance *Acipenser* and *Lepidosteus*‡.

The only remaining absolute characters by which we can distinguish the Dipnoi from the Ganoids and Chondropterygians are the position of the nostrils within the mouth, the dentition consisting of two pairs of molars and one of vomerine teeth, and the lobate fins or paddles, supported by an *axial* cartilaginous skeleton.

These three points cannot be considered to constitute the characters of a subclass equivalent to *Teleostei*, *Ganoidei*, &c.; and therefore the *Dipnoi* are better united with the Ganoids.

But it appears to me that also the Ganoids and Chondropterygians are much more closely allied to each other than either of them to the Teleosteans. It would be beyond the limits of this paper to enter into all the various points of organization which have to be considered with regard to this question. But I may urge, as the most important fact in favour of this view, the treble partition of the heart of all these fishes—the bulbus aortæ of the Teleosteans being, as MÜLLER has shown, simply the thickened origin of the aorta, separated from the pulsating heart by the (almost always) double valve. A heart with a true *conus arteriosus* is always accompanied by a spiral valve of the intestine (which only exceptionally, as in *Lepidosteus*, remains in a rudimentary condition), and by non-

* The similarity between vertical sections of the cartilaginous brain-capsule of *Acipenser* and *Ceratodus* is surprising.

† In the latter genus only recently discovered by STEINDACHNER; they may be found also in young *Ceratodus* and other Ganoids.

‡ The male organs of *Lepidosiren* are not known, and those of Ganoids only imperfectly; Plagiostomes have a deferent duct, but it is in direct communication with the sperm-gland.

decussating optic nerves. The *Holocephala*, which differ in several important points from the other Chondropterygians, approach the Ganoids by these very characters, and are an intermediate form. Furthermore, all those modifications which show an approach of the ichthyic type to that next above it, are found in Ganoids and Chondropterygians, none in Teleosteans; and, finally, the early coexistence and development of Ganoids and Chondropterygians in geological epochs, when no (or only very few) Teleosteans existed, is a circumstance which seems to confirm a conclusion arrived at from an anatomical point of view only.

Therefore I would propose, after the separation of the *Cyclostomata* and *Leptocardii*, to refer the remaining host of living fishes to two subclasses only, viz. the subclass of *Teleostei*, and one for which the name *Palæichthyes* may be used. In order to put the preceding statements into a readily comprehensible form, I have prepared the following synoptical table, in which, at present, reference is made to those fossil genera only which approach most closely the *Dipnoi*:—

First subclass: LEPTOCARDII.
Second subclass: CYCLOSTOMATA.
Third subclass: TELEOSTEI.
Fourth subclass: PALÆICHTHYES. Heart with a contractile conus arteriosus; intestine with a spiral valve; optic nerves non-decussating.

Order I. CHONDROPTERYGII.

Suborder 1. *Plagiostoma.*
Suborder 2. *Holocephala.*

Order II. GANOIDEI.

Suborder 1. *Amioidei.*
Suborder 2. *Lepidosteoidei.*
Suborder 3. *Polypteroidei.*
Suborder 4. *Chondrostei.*

Fam. *a. Acipenseridæ.*
Fam. *b. Polyodontidæ.*

Suborder 5. *Dipnoi.* Nostrils two pairs, within the mouth. Limbs with an axial skeleton. Lungs and gills. Skeleton notochordal. No branchiostegals.

Fam. *a. Sirenidæ.* Caudal fin diphycercal; no gular plates. Scales cycloid. Two pairs of molars and one pair of vomerine teeth.

Subfamily *Ceratodontina.* Conus arteriosus with transverse series of valves. Ovaries transversely lamellated. One continuous vertical fin: *Ceratodus* [*Cheirodus*?].

Subfamily *Protopterina.* Conus arteriosus with two longitudinal valves. Ovaries closed sacs. One continuous vertical fin: *Lepidosiren, Protopterus.*

Fam. *b. Ctenododipteridæ.* Caudal fin heterocercal; gular plates. Scales cycloid. Two pairs of molars and one pair of vomerine teeth: *Dipterus.*

?(Fam. *c. Phaneropleuridæ.* Caudal fin diphycercal; gular plates. Scales cycloid. Jaws with a series of minute conical teeth on the margin: *Phaneropleuron.*)

On the Affinities of Ceratodus *to certain Fossil Fishes.*

Those who have followed the researches made of late years into the affinities of Ganoid fishes, must have remarked that in the conclusion of the preceding chapter I have touched upon a subject which had been approached by Professor HUXLEY from a palæontological point of view*. One result of his examination of the Devonian fishes was the establishing a separate suborder of Ganoids, *Crossopterygidæ*, which comprises the Ganoids provided with fringed or lobate fins and generally with gular plates, branchiostegals being absent. This suborder was divided by him into five families of extinct fishes, chiefly from the Devonian epoch, viz. *Saurodipterini*, *Glyptodipterini*, *Ctenododipterini*, *Phaneropleurini*, and *Cœlacanthini*; *Polypterus* was associated with them as

* "Preliminary Essay upon the Systematic Arrangement of the Fishes of the Devonian Epoch," by T. H. HUXLEY, F.R.S., in Mem. Geolog. Survey, Dec. 10, 1861. At that time, when *Lepidosiren* was still the only representative of MÜLLER'S subclass *Dipnoi*, Professor HUXLEY pointed at its affinity to certain Ganoids:—"Without wishing to lay too much stress upon the fact, I may draw attention to the many and singular relations which obtain between that wonderful and apparently isolated fish, *Lepidosiren*, sole member of its order, and the cycloid Glyptodipterine, Ctenododipterine, Phaneropleurine, and Cœlacanth Crossopterygidæ. *Lepidosiren* is, in fact, the only living fish whose pectoral and ventral members have a structure analogous to that of the acutely lobate, paired fins of *Holoptychius*, of *Dipterus*, or of *Phaneropleuron*, though the fin-rays and surface-scales are still less developed in the modern than in the ancient fish. The endoskeleton of *Lepidosiren*, again, is as nearly as possible in the same condition as that of *Phaneropleuron*, and is more nearly similar to the skeleton of the Cœlacanths than that of any other recent fish [*quære Acipenser*? A. G.]; while, perhaps, it is not stretching the search for analogies too far to discover in the stiff-walled lungs of *Lepidosiren* a structure more nearly representing the ossified air-bladder of the Cœlacanths than any with which we are at present acquainted, among recent or fossil fishes. Furthermore, *Lepidosiren* is the only fish whose teeth are comparable in form and arrangement to those of *Dipterus*. Though *Lepidosiren* may not be included among the *Crossopterygidæ*, nor even in the order of *Ganoidei*, the relations just pointed out are not the less distinct; and perhaps they gain in interest when we reflect that while *Polypterus*, the modern representative of the rhombiferous *Crossopterygidæ*, is that fish which has the most completely lung-like of all air-bladders, *Lepidosiren*, which has just been shown to be, if not the modern representative of the cycliferous *Crossopterygidæ*, yet their 'next of kin,' is the only fish which is provided with true lungs. These are unquestionable facts. I leave their bearing upon the great problems of zoological theory to be developed by every one for himself."

the type of a sixth family. *Ceratodus*, having the fins lobate in an eminent degree, would appear to belong to this suborder; and therefore I shall attempt to examine the affinities of the *Dipnoi* to some Crossopterygian fishes, and finally, perhaps, arrive at a conclusion regarding the limits of this suborder.

I take first the type of the family *Ctenododipterini*, viz. the genus *Dipterus*.

The materials on which the following observations on *Dipterus* are based, are examples in the British Museum, and especially some most instructive specimens belonging to the Museum of the School of Mines in Jermyn Street, and kindly offered for my examination by Professor HUXLEY. PANDER's excellent Monograph, "Ueber die Ctenodipterinen des devonischen Systems," St. Petersb. 1858, 4to, was my guide in examining these materials.

The characters of *Dipterus* which appear to indicate an affinity to *Ceratodus* may be described thus:—They are fishes with an elongate body covered with cycloid scales; head depressed, snout obtuse; dorsal and anal fins thrown back, belonging to the caudal portion of the vertebral column; paired fins acutely lobate; opercular apparatus well developed, consisting of two, possibly three pieces. No separate maxillary or intermaxillary. The bones of the lower part of the skull and the mandible essentially as in *Ceratodus*, only the pterygo-palatine shows a longitudinal groove as if it had consisted of two bones. Vertebral column not divided into vertebræ*. The palate and mandible armed with a pair of large flat tubercular dental plates, placed as in *Ceratodus*. To these characters, which were previously known, I may add that a specimen in the Jermyn-Street Museum (marked $\frac{\text{R.D.}}{156}$), figured on Plate XXXIV. fig. 4, shows clearly the *presence of a pair of vomerine teeth in* Dipterus. These teeth, indeed, are not preserved themselves; but two small roundish cavities (*v*), one on each side of the median line of the skull, in front of the palatine teeth, indicate distinctly the place where a pair of vomerine teeth were implanted in the cartilaginous substance of the vomer. In the same specimen the position of the nostrils (*n*) is indicated by an accumulation of the matrix; it is the same as in *Ceratodus* and *Lepidosiren*.

On the other hand, *Dipterus* differs from *Ceratodus* in being eminently heterocercal; there are two dorsals and one anal fin separate from the caudal; the scales are covered with a layer of enamel, with smooth, porous surface. Head covered with numerous scutes with the same surface as the scales; gular plates behind the mandibulary symphysis. Interneural and interhæmal spines branched at their distal end, to which the dermal rays are joined, the latter being branched and jointed. Finally, as regards the microscopical structure of the dental plates, the medullary canals are more irregular in their course than in *Ceratodus*; they frequently anastomose with one another; the dentinal tubes are coarser, dendritically branched, and less numerous.

Can any thing be more singular than a combination of such characters? If we had had the head only of *Dipterus* with the trunk, we should certainly have seen in it

* No vertebra has ever been found in British specimens; and PANDER, who figures vertebræ (which he found in Russia) in connexion with this genus, is by no means certain of their correct determination.

nothing but a genus most nearly allied to *Ceratodus*, as shown by the arrangement of its three pairs of teeth, form of the snout, position of nostrils, acutely lobate fins, &c.; the presence of the gular plates alone might have offered ground for hesitation. Yet, when we examine the tail, we find that the one genus is truly diphycercal, the other eminently heterocercal, so that they cannot remain even in the same family. I do not attach much importance to the separation of the vertical fins, or to the division of the dorsal into two; such modifications are common in the Gadoid fishes, where they are considered to be scarcely of generic value. Of much greater importance is the junction of the dermal fin-rays with the supporting spines, and the position of the dorsal fin or fins. In the former point *Dipterus* differs from *Ceratodus*; in the latter both genera agree.

Weighing the points of affinity and difference against each other, we must come to the conclusion that *Dipterus* has a better right to be associated with the living *Dipnoi*, than with *Polypterus*.

Wherever *Dipterus* and *Ceratodus* are placed, thither *Cheirodus* (M'Coy, Pander) or *Conchodus* (M'Coy) must follow. But it is probable that this genus is more nearly allied to *Ceratodus*.

At first I thought that *Holodus* (Pander) was another Dipnoous genus; but I changed this opinion after having compared it with *Palædaphus* of Van Beneden and De Koninck (Bull. Ac. Roy. Belg. 2nd ser. xvii. p. 151, pls. 3 & 4). These two genera are evidently closely allied; and the position of their nostrils (as far as we can judge from the fragmentary remains) appears to have been different from that of the Dipnoi; these openings were more lateral, and outside of the mouth. It seems also that there would not have been room for a pair of vomerine teeth, at least not in *Palædaphus*.

There are two other genera of fossil fishes of the Devonian epoch which have been referred to the Crossopterygians, and which appear to approach the Dipnoi more closely than the other fringe-finned fishes, viz. *Phaneropleuron* and *Tristichopterus*. With regard to the former genus I refer to Professor Huxley's description and figures in Anderson's 'Dura Den,' p. 67 *et seqq.*, and Mem. Geol. Surv. Dec. 10, p. 24 *et seqq.* The structure of the skeleton of the trunk and tail and of the fins is extremely similar to that of *Ceratodus*. In those specimens in which I found the foremost part of the snout tolerably well preserved, no interruption in the surface of the osseous substance could be discovered; and therefore it is not improbable that the nasal openings were inside the mouth as in *Dipnoi*. On the other hand, minute conical teeth in rather small number were visible in the margins of the upper as well as lower jaw, thus indicating by their presence a development of maxillary elements which are entirely missing in the *Dipnoi*.

The genus *Tristichopterus* was established and described by Sir Philip Egerton in Mem. Geol. Surv. Dec. 10, p. 51 *et seqq.*, where already the affinities of this highly interesting fish to *Dipterus* were fully considered by its author. The fins were composed of innumerable fine fin-rays closely placed together, and overlapping with their proximal

ends the extremities of the interspinous bones as in the *Dipnoi*. But these bones are very much reduced in number, and enlarged, the vertical fins themselves being entirely separate from one another. The peculiar termination of the vertebral column, with an unequal development of the caudal lobes, represents a most curious intermediate condition between the diphycercal tail of the *Sirenidæ* and the heterocercal of *Dipterus*. To this is added "the complete ossification and segmentation of the vertebral column, in which respect this genus stands alone among the contemporaneous fishes." Unfortunately the head and base of the paired fins are destroyed in the only two specimens known; and it is chiefly the last-named character which prevents me from associating this genus with the *Dipnoi*.

However uncertain the affinities of the last-named genera must appear from a zoological point of view, I believe that I have shown that *Ceratodus* clearly proves the correctness of Professor HUXLEY's view regarding the similarity of the *Lepidosiren*-limb with the fringed fin of certain Crossopterygians, and that *Ceratodus*, *Lepidosiren*, and *Dipterus* are most closely allied forms, and must remain together in the same suborder. Consequently if we retained the suborder Crossopterygians with the limits assigned to it by its author, it would comprise four recent Ganoid genera, viz. *Polypterus*, *Ceratodus*, *Protopterus*, and *Lepidosiren*; or, in other words, these four genera would require to be regarded as more nearly allied to each other than to the other recent Ganoids. I am not prepared to adopt this view. MÜLLER was certainly right in regarding the condition of the skeleton as a character of primary importance for the systematic division of Ganoids, and in supporting his opinion by the analogous case of the Chondropterygians, of which the Plagiostomes have the skeleton completely divided into segments, whilst the "Holocephala with their notochordal skeleton form a division distinguished in a marked manner also in other respects" (Ganoid. p. 150). Thus also I have had repeated occasion to draw attention to identity of structure in *Ceratodus* and *Acipenser*; and when we consider that the notochordal or segmented condition of the skeleton is systematically of as great importance at least as the state of development of an air-bladder into a lung, we must admit that the points of affinity of *Ceratodus* to *Acipenser* counterbalance those existing between *Ceratodus* and *Polypterus*, and consequently that the fishes named are representatives of *cosubordinate* divisions of *Ganoidei*.

Thus, from an examination of the living forms, I am inclined to withdraw from the Crossopterygians some of the component parts of this suborder; and I am encouraged in this by the following additional consideration. With the knowledge obtained by means of *Ceratodus*, we are now able to define more distinctly two types of "fringed fin," already indicated by Professor HUXLEY, who terms them "acutely lobate" and "obtusely lobate." Fringed fins of the former type are long, pointed, like those of *Ceratodus*, or even narrower; they are covered with small scales along the middle, and surrounded by a cutaneous fringe containing innumerable fine fin-rays. There can scarcely be any doubt that such fins in fossil fishes were supported, as in *Dipnoi*, by an axial cartilaginous skeleton, extending from the base to the extremity. I have never

been able to find any trace of it preserved in the fossil remains, as it is entirely destroyed like the remainder of the cartilaginous portions. In the fringed fins of the second type, termed "obtuse" or "subacutely lobate," the scaly covering is limited to the base; the fin-rays are very distinct, comparatively few in number, and joined to a simple transverse series of cartilaginous rods. Such are the fins of *Polypterus*, of the Cœlacanths as restricted by HUXLEY, and probably also of the *Saurodipterini*. Having drawn attention to this, as I believe, important diversity in the only character on which the suborder *Crossopterygii* is founded, I leave the bearing of this fact upon the further classification of these fossils to be developed by those who are more intimately acquainted with the details of their structure than myself.

I should be found wanting in respect towards the founder of Palæichthyology, were I to pass over in silence the opinion expressed by him with regard to the affinities of *Ceratodus*. In a letter addressed to Sir PHILIP EGERTON ('Nature,' iii. No. 61, p. 166), Professor AGASSIZ states that this fish is clearly a member of the family named by him *Cœlacanthini*. I am not aware how far AGASSIZ has modified the characters and limits of this family since the publication of the 'Recherches' (vol. ii. p. 168) and 'Vieux Grès Rouge' (p. 64); at that time he pointed out as the principal characters the hollow condition of the fin-rays and bones (as in birds), the presence of interspinous bones in the caudal fin, the continuation of the vertebral column between the two lobes of that fin, and the prolongation of the caudal extremity beyond it as a filamentary appendage. *Arapaima* (*Sudis*) *gigas* was stated to be one of the living representatives of this family. Now it is scarcely correct to describe the fin-rays and bones of these fishes as hollow like "those of birds;" before they underwent the alterations during the process of decomposition and fossilization, they were *thoroughly solid*, without hollow space in the interior, like the fin-rays and bones of *Ceratodus* and other notochordal fishes, with or without a bony covering. With regard to the second and third characters, they are common to so many other fishes not reckoned among the Cœlacanths, that no safe conclusion can be drawn from them regarding the affinities of a fish. Finally, the prolongation of the caudal extremity as a filamentary appendage is not observed in *Ceratodus*. No student of recent ichthyology has followed AGASSIZ in placing *Arapaima* among the Cœlacanths, or indeed among the Ganoids; MÜLLER has shown it to be a true Teleostean; and the degree of affinity between this genus and *Ceratodus* is not greater than that between a Salamander and a Lizard*.

THIOLLIÈRE and HUXLEY have independently come to the conclusion that AGASSIZ's family of Cœlacanths comprises too many heterogeneous forms to allow us to regard the affinities of a fish as determined by its being referred to it. Both have limited the term to the genus *Cœlacanthus* as type, and a few other forms closely allied to it. The family thus restricted appears to me even more remote from the *Dipnoi* than the

* Botanical science is in this respect more advanced than Zoological: no botanist would allow himself to be influenced by merely external similarity; and a system in which the African *Euphorbia* were associated with the American *Cactus* would be repudiated by all.

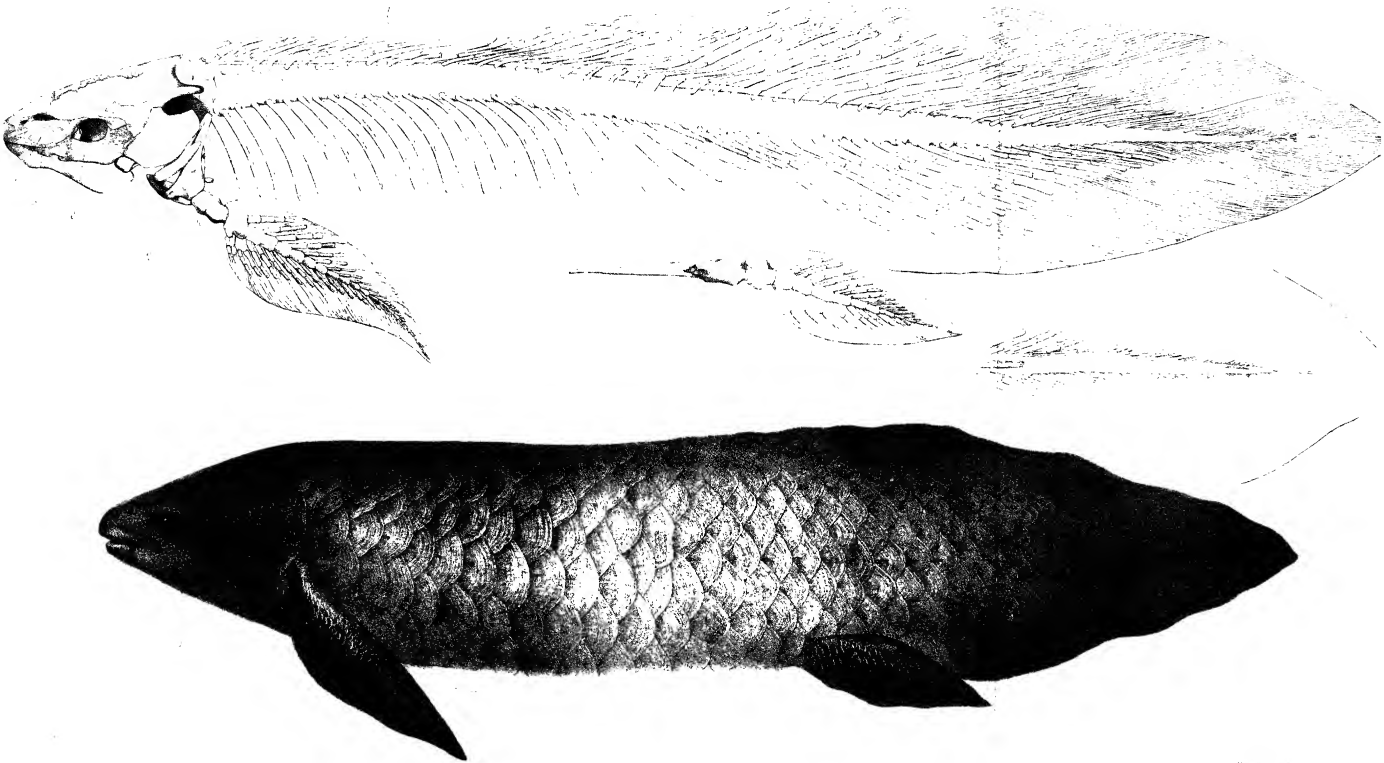
other "Crossopterygians." Fin-rays in definite numbers, joined to the interspinous bones, obtuse paired fins as in *Polypterus*, a double pelvis, developed upper jaws with small conical teeth, external nostrils, are characters sufficient to prevent us from associating *Ceratodus* and its allies with the Cœlacanths, which form not only a distinct family, but belong, according to my view, to a distinct suborder*.

Concluding Remarks.

The great interest attached to the discovery of *Lepidosiren* lay in the combination of elements of its organization denoting an approach to the Amphibian type, with others which remind us of embryonic stages of development. The supposition of some zoologists, who saw in *Lepidosiren* an instance of the latest step of advancement attained by the struggling ichthyic type towards the higher class, that of Amphibians, is not confirmed; for we find that the *Dipnoi* reach back, with comparatively insignificant modifications, into one of the oldest epochs from which fish-remains are preserved. The modification which would appear to demand our special attention is the condition of the tail, heterocercy being generally taken to be a lower degree of development than diphycercy. Thus one might suppose that the heterocercal *Dipterus* of the Devonian epoch is the less highly organized ancestor of the diphycercal *Lepidosiren* and *Ceratodus*; but then, on the other hand, *Phaneropleuron*, and also *Tristichopterus*, prove that diphycercy is a condition attained even at that early time by fishes more or less closely allied to the Dipnoi.

The defenders of the doctrine of evolution hold that a space of time like that which elapsed from the Devonian epoch to our period, is as a drop in the sea, when compared with the time required for the development of organic life. From this point of view the Ganoid fauna of the Devonian and even the *Pteraspis* of the Upper Ludlow formation, must have been preceded by long and varied ichthyic series. *Tristichopterus* with its osseous vertebral column may have been the surviving representative of a Ganoid suborder then extinct, as *Polypterus* has been regarded as the sole survivor of Crossopterygians; and the Dipnoous type, as it appears to us for the first time in the Devonian epoch, was not the beginning of a series, but the last of many preceding developmental stages. Future discoveries in hitherto unexplored formations may prove these suppositions to be quite correct; but when we limit ourselves for the present to the actual evidence before us, we find it consists of the following facts only:—The Dipnoous type is represented in the Devonian and Carboniferous epochs by several genera (*Dipterus, Cheirodus, Conchodus, Phaneropleuron*); it is then lost down to the Trias and Lias, where the scanty remains of a distinct genus, *Ceratodus*, testify to its presence; no further trace of it has been found until the present period, where it reappears in three genera, one of which is identical with that of the Mesozoic era. Now at present scarcely any

* I do not attach any value to the terms subfamilies, families, suborders, &c., except as expressions of the relative degree of affinity; and in the preceding notes I have used them in accordance with the synoptical table published by Professor HUXLEY.



zoologist will deny that there must have been a continuity of the Dipnoous type; and it is only a proof of the incompleteness of the palæontological record, that we have to derive all our information regarding it from only three so very distant periods of its existence.

The *Dipnoi* offer the most remarkable example of persistence of organization, not in Fishes only, but in Vertebrates. On a former occasion I have shown that numerous recent *species* of fishes have survived from the period of the geological changes which resulted in the separation of the Atlantic and Pacific by the Central-American isthmus. In *Ceratodus* we have now found a *genus* which, as far as evidence goes, persisted unchanged from the Mesozoic era; and in the *Sirenidæ*, a *family* the nearest ally of which lived in the Palæozoic epochs.

Perhaps future palæontologists will be able to demonstrate as complete a series of transitional stages from the Fish to the Amphibian as that obtained by the study of the living and therefore more accessible forms of Hæmatocrya. Zoologists have had to abandon the attempt to separate the two classes by one or several absolute characters; and it is only the concurrence of either decidedly ichthyic or amphibian characters by which they refer a creature to the one or the other class. However, this is a problem upon which *Ceratodus* has only a distant bearing; and I am satisfied if I have succeeded in showing its relations to *Lepidosiren*, and the connexions of both genera with the Palæichthyic type.

Explanation of the Plates.

PLATE XXX.

Fig. 1. *Ceratodus miolepis*, male, $\frac{1}{3}$ the natural size.

Fig 2. Skeleton of *Ceratodus forsteri*, $\frac{1}{3}$ the natural size.

In the side view of the skull, the facial cartilages are preserved; their extent will be apparent from a comparison of this figure with fig. 1 of Plate XXXV. The three slits on the side of the snout, in front of the eye, are places over which the cartilage does not extend; they are perfectly closed by the skin.

Fig 3. Termination of the vertebral column of another example.

a. Entrance into labial cavity.

b. The extent of the labial cavity is indicated by dotted lines.

c. Cavity in the interior of the infraorbital cartilage.

l. Ligamentum longitudinale.

m. Antebrachial.

n. Termination of the notochord.

t. Last segments of the confluent neural and hæmal elements.

PLATE XXXI.

Structure of the Scales.

Fig. 1. A scale taken from the lateral line of *Ceratodus forsteri*, outer surface. Natural size.

Fig. 2. Inner surface of the same scale.

Fig. 3. One half of a vertical transverse section across the middle of the scale, magn. 8. The vertical divisions (*a*, *a*) indicate the place where the section passed through the sutures dividing the outer surface into trapezes.

Fig. 4. A portion of the same section, magn. 216.
- *a.* A layer of fibres cut transversely.
- *b.* A layer of fibres cut obliquely.
- *c.* A layer of fibres cut longitudinally.
- *d.* Outer calcified stratum.

Fig. 5. Three layers of fibres crossing each other at angles of 90° or 45°, magn. 216.

Figs. 6–8. Portions of the calcareous stratum, after removal of the organic matter by burning, magn. 216: 6, from the middle of the lateral area of the scale; 7, from the exposed area; 8, from the posterior area.

Fig. 9. Right upper tooth of *Ceratodus forsteri*, nat. size.

Fig. 10. Right upper tooth of *Ceratodus runcinatus*, from the Muschelkalk, nat. size. From a specimen in the British Museum.

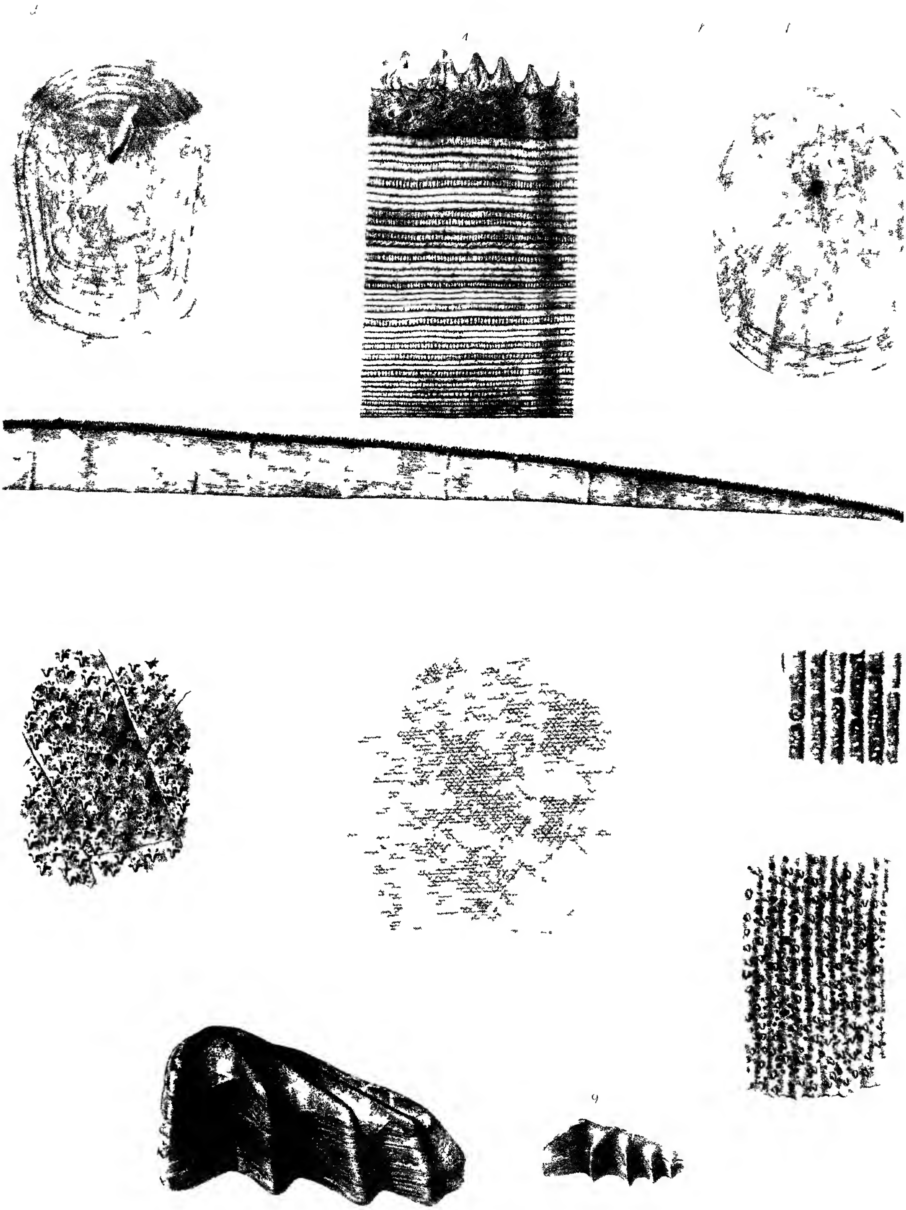
PLATE XXXII.

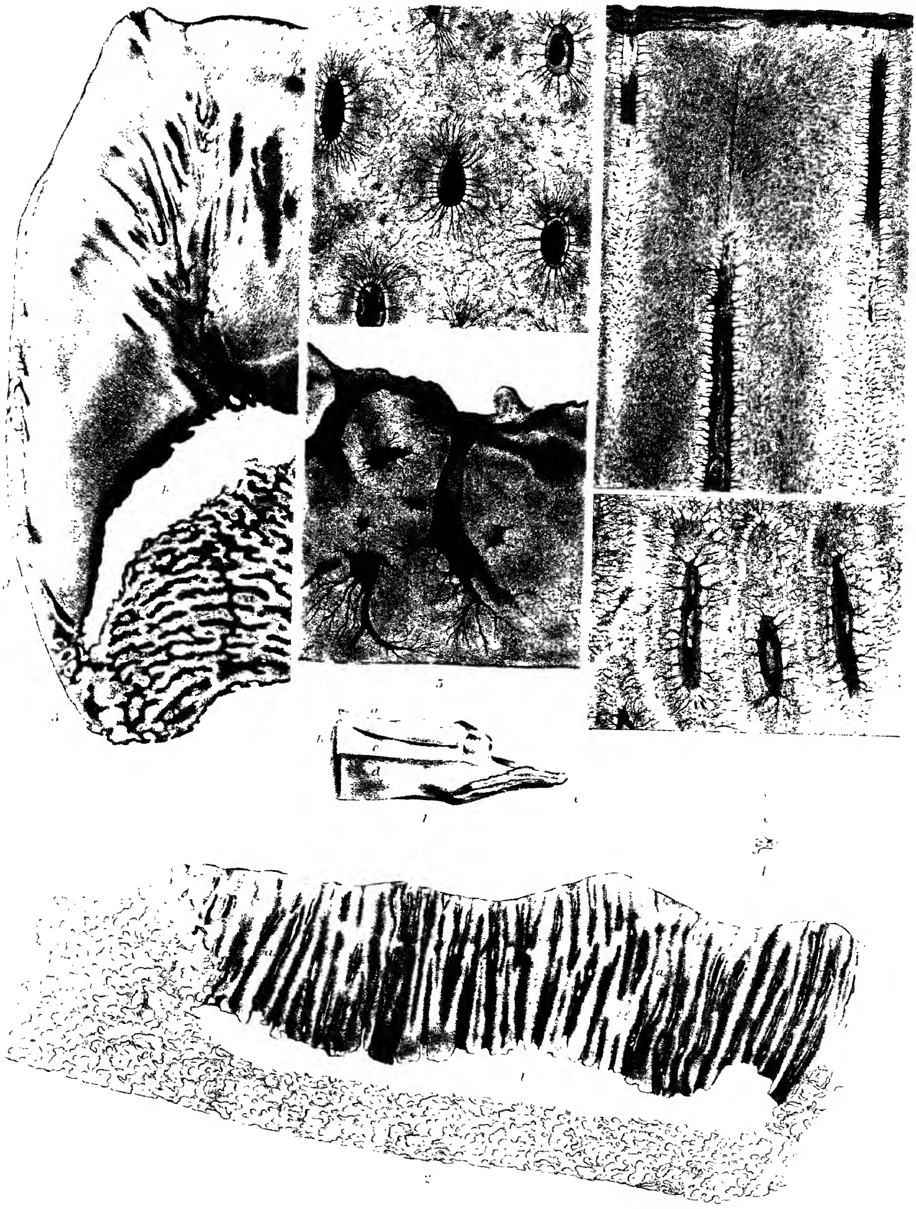
Microscopical Structure of the Teeth, compared with that in Fossil species, Protopterus *and* Psammodus.

Fig. 1. Front part of one half of the lower jaw, with a vertical section through the tooth (nat. size).
- *a.* Dentine.
- *b.* Pulp-cavity.
- *c.* Osseous base of tooth.
- *d.* Dentary bone.
- *e.* Cartilaginous symphysis.

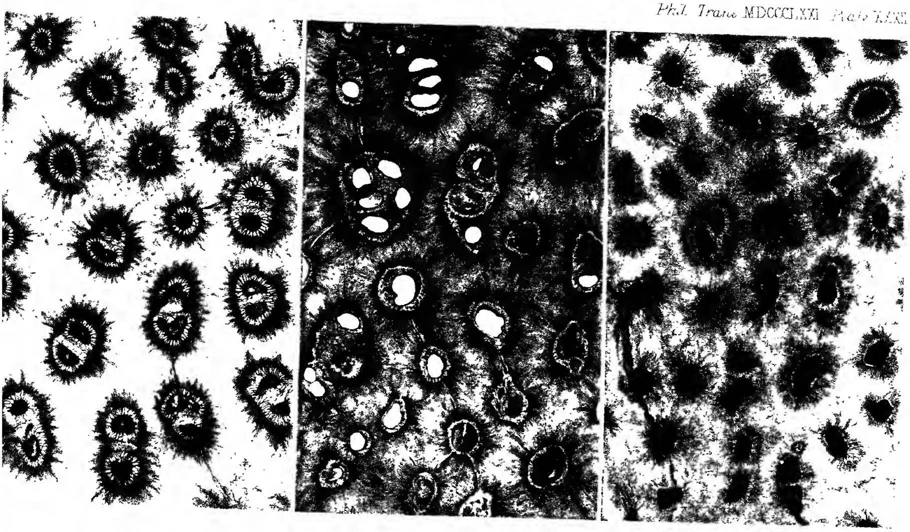
Fig. 2. The same section through the lower tooth, magnified: the clear vertical line indicates that a portion of the length of the tooth is left out.
- *a.* Dentine with medullary canals.
- *b.* Pulp-cavity.
- *c.* Spongious osseous base of the tooth.

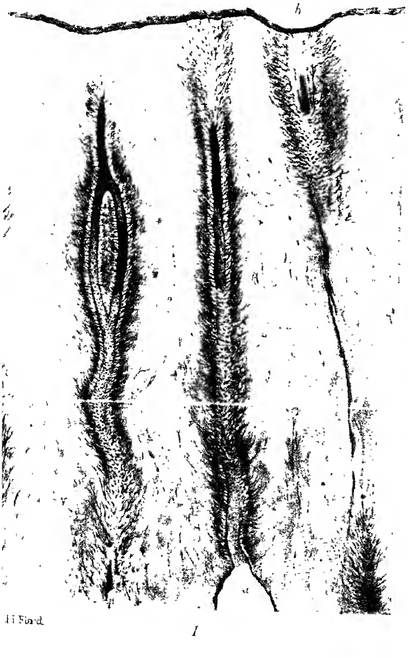
Fig. 3. Vertical section through the front prong of the lower tooth of *Protopterus*, magn. 20.
- *a.* Spongious osseous base of tooth.
- *b.* Pulp-cavity.





Gunther





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H Ford

Mintern Bros imp.

c. Origin of central medullary canals, which ramify in an irregular manner, chiefly in the direction towards the point of the prong.

d. Narrower superficial medullary canals, running in an oblique or vertical direction, and emitting horizontal branches.

Fig. 4. Horizontal section through the lower tooth of *Protopterus*, of natural size, the darkened part being that of which a magnified view is given in figure 5.

Fig. 5. Part of a horizontal section through the front prong of the mandibulary tooth of *Protopterus*, magn. 216. It comprises several of the smaller medullary canals represented in fig. 3, *d.* Two are nearly horizontal, so that their whole extent is visible; others run in a vertical direction. They ramify in a very irregular manner, and are not surrounded by a dark ring of calcigerous tubes.

Fig. 6. Vertical section through the tooth of *Psammodus*, magn. 216. The horizontal white line indicates that a portion of the depth of the tooth is left out. The upper portion contains three medullary canals, two penetrating to the surface, where they form the "pores;" in the lower portion the canals have a more oblique direction, and consequently the vertical section passes slantingly through them; the concentric lamellated arrangement of the dentinal substance round the canals is also conspicuous, whilst it has disappeared nearer to the surface of the tooth.

Fig. 7. Horizontal section through the tooth of *Psammodus*, magn. 216, made nearer to the root than to the surface.

PLATE XXXIII.

Microscopical structure of the teeth (continued).

Fig. 1. Vertical section through the medullary canals of *Ceratodus forsteri*, magn. 216. The clear horizontal line indicates that a portion of the length of the tooth is left out.

a. Termination of a medullary canal in the pulp-cavity.

b. Termination of a medullary canal in one of the punctate impressions on the surface of the tooth.

Fig. 2. Horizontal section through the same tooth, near the surface of the crown, magn. 216.

Fig. 3. Horizontal section through the same tooth, near its base.

Fig. 4. Portion of a vertical section through a tooth of *Ceratodus runcinatus* (magn. 20), for comparison with the recent species (Plate XXXII. fig. 2).

Fig. 5. Part of the same section (magn. 216), for comparison with the structure of the same part in the recent species (fig. 1). Three medullary canals open on the surface of the crown, producing the appearance of punctate impressions.

Fig. 6. Horizontal section through a tooth of *Ceratodus runcinatus* (magn. 216), for comparison with fig. 2.

PLATE XXXIV

Structure of the Skull, compared with that of Dipterus.

The cartilaginous parts are coloured blue; figs. 1–3 are $\frac{2}{3}$ the nat. size.

Fig. 1. Upper view of the skull, with the facial cartilages preserved.

Fig. 2. Upper view of the same, after removal of the greater part of the osseous covering and of the temporal muscle, to show the cartilaginous skull.

Fig. 3. Lower view of skull; the soft parts surrounding the upper jaw have been preserved, to show the position of the nostrils.

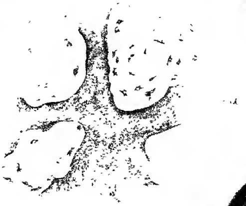
a. Scleroparietal.
b. Ethmoid.
c. Frontal.
d. Tympanic lamina.
e. Infraorbital ossifications.
f. Infraorbital cartilage; *f'*, facial cartilage.
g, g, g. Vacuities within the cartilage.
h. Operculum (removed in fig. 3).
i. Part of scapular arch.
k. Præoperculum.
l. Pterygo-palatine.
m, m'. Foramina for the 1st and 2nd rami of *nervus trigeminus*.
n. Nostrils.
o. Basale.
p. Cartilaginous roof of gill-cavity.
q. Os quadratum.
r. Tubercle for hyoid.
s. Suspensory pedicle.
t. Foramina for 3rd ramus of *nervus trigeminus*.
u. Foramen for *carotis posterior*.
v. Vomer, with pair of vomerine teeth.
w. Foramina for *nervus vagus*.
x. First rib.
x'. Cartilaginous tubercle attached to the first rib.

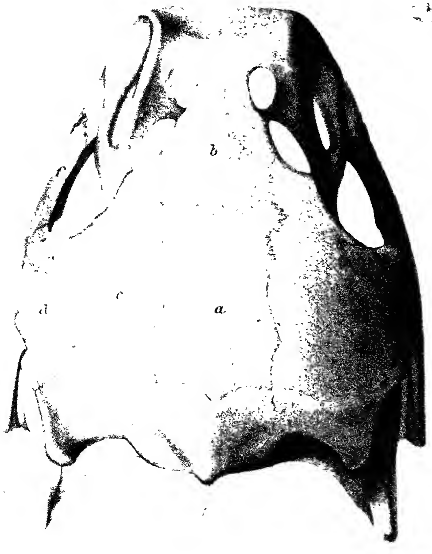
Fig. 4. Lower view of the skull of *Dipterus*, of the natural size, from a specimen in the Geological Museum in Jermyn Street, No. $\frac{\text{R.D.}}{156}$.

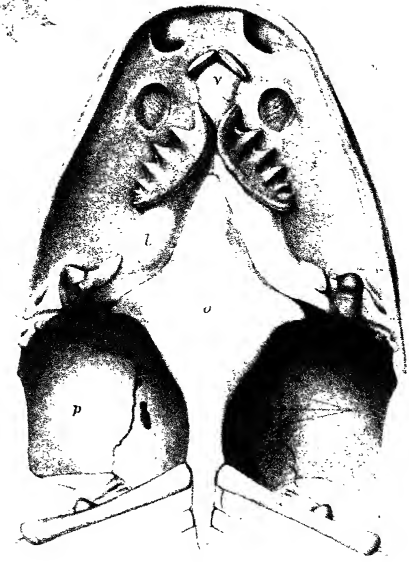
l. Pterygo-palatine.
o. Basale.
n. Excavated place where the two nostrils were situated.
v. A pair of round holes in which the vomerine teeth were fixed.

Figs. 5 and 6. Horizontal section of the anterior (fig. 5) and posterior (fig. 6) portion of the sclero-parietal bone, magn. 216.

a. Medullary canals.







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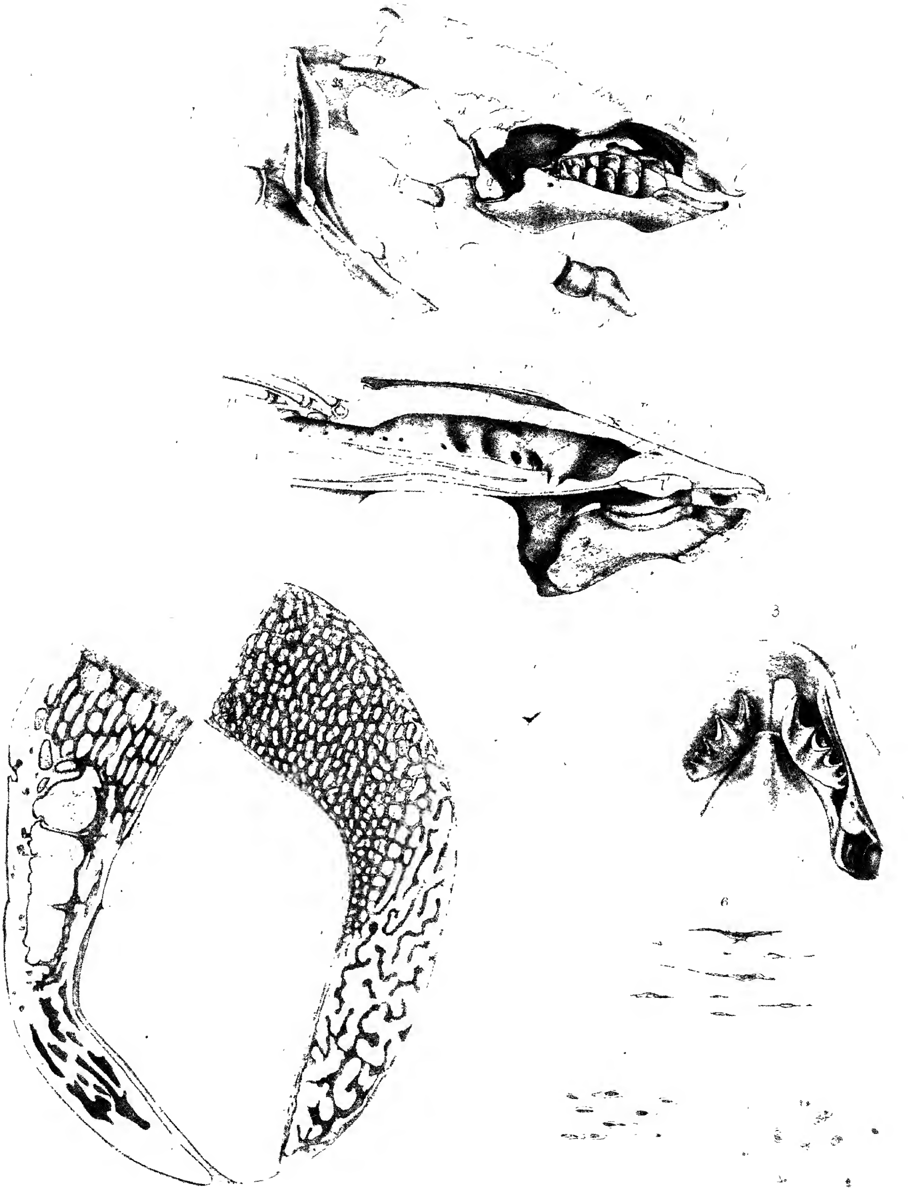
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PLATE XXXV.

Structure of the Skull (continued). Figures 1–3 are $\frac{2}{3}$ the natural size.

Fig. 1, Lateral view (the facial cartilages are removed).

Fig. 2. Vertical section of the skull made along the median line, showing also the inner surface of the mandible and molars.

To facilitate a comparison with figs. 1–3 of Plate XXXIV., the same letters have been used in these figures whenever possible.

a. Scleroparietal.
ac. Acoustic membrane.
b. Ethmoid.
b'. Foramen olfactorium.
bh. Basihyal.
c. Frontal.
c'. Process of frontal for its connexion with the pterygo-palatine bone.
cd. Section through notochord; *cd'*, end of central cylinder.
ch. Ceratohyal.
co. Upper coracoid; *co'*, lower coracoid.
d. Tympanic lamina.
e. Infraorbital ossifications.
f. Facial cartilage, turned upwards in fig. 1, and with its end longitudinally cut through in fig. 2.
gh. Glossohyal.
h. Operculum.
h'. Suboperculum.
hc. Humeral cartilage.
l. Pterygo-palatine; *l'*, symphysis of the pterygo-palatine bones.
la. Lower labial cartilage.
ll. Ligamentum longitudinale.
m. Foramen for 1st and 2nd ramus of *nervus trigeminus.*
m''. Foramen for 3rd ramus of *nervus trigeminus* and part of *nervus acusticus.*
mc. Median cartilage of scapular arch, cut through its median line.
ml. Molar teeth, viewed from the inside of the oral cavity.
mt. Insertion of *musculus temporalis.*
o. Basale.
ol. Passage for olfactory nerve.
op. Foramen opticum.
p. Cartilaginous roof of gill-cavity.
pt. Pituitary gland.
q. Os quadratum.

s. Suspensory pedicle.
ss. Suprascapula.
sy. Symphysis mandibulæ.
v'. Vomerine tooth.
w. Foramen for *nervus vagus.*

Fig. 3. Lower jaw, viewed from the oral cavity; on the left side the soft parts are preserved; on the right the osseous and cartilaginous parts are dissected.
ar. Articulary.
ar'. Large foramen perforating the lower jaw from below.
dt. Dentary.
la. Labial cartilage.
to. Tongue.

Fig. 4. Vertical section through a ramus of the mandible, of the natural size. The centre is entirely cartilaginous, the sides being covered by the articulary (*ar*) and dentary (*dt*). *a* is the section through the tooth which is anchylosed to the dentary, but separated from the articulary by connective tissue (*ct*); pulp-cavity = *pc*.

Fig. 5. The same magnified, without the tooth. This figure shows the porous nature of the bones. *ct* is a strip of connective tissue between the central cartilage and articulary.

Fig. 6. Bone-corpuscles of articulary.

Fig. 7. Cartilage-cells from the margin of mandibulary cartilage.

Fig. 8. Cartilage-cells from its centre.

PLATE XXXVI.

Fig. 1. Termination of the notochord, and its junction with the skull; a vertical longitudinal section, magn. 216.
n. Cartilage of the notochord.
c. Cartilage of the base of the skull.

Fig. 2. Scapular arch (right half), anterior view, of the natural size.

Fig. 3. The same, posterior view.
a. Median cartilage (right half).
b. Humeral cartilage.
c. Condyle for the articulation of the fore arm.
d. Coracoid (scapula).
e. Suprascapula.

Fig. 4. Pelvis (outer view), with right ventral paddle; $\frac{2}{3}$ nat. size.
a. Fibrous sheath of process slit open.

Fig. 5. Longitudinal section through the pelvis, to show the cavity (*b*).

Fig. 6. Pelvis of *Protopterus*, of the natural size, for comparison with fig. 4.

Gifford

Mintern Bros imp.

Günther

G. H. Ford

Mintern Bros. lith.

Fig. 7. A longitudinal section of a dermo-neural cartilage, magn. 216.
Fig. 8. Cartilage-cells from a joint of a fore paddle, magn. 216.

PLATE XXXVII.

Heart and Gills. All figures of the natural size.

Fig. 1. Front view of heart of *Ceratodus miolepis.*
Fig. 2. The ventricle opened.
Fig. 3. The ventricle and atrium opened.
Fig. 4. The ventricle and transverse portion of the conus arteriosus opened
Fig. 5. The right side of the heart opened, to show the spiral valve within the conus.
Fig. 6. Ganoid valves in a normal state of development.
Fig. 7 Ganoid valves partly rudimentary.

a. Atrium.
b. Conus arteriosus.
c. Large valve between atrium and ventricle.
d. Papillary valve within the conus, with its muscle (*d'*).
e Transverse series of large ganoid valves.
f Raised stripes (? rudimentary valves).
g. Transverse series of small ganoid valves.
g'. The same reduced to a pair of small papillary prominences
h, i Anterior arcus aortæ
k. l. Posterior arcus aortæ.
m. Papillary muscle. with its cartilage (*m'*).
v Ventricle.

Fig 8 Gills of the right side. The scapular arch has been removed, and only a small portion of the left coracoid (*co*) is preserved; the ceratohyal (*ch*) is pulled forwards, beyond the mandibulary joint of the suspensory pedicle (*sp*). The glossohyal (*gl*) and suboperculum (*so*) are also preserved in this preparation

h, i. Anterior arcus aortæ.
k, l. Posterior arcus aortæ.
u Attachment of the first gill to the roof of the gill-cavity.
u'. Attachment of the fourth gill to the suprascapula
x. Pseudobranchia.
x'. One series, and *x''* the other series of gill-rakers belonging to the pseudobranchia.

Fig. 9. Anterior parts of the aorta The branchial arches of the left side are drawn outwards, to show the arrangement of the gill-rakers. The ceratohyal (*ch*) is drawn forwards. Basal bone=*ba*

h, i. Anterior arcus aortæ.
k, l. Posterior arcus aortæ.

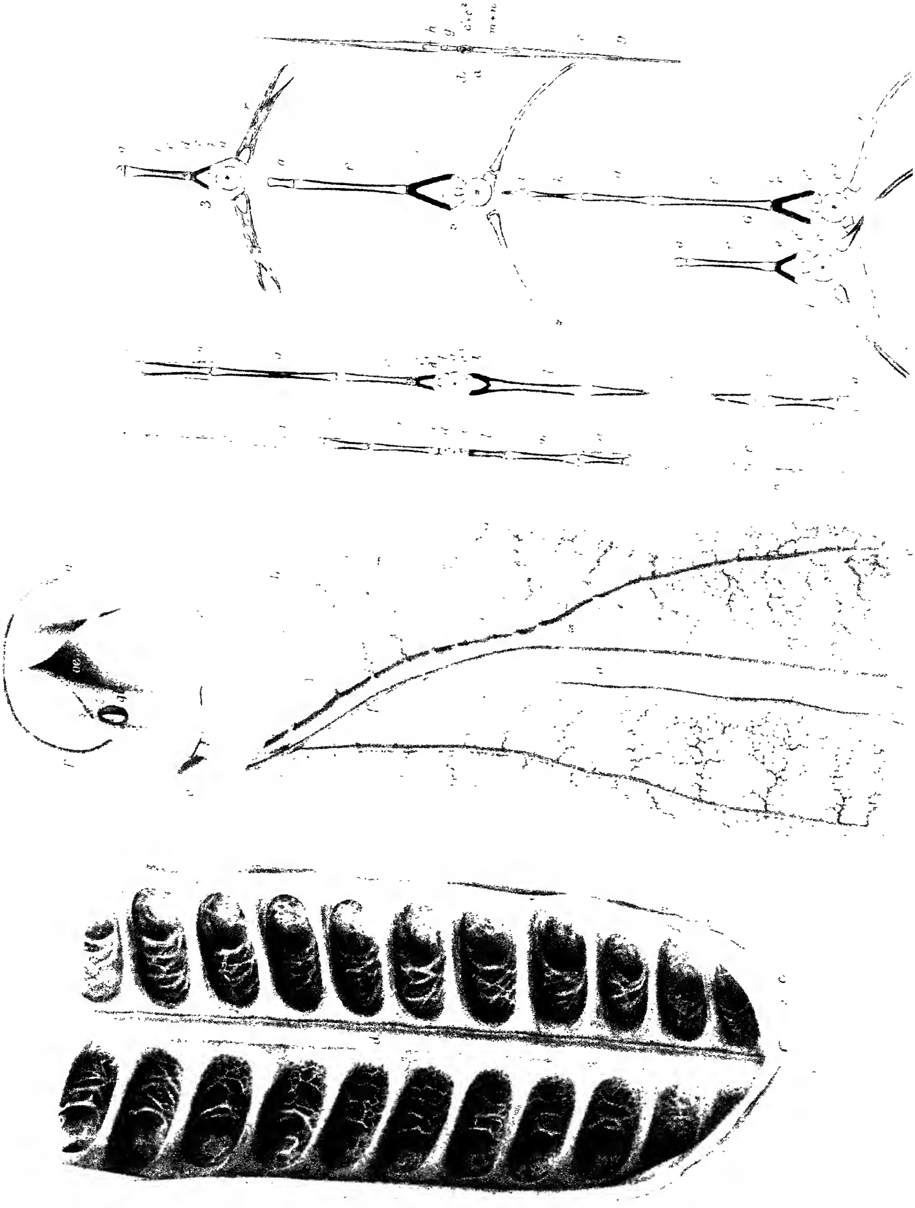
n. Carotis.
o. Ramus recurrens.
p. Arteria cœliaca.
q, *q'*. Arteria subclavia.
r. Anastomosis between arteria subclavia and vena cava superior.
s. Vena cava superior dextra.
s'. Vena cava superior sinistra.
t. Semiobliterated branch of vena cava superior.
w. Fifth branchial arch.
x' and *x''*. As in fig. 8.

PLATE XXXVIII.

Fig. 1. Ventral aspect of upper half of lung, with the system of the pulmonal vein blood-injected. Natural size.
a. Smooth median band.
b. Anterior end of left half of lung.
e. Vena pulmonalis; *e'*, right branch.
f. Arterial (?) blood-vessel, *f'*, right branch
g. Place where this vessel breaks up into several terminal branches
gl. Glottis.
œ. Part of the œsophagus slit open.

Fig. 2. Lower half of the lung, slit open along the ventral median band, to show the internal structure.
c. Posterior end of left half of lung.
d Dorsal median line of lung running along the aorta.

Figs. 3–9. Vertical sections through the first (3), fourth (4), fifth (5), fifteenth (6), twenty-ninth (7), forty-eighth (8), and sixtieth (9) sets of apophyses, to show the relative extent of bone and cartilage (half the natural size). The ossified portions are represented black, the cartilaginous only slightly shaded.
a. Central cylinder of notochord.
b. Notochord.
c. Confluent neur- and hæmapophysis of first set.
c' Neurapophysis, cartilage.
c². Hæmapophysis.
d. Medullary canal.
e. Neurapophysis, bony part.
e'. Canal for ligamentum longitudinale.
f. Neural spine.
g. Interneural first.
h. Interneural second.



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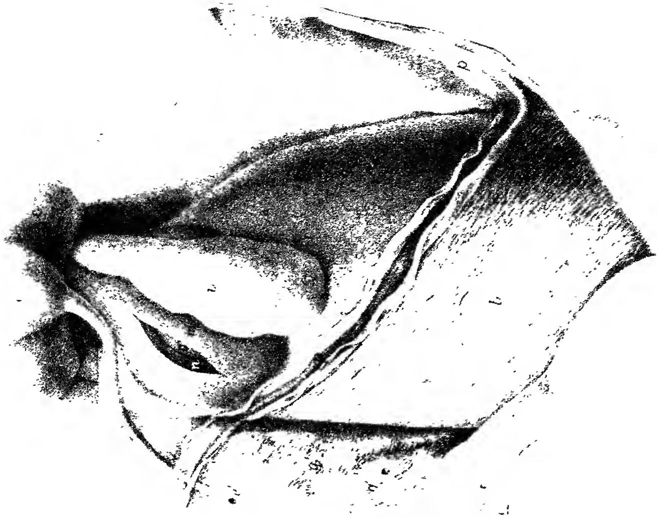

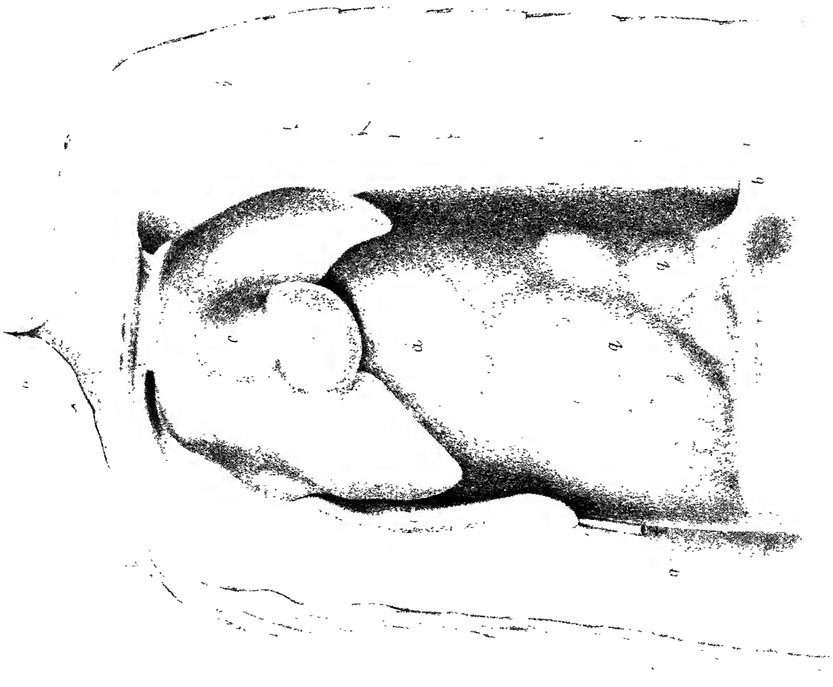

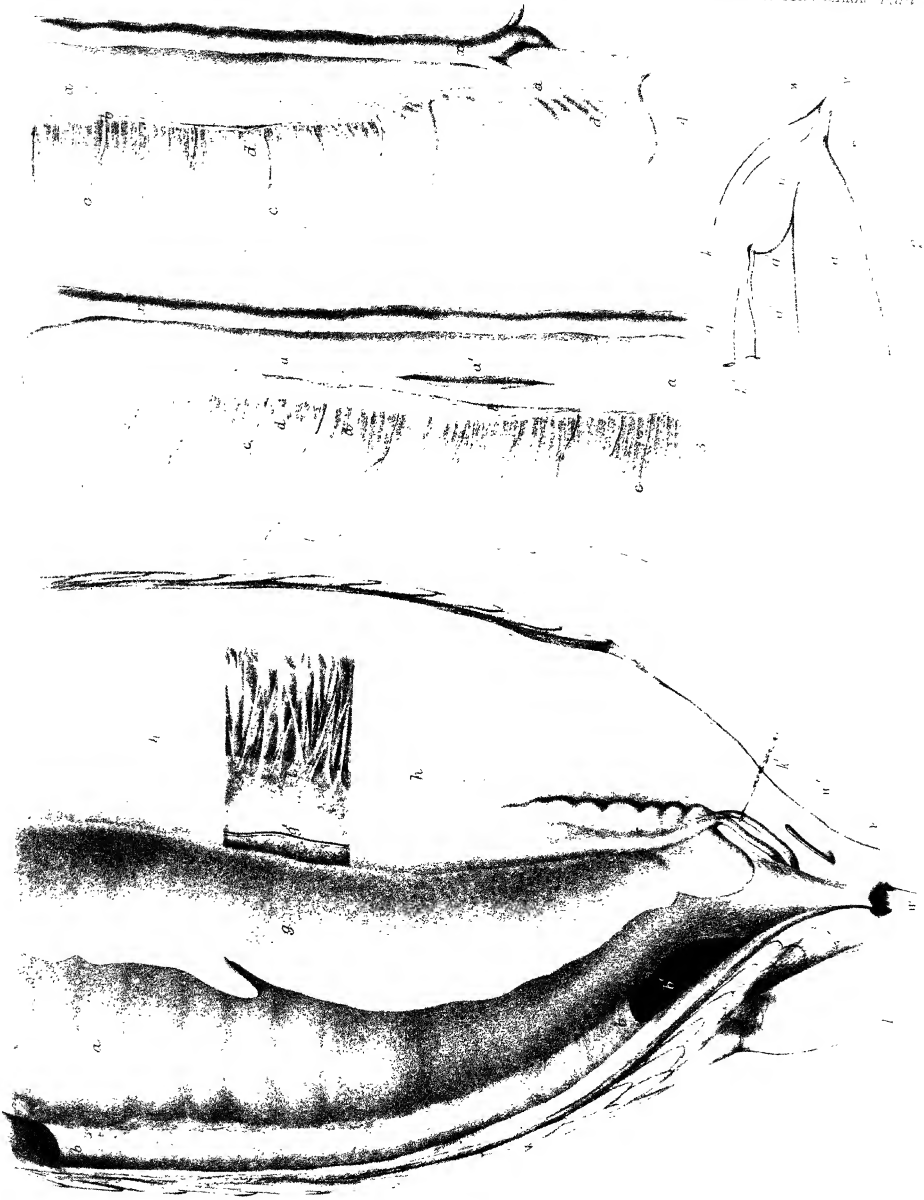

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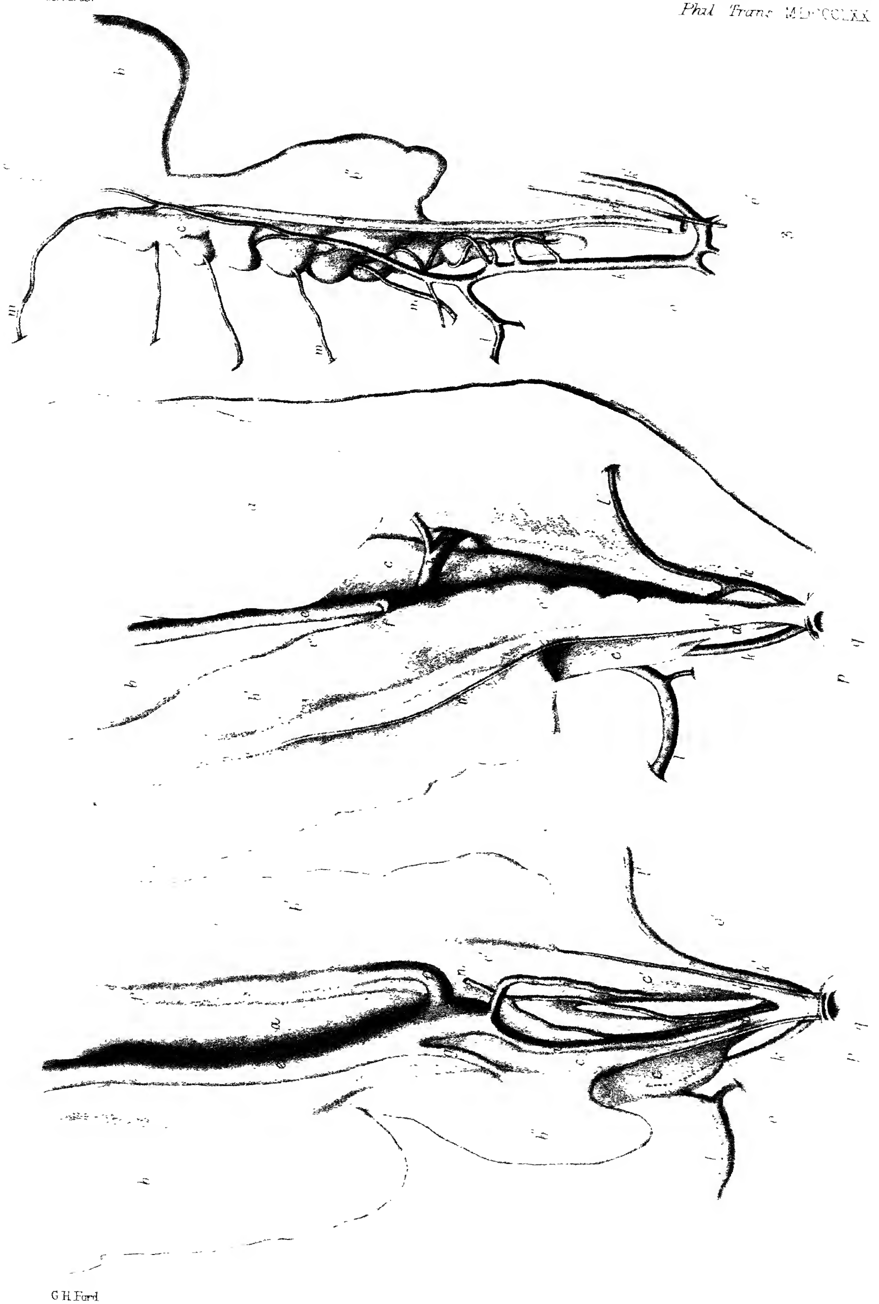

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i. Dermo-neurals.
k. Hæmal canal.
l. Hæmal spine.
m. Interhæmal first.
n. Interhæmal second.
o. Dermo-hæmals.
r. Rib.

PLATE XXXIX.

Upper part of intestinal tract (¾ nat. size).

Fig. 1. Ventral aspect.

Fig. 2. The anterior part of the intestine is opened, the liver (*c*) and gall-bladder (*e*) being drawn forward. A slit is made at *m*, through which part of the next compartment (*o*) may be seen.

a. Intestine.
b. Ligament fixing the intestinal sac to the wall of the abdomen.
c. Liver; *c'*, lateral lobe of liver.
d. Diaphragm.
e. Gall-bladder; *e'*. mouth of ductus choledochus.
f. Stomach.
g. Right testicle drawn away from the side of the intestine.
i Adipose layer descending from the right side of the œsophagus.
l First compartment of intestinal spire.
m. Spleen. A slit shows the thickness of this body by *n*, *o* being part of the next following compartment.
a. Part of œsophagus, opened.
p Double pyloric fold.
q, *q*. Glandular patches in the wall of the foremost compartment of the spire, as seen from without: and *q'*, *q'*, as seen from within

PLATE XL.

Fig. 1. Lower half of the abdominal organs (¾ nat. size). The intestine and left testicle are a little pushed towards the right. to render the layer of cellular tissue (*h*, *h*) conspicuous, this tissue has been removed at *l*, to show the position of the lung.

a. Intestine.
b. Ligament fixing the intestine to the wall of the abdomen.
b'. Slit in the ligament.
g. Left testicle; *g'*, vas deferens

h. Cellular tissue.
k. Kidney.
k'. The two ureters
l. Lung.
v. Vent.
w. External peritoneal opening.
w'. Probe passed through the peritoneal canal.

Fig. 2. Vertical section through the end of the intestine.
a. Cavity of the rectum.
g'. Vasa deferentia; *g''*, their common orifice.
k'. Ureters; *k''*, their common orifice.
r. Fold of the dorsal wall of the rectum, separating it from
u, the Urinary bladder
v. Vent.
w. External peritoneal opening.

Figs. 3 & 4. Upper and lower halves of the right testicle, with portions of the circulatory and seminiferous systems injected (natural size).
a. Vas longitudinale; its course is indicated by a dotted line: it is opened at *a'*, to show the pore-like openings of the canaliculi seminiferi.
b. Line of attachment of peritoneum.
c, *c*. Small veins injected from the vena cava.
d, *d*. Canaliculi seminiferi injected from the vas longitudinale
x. Vena cava.

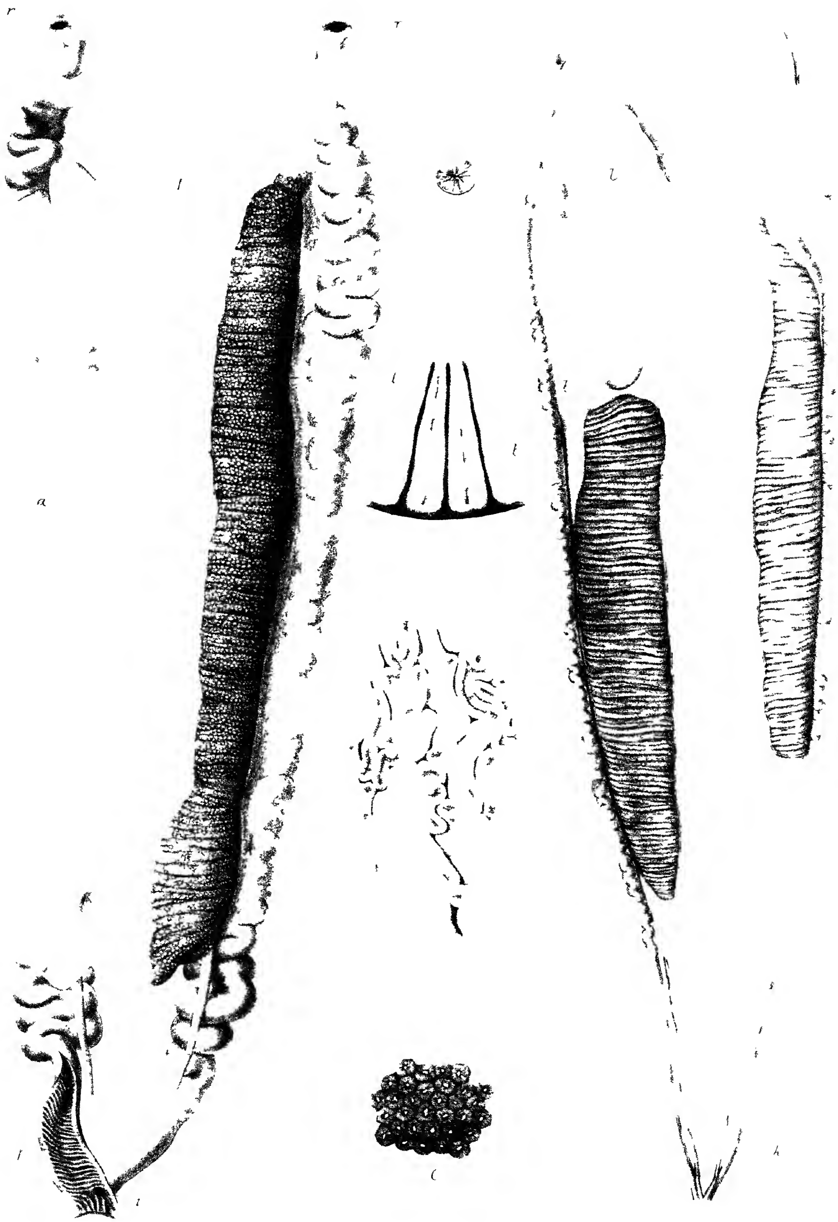
PLATE XLI

Uropoetic organs, with their veins. Figures of the natural size

Fig. 1. Front view after removal of the intestine

Fig. 2. The left testicle is turned over to the right, to show the entrance of the *vena caudalis* (*h*) into the *vena cava*.

Fig. 3. Right kidney, with portal system, the tissue in which the organ was imbedded is removed, and the right testicle drawn to the left.
a. Lower part of lung.
b. Lower part of right testicle, with its terminal lobe (*b''*).
b'. Lower part of left testicle.
c. Right kidney; *c'*, left kidney
d, *d'* Ureters.
e Vena cava.
f. Vena testicularis sinistra.
g. Vena testicularis dextra.
h. Vena caudalis.
i. *i*. Venæ renales revehentes.



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k, *k'*. Venæ renales advehentes.

l, *l'*. Posterior venæ intercostales.

m, *m*. Anterior venæ intercostales.

n. Arteria renalis.

o. Right, and *o'*. left vas deferens.

p. Orifice of the vasa deferentia.

q. Uretral orifice.

PLATE XLII.

Female Sexual Organs.

Fig. 1. In a fully developed condition. The liver has been removed; only a part of its lateral lobe (*l*) remains. The right ovary (*a*) is shown from the inner surface covered by the peritoneum; the left (*a'*) is turned inwards, so as to show its outer surface and the oviduct. $\frac{2}{3}$ nat. size.

Fig. 2. In an undeveloped condition. The liver (*l*) has been preserved. Of both ovaries the inner side is shown. $\frac{3}{4}$ natural size.

a. Right, and *a'*. left ovary.

b. Adipose anterior termination of ovary of the right side.

b'. The same of the left side; its end is split open in fig. 2 to show the blind end of the oviduct (*r'*).

c, *c'*. Posterior termination of ovary.

d, *d'*. Groove at the base of the ovary.

e, *e'*. Reverted fold of the peritoneum.

k, *k'*. Ureter.

l. Liver.

o. Right, and *o'*. left oviduct.

p. The lower part of the right oviduct is opened, to show the oblique transverse folds.

q. Opening of the left oviduct into the right.

r, *r'*. Abdominal orifices of the oviduct.

Fig. 3. Transverse section through the oviduct, of the natural size.

Fig. 4. A portion of the same, containing three mucous lamellæ (magnified).

a. Fibrous outer membrane of the oviduct, sending off

b. Processes into the lamellæ.

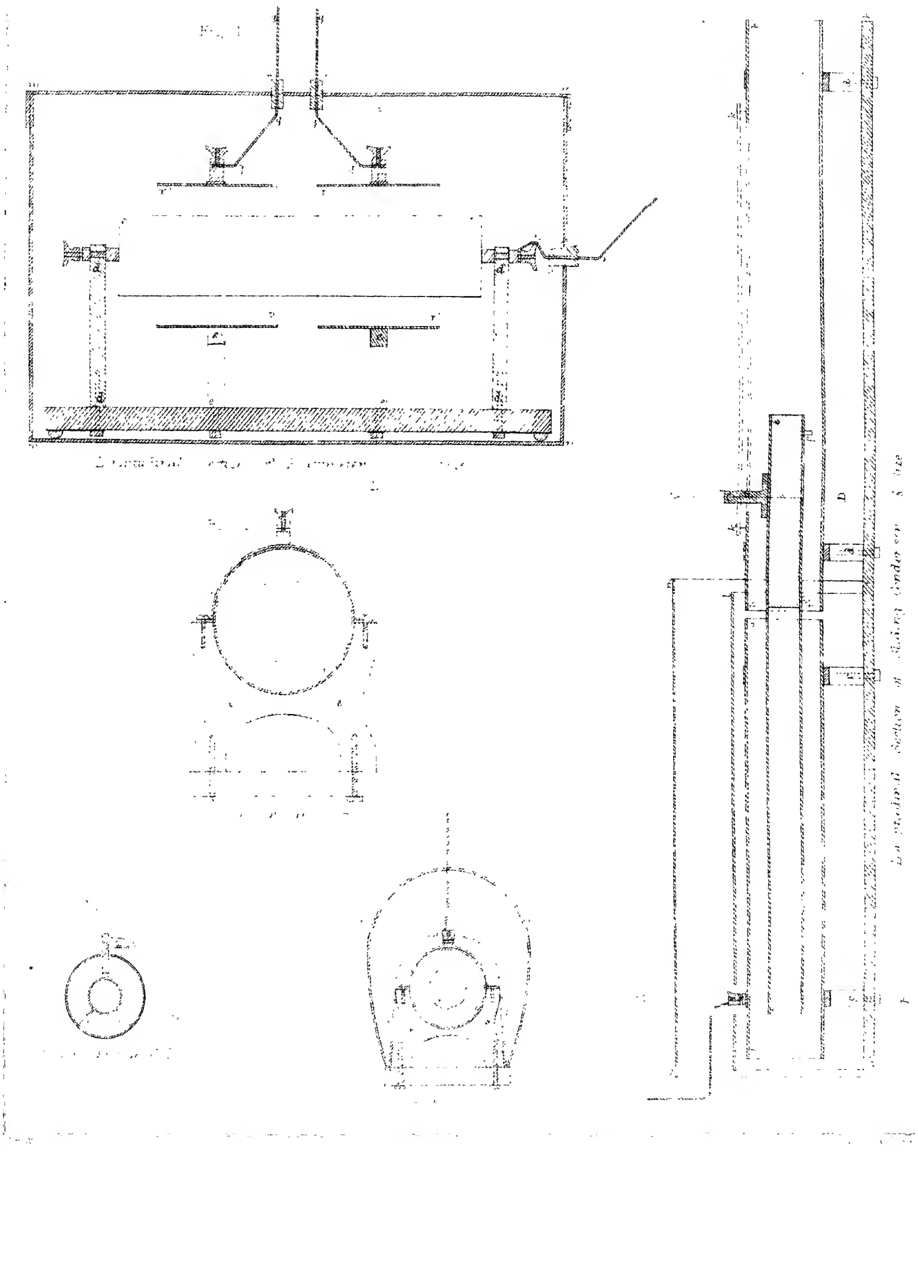
c. Epithelial stratum.

Fig. 5. Transverse section through the mucous lamellæ, made in the longitudinal axis of the oviduct (magnified).

b. Central fibrous tract.

c. Epithelial stratum.

Fig. 6. Transverse section through the mucous membrane of the oviduct of *Menopoma*. made in the longitudinal axis of the duct (magnified).



XX. *Measurements of Specific Inductive Capacity of Dielectrics, in the Physical Laboratory of the University of Glasgow.* *By* John C. Gibson, *M.A., and* Thomas Barclay, *M.A.* *Communicated by Sir* William Thomson, *F.R.S.*

Received November 23, 1870,—Read February 2, 1871.

The object of this paper is to describe the instruments and processes employed in a series of experiments on the specific inductive capacity of paraffine, and the effect upon it of variations of temperature.

The chief instruments employed in this investigation were the Quadrant Electrometer, and two others which will now be described.

I. The Platymeter, so named by suggestion of the late Dr. Whewell from its use in the comparison of electrostatic capacities, was, in a rudimentary form, shown to the Mathematical and Physical Section of the British Association at its Glasgow Meeting in 1855, by Sir William Thomson. It consists of two condensers of equal capacity, each of which has one of its opposed surfaces in metallic connexion with the corresponding surface in the other.

In the instrument employed (see Plate XLIII. fig. 1) the conductor constituting these two connected surfaces is a brass cylinder (*c c*) 22·94 centimetres in length and 5·1 centimetres in diameter, supported at its ends in a horizontal position by two vulcanite stems (*d d*, *d d*). Round it are placed two equal and similar pieces of brass tubing (*p p*, *p′ p′*), 7·68 centimetres in length and 8·6 centimetres in diameter, each supported by a vulcanite stem (*e e*, *e e*), so as to be concentric with the cylinder and at equal distances from its ends. The four supports (*d d*, *d d*, *e e*, *e e*) are fixed in a massive plate of iron (*i i*); and the whole is enclosed in a stout metal box (*m m*, *m m*), an electrode (*n n*) from the inner cylinder being carried through an insulating plug (*o*) of paraffine in the end of the box, and another (*q q*, *q q*) from each of the tubes through similar plugs (*r r*) in the top of it. By this means shreds and dust are excluded, and paraffine is found to be so good an insulator that no inconvenience is experienced by employing it in this way.

II. The Sliding Condenser is a condenser the capacity of which may be varied by altering the effective area of the opposed surfaces. The following form of this instrument was employed (see fig. 2). Two pieces of brass tubing (*a a*, *b b*), 2·48 centimetres in diameter and 26·58 and 35·3 centimetres in length respectively, are supported end to end upon vulcanite stems (*c c*, *d d*) securely fixed in a massive iron plate (*h h*). Within these tubes, and coaxially with them, moves a brass cylinder (*e e*) 36·6 centimetres in length and 1·15 centimetre in diameter, supported at one end upon four brass feet (*f*, *f*) which slide on the inner surface of the longer tube, and loaded at that end so as to rest

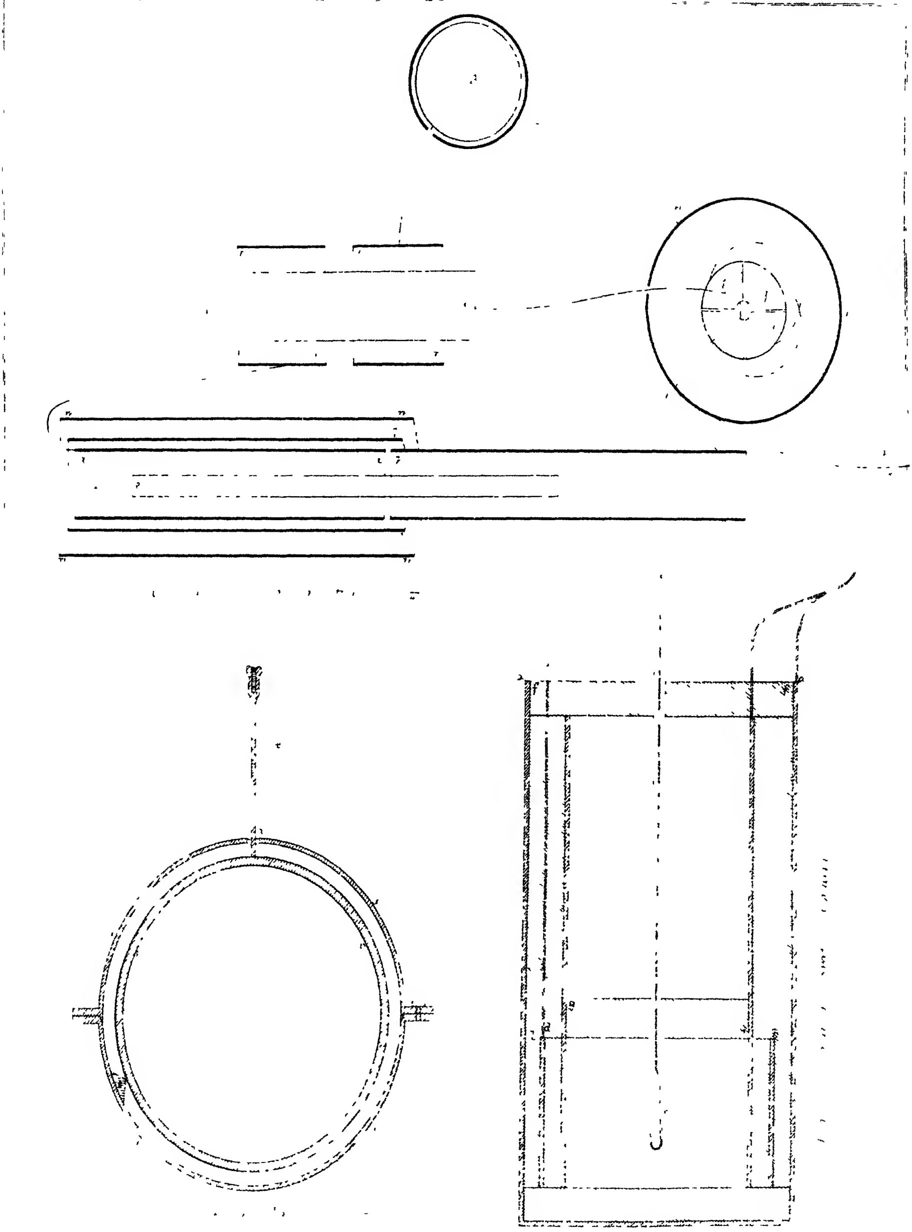
stably upon them. Attached to this cylinder is a vertical arm (g) which projects through a slot extending along the upper part of the surrounding tube. This arm carries an index (i) by which the position of the sliding cylinder or core is indicated on an attached scale ($k\,k$). This scale is 22·86 centimetres long, and is divided into 360 divisions, each $\frac{1}{40}$ of an inch in length. The tube ($b\,b$) which supports these parts of the instrument, though placed on vulcanite stems for convenience in testing, is in working kept in connexion with the earth. The other ($a\,a$), which may therefore be called the insulated tube, is closely surrounded by a sheet of metal ($l\,l$), which is securely fastened to the iron base of the instrument. This sheet, being connected with the earth, guards the insulated tube against electric disturbance, and adds largely to the capacity of the condenser. As an additional security against disturbance, another sheet of metal ($m\,m$) is fastened round the former at some distance from it.

In using these instruments the following connexions are made (see Plate XLIV. fig. 3). The inner cylinder or core ($c\,c$) of the platymeter is connected with two (q, q'') of the quadrants of the electrometer, the other two (q', q''') being connected with the case of the electrometer ($n\,n\,n$) and with the earth. The condenser to be measured is connected with one of the tubes or sides of the platymeter, and the insulated tube of the sliding condenser with the other. All other parts of the sliding condenser and the box and base-plate of the platymeter are kept in metallic communication with one another, and with the disinsulated pair of quadrants of the electrometer.

Now suppose the two sides (p and p') of the platymeter to be exactly equal, and the condensers to be compared, say A and B, also equal. Let the insulated quadrants (q, q'') be temporarily connected with the disinsulated pair (q', q'''), and let the condenser A, together with p, the side of the platymeter with which it is connected, receive a positive charge. Then c, the core of the platymeter, becomes by induction negatively electrified. The earth-connexion of the quadrants (q, q'') is now broken, and the charge thus insulated upon c remains masked by the action of p so as to cause no deflection of the needle. A connexion is now established between p and p', the two sides of the platymeter, so that on the supposition of equality already made, the charge is now distributed over a system of double capacity, and the potential of the whole becomes half of that of A and p before distribution. Both sides of the platymeter are now acting upon c, each with half the original intensity of p, so that the resulting effect upon the core being unaltered, no deflection of the needle takes place.

If the condenser A be of greater capacity than the condenser B, then, on connecting p and p', the capacity of the whole charged system is less than double that of A and p, and its potential, therefore, greater than half the original potential of A and p. Each side of the platymeter, therefore, is now acting with more than half the original intensity of p. The result is an increased action on the core of the platymeter, and the consequent liberation of positive electricity to act on the needle, which, in the ordinary arrangement of the electrometer, is deflected to the right.

If, on the other hand, the capacity of the condenser A be less than that of the con-



denser B, the potential of the system after the distribution of the charge is less than half the original potential of A and p. The combined action of the two sides p and p' is now less than the original action of the one side p. Negative electricity is thus set free, and this, acting on the needle, deflects it to the left.

If B be the charged condenser, or if A be charged negatively, the effect in the last two cases will be reversed. If both the condensers be charged, one positively and the other negatively, the effect will be greater in degree, but precisely similar in kind. This is the method usually adopted, as it gives more marked indications without increasing the risk of failure of insulation.

In measuring any condenser by comparing it in this way with the sliding condenser, the latter is, in accordance with the indications thus given by the electrometer, so adjusted that its capacity becomes equal to that of the condenser to be measured.

In order to find from this the true value of the measured condenser, it is necessary to know the value of the sliding condenser in scale-divisions when its index is at zero on the scale.

For this purpose a condenser was employed of such a form that its capacity could be very accurately determined in absolute measure* (see fig. 4). This consists of two metallic spheres of different diameters placed one within the other. The outer consists of two brass hemispheres ($a\,a$, $a'\,a$), having their inner surfaces accurately turned, and having projecting flanges which are firmly fastened together by three brass screws (one of which is shown at c). In the top of the upper hemisphere is a small hole through which projects an electrode ($d\,d$) from the inner sphere ($b\,b$), which rests upon three vulcanite pins (one of which is shown at e), so as to present an accurately spherical surface truly concentric with the inner surface of the outer sphere.

In order to determine in absolute measure the electrostatic capacity of this condenser, its dimensions were carefully ascertained in the following manner. The quantity of water contained by the outer sphere was measured, and found to be 7394·8 grains at temperature 15°·5 Centigrade, or 479·66 cubic centimetres, which, with the necessary correction for the vulcanite pins, ·287 cubic centimetre, gave as the radius 4·857 centimetres. The radius of the inner sphere was obtained from the content of the outer sphere and that of the space between the two spherical surfaces. Of the content of this space five determinations were taken. These are shown in the following Table. The values in the second column include the correction for the volume of the vulcanite pins.

Grains at 15·5 C.	Cubic centimetres
1465	95·313
1467	95·443
1471	95·702
1469	95·573
1468·5	95·540

* The absolute unit of electrostatic capacity referred to in this paper is the capacity of an insulated spherical conductor of one centimetre radius placed at an infinite distance from all external objects.

The mean of these values, and the previously measured volume of the containing spherical surface, give 4·5107 centimetres as the radius of the inner sphere.

The capacity of a spherical condenser is calculated by the formula

$$\text{Capacity}=\frac{rr'}{r'-r},$$

where r' is the radius of the outer, and r the radius of the inner sphere.

In this case $r'=4·857$ centimetres and $r=4·5107$ centimetres. Hence the capacity is equal to 63·264 centimetres. The specific inductive capacity of the vulcanite of the pins which support the inner sphere being greater than that of air, causes an increase of the whole capacity of about ·19 centimetre. The hole in the top causes a diminution of ·085 centimetre, and the electrode passing through it an increase of ·15 centimetre. The actual value of the condenser is therefore 63·519 centimetres.

To reduce this to scale-divisions of the sliding condenser, the value of one division in absolute measure is calculated by the formula for a cylindrical condenser.

$$\text{Capacity}=\frac{1}{2}\cdot\frac{l}{\log\frac{r'}{r}},$$

where r' is the radius of the outer surface, r the radius of the inner surface, and l the length of the condenser whose capacity is to be calculated.

In this case $r'=2·4837$ centimetres, $r=1·1515$ centimetre, and $l=\frac{1}{40}$ of an inch or ·063499 centimetre. Here r' was determined from a measurement of the volume of water contained by the tube, the length of which was accurately measured. To determine r, the circumference of the core was measured by winding fine wire round it, and measuring the length of a certain number of turns, the necessary corrections being made for the thickness of the wire and its spiral arrangement. The value for electrostatic capacity of one scale-division is therefore ·0413 centimetre; and hence the spherical condenser is equal in capacity to 1538 scale-divisions.

[Direct electrical measurements taken subsequently on a sliding condenser of greater range gave 1607 scale-divisions as the value of the spherical condenser. This is probably greater than the actual value, while that derived from calculation is too small by a quantity due to the action of parts whose capacity could not be numerically determined. The mean of these values, 1572 scale-divisions, may therefore be taken as the value of this condenser.]

When the capacity of the spherical condenser was measured electrically while in connexion with the side p of the platymeter, the reading obtained on the scale of the sliding condenser was 211. When the connexions were reversed, that is, when the spherical condenser was connected with the side p' of the platymeter, the reading obtained was 183. The difference of the two readings thus obtained shows some inequality of the sides of the platymeter.

Now to find the true reading in any case from two such readings, suppose the side p to be equal to $n\times p'$. If no deflection of the needle takes place on the distribution of

the charge, this can only arise from the action of the two sides of the platymeter being equal to that of the one charged side, say p, before the connexion was made. But the surface of action is now equal to $\left(1+\frac{1}{n}\right)p$, and the potential of the system, therefore, is now $\left(\frac{1}{1+\frac{1}{n}}\right)\times v$, where v is the original potential of A and p. But the quantity of electricity taken from A and p is such as to raise B and p' to the same potential as that to which A and p, with which they are now connected, are reduced. But in condensers at equal potentials the capacities are proportional to the amounts of the charges. Suppose the original charge of A and p to be unity. Then their charge after distribution is $\frac{1}{1+\frac{1}{n}}$, and that of B and p' is $1-\frac{1}{1+\frac{1}{n}}$. Then

$$\mathrm{A}+p:\mathrm{B}+p'::\left(\frac{n}{n+1}\right):\left(\frac{1}{n+1}\right)::n:1;$$

but

$$p:p'::n:1,$$

and therefore

$$\mathrm{A}:\mathrm{B}::p:p'.$$

That is, generally, when there is no deflection, the condensers compared are to one another as the sides of the platymeter with which they are connected. But the reading obtained with the platymeter connexions arranged normally was greater than that obtained when they were reversed. The side p' is therefore greater than the side p.

Now let a be the greater and b the less reading with the value of the sliding condenser when its index is at zero added to them, and let x be the true value to be deduced from them. Then we have the ratios

$$p:p'::x:a \text{ and } p':p::x:b,$$

therefore

$$x:a::b:x, \text{ and } x=\sqrt{ab}.$$

That is, the true value of a condenser measured by an imperfect platymeter is the geometric mean of the two values obtained by the different arrangements of the connexions.

In practice, when the error of the platymeter is so small as it is in the present case, the arithmetic mean may be taken instead of the geometric. For let M be the arithmetic mean of a and b, and D the difference between it and either of these values. Then $x=\sqrt{\mathrm{M}^2-\mathrm{D}^2}=\mathrm{M}\left(1-\frac{\mathrm{D}^2}{2\mathrm{M}^2}-\frac{\mathrm{D}^4}{8\mathrm{M}^6}-\ldots\ldots\right)$. In the present case, where $\frac{\mathrm{D}^2}{2\mathrm{M}^2}$ is about $\frac{1}{25000}$, M does not differ sensibly from x. If α be the greater and β the less reading, then z, the value (in scale-divisions) of the sliding condenser with its index at zero, is $x-\frac{\alpha+\beta}{2}$. When z has been determined, the true value of any condenser may be got by adding to it the mean of the two readings obtained; and when the error of the platy-

meter used has been once ascertained, this mean may be obtained from a single reading without reversing the connexions.

The following Table shows the values obtained for α and β in the measurement of the spherical condenser:—

Date.	α.	β.	$\frac{\alpha+\beta}{2}$.
Nov. 17, 1869......	211	183	197
Nov. 17, 1869......	211	186	198·5
Nov. 19, 1869......	211	......	198

Taking 198 as the mean reading and 1572 as the value of x, the value of the sliding condenser with its index at zero was 1374 scale-divisions.

A condenser was now prepared having paraffine instead of air as the dielectric. This consisted of a flat circular brass box with a tube projecting from the centre of the lid. through which an electrode was carried from a brass disk imbedded in paraffine, with which the box and tube were filled, midway between the top and bottom of the box. The box was placed in water reaching nearly up to the top of the tube. For the value of this condenser at different temperatures the following results were obtained:—

Date.	Temp.	α.	β.	$\frac{\alpha+\beta}{2}$.	Value $=\frac{\alpha+\beta}{2}+x$.
Nov. 17, 1869......	11°·5 C.	180	......	168	1542
Nov. 18, 1869......	13·3	215	192	203	1577
Nov. 18, 1869......	17	......	212	224	1598
Nov. 19, 1869......	12·5	......	220	232	1606
Nov. 19, 1869......	15·8	228	206	217	1591
Nov. 22, 1869......	17·9	235	......	223	1597

Crevices which had formed in the paraffine when cooling, gradually admitted water so as ultimately to destroy the insulation of the disk.

The increase of capacity thus caused is probably sufficient to account for the variations shown in the Table, which, therefore, are not to be attributed to any alteration of inductive capacity due to change of temperature. Taking the first of the values, 1542 scale-divisions, or 63·687 centimetres, as probably the most accurate, the following determination of specific inductive capacity of paraffine is obtained. The value of the box condenser with air instead of paraffine as the dielectric, obtained from measurement of it on the sliding condenser described in the Appendix, is 32·306 centimetres. These two values give 1·975 as the specific inductive capacity of paraffine when that of air is taken as unity, a correction having been made on account of three vulcanite pins used to support the insulated plate.

The following form of condenser (see fig. 5) was next employed as being most likely to show effects of temperature undisturbed by other causes. Into a cylindrical brass vessel ($a\,a$) 15·5 centimetres in depth and 8·61 centimetres in internal diameter, melted

paraffine was poured to a depth of about one centimetre. Upon this, when solid, a piece of brass tubing ($b\,b$), 4·3 centimetres long and 7·24 centimetres in internal, and 7·47 in external diameter, was rested so as to be concentric with the outer tube. Inside of this, and concentric with it, was placed another piece of brass tubing ($c\,c$), 13·1 centimetres long and 6·1 centimetres in external diameter. The space between this and the outer tube was then filled with paraffine, and from the middle tube thus imbedded a fine wire electrode ($d\,d$) was carried up. The condenser was put into a vessel containing water in which was placed a thermometer; and another thermometer ($e\,e$) was supported in the centre of the inner tube by a paraffine plug ($f\,f$) about a centimetre thick, which rested upon the top of this tube. This plug prevented the communication downwards of the temperature of the air. As an additional security, another paraffine plug ($g\,g$) of the same thickness was inserted in the inner tube a little above the bulb of the thermometer.

The outer and inner tubes were connected with the earth, and the electrode from the other tube, which was insulated, was connected with the one side of the platymeter.

An addition to the capacity of the sliding condenser had been made, increasing its zero-value, so that the readings for the spherical condenser were:

Date.	α.	β.	$\frac{\alpha+\beta}{2}$.
Dec. 13, 1869.........	28	2	15

This gives for the zero-value of the sliding condenser $1572-15=1557$ scale-divisions.

With this value the results shown in the following Table were obtained for the value of the cylindrical paraffine condenser at different temperatures.

Date.	Temperature.			Condenser Scale.		Value.
	Outside.	Inside.	Mean.	α.	$\frac{\alpha+\beta}{2}$.	
Dec. 22, 1869.........	7·5	7·5	7·5	158	145	1702
Dec. 22, 1869.........	17·6	17·6	17·6	158	145	1702
Dec. 29, 1869.........	2·7	2·7	2·7	150	137	1694
Dec. 30, 1869.........	5·7	5·7	5·7	147	134	1691
Dec. 30, 1869.........	15·4	15·7	15·55	145	132	1689

Without the addition to the value of the sliding condenser the following measurements of the spherical condenser were taken:—

Date.	α.	β.	$\frac{\alpha+\beta}{2}$.
Dec. 10, 1869.........	202	176	189
Jan. 5, 1870.........	201	......	188
Jan. 7, 1870.........	200	......	187
Jan. 8, 1870.........	199	......	186
Jan. 12, 1870.........	201	......	188
Jan. 17, 1870.........	202	......	189
Jan. 17, 1870.........	202	173	187

These readings give the zero value $=1572-188=1384$. With this value of z the following measurements of the cylindrical paraffine condenser were obtained:—

Date.	Temperature.			Condenser Scale.		Value.
	Outside.	Inside.	Mean.	α.	$\frac{\alpha+\beta}{2}$.	
Jan. 6, 1870	°	° 13	°	322	309	1693
Jan. 6, 1870	10	10·7	10·35	316	303	1687
Jan. 6, 1870	26·2	13·4	19·8	314	301	1685
Jan. 7, 1870	21·2	20·5	20·85	318	305	1689
Jan. 8, 1870	8·2	8·2	8·2	322	309	1693
Jan. 8, 1870	17·7	14·9	16·3	319	306	1690
Jan. 8, 1870	16·4	16·2	16·3	319	306	1690
Jan. 11, 1870	7·1	7·1	7·1	320	307	1691
Jan. 12, 1870	24·7	24	24·35	316	303	1687
Jan. 12, 1870	23	24·3	23·65	317	304	1688

In order to vary the conditions of the experiment so as to allow the paraffine to expand more freely, a condenser of the following form was employed. Its metallic surfaces consisted of circular pieces of tinfoil. These were arranged horizontally with plates of paraffine half a centimetre thick between them. The following Table shows the values obtained for this condenser at various temperatures, the zero-value of the sliding condenser being still the same:—

Date.	Temp.	Condenser Scale.		Value.
		α.	$\frac{\alpha+\beta}{2}$.	
Jan. 18, 1870	° 11·4	269	256	1640
Jan. 18, 1870	21·8	263	250	1634
Jan. 19, 1870	14·5	267	254	1638
Jan. 25, 1870	7·2	271	258	1642

These values arranged according to temperature are as follows:—

Temp.	Value.
° 7·2	1642
11·4	1640
14·5	1638
21·8	1634

Some experiments were now made upon the expansion of paraffine with temperature, with a view to determine the alterations produced by it in the capacity of paraffine condensers. This was done by weighing a quantity of paraffine in thin plates, with a platinum sinker attached, in distilled water at different temperatures. The weight of the paraffine in air was 48·358 grammes, that of the platinum was 8·625 grammes, or

altogether 56·983 grammes. The following Table shows the results of the weighings in water. The weights are given in grammes, and the volumes in cubic centimetres.

Temperature.	Weight of paraffine and platinum in water.	Weight of water displaced by both.	Volume of both.	Volume of platinum.	Volume of paraffine.
1°	3·698	53·285	53·289	·4077	52·881
8·8	3·413	53·570	53·579	·4078	53·171
10·1	3·372	53·611	53·626	·4078	53·218
17·2	3·139	53·844	53·909	·4079	53·501

These numbers give for the expansion of paraffine the results shown in the following Table*:—

Temperature.			Difference of volume.	Mean volume.	Total expansion.	Cubic expansion per degree.	Linear expansion per degree.	Mean temperature.
From	To	Diff.						
1°	8°·8	7°·8	·290	53·026	·00547	·000701	·000234	4°·9
1	10·1	9·1	·337	53·049	·00635	·000698	·000233	5·55
1	17·2	16·2	·620	53·191	·01166	·000720	·000240	9·1
8·8	10·1	1·3	·047	53·195	·00088	·000677	·000226	9·45
8·8	17·2	8·4	·330	53·336	·00619	·000737	·000246	13
10·1	17·2	7·1	·283	53·360	·00530	·000746	·000249	13·65

From these determinations of the expansion of paraffine the effect produced by variations of temperature upon the capacity of the tinfoil condenser may be easily calculated. The mean value of the condenser 1638·5 scale-divisions may be taken as its value at the mean temperature 13°·7. Its values at the different temperatures estimated from this, together with those previously given as obtained from actual measurements, are shown in the following Table:—

Temperature.	Measured value.	Estimated value.
7°·2	1642	1641
11·4	1640	1639·4
14·5	1638	1638·2
21·8	1634	1635·1

The values given in the last column of this Table were estimated on the supposition that the expansion of the paraffine did not produce any sensible stretching of the tinfoil; and this, from the manner in which the condenser was formed, was probably the case. From this it appears that the regular alteration in the values obtained for this condenser at different temperatures follows so nearly that resulting from the variations of the

* After these experiments were made it was found that the French "Bureau des Longitudes," in the 'Annuaire' for 1870 published by them, give, on the authority of M. Fizeau, ·00027854 per degree Centigrade as the coefficient at 40° Centigrade of linear expansion of "paraffine de Rangoon," and ·0000009926 as the "variation of coefficient" per degree. This gives ·00024787 as the expansion per degree at temperature 9°·1, that given in the Table at the same temperature being ·000240.

distance between the plates due to the expansion of the paraffine that no change of specific inductive capacity can be inferred from it.

The following further measurements of the Cylindrical Paraffine Condenser were now made, the zero-value of the sliding condenser being still equal to 1384 scale-divisions.

Date.	Temperature.			Condenser Scale.			Value.
	Outside.	Inside.	Mean.	α.	β.	$\frac{\alpha+\beta}{2}$.	
Feb. 10, 1870.........	−17°·8	− 6°·4	−12°·1	......	282	294	1678
Feb. 11, 1870.........	......	− 0·2	......	......	280	292	1676
Feb. 15, 1870.........	5·4	4·85	5·125	307	......	295	1679
Feb. 15, 1870.........	−15·4	− 4·65	−10·025	306	......	294	1678
Feb. 15, 1870.........	−14·1	−10·2	−12·15	305	......	293	1677
Feb. 15, 1870.........	−11·5	−11·25	−11·375	......	285	297	1681
Feb. 15, 1870.........	−10·25	−11·125	−10·79	310	......	298	1682
Feb. 16, 1870.........	0·5	0·425	0·46	309	......	297	1681
Feb. 17, 1870.........	5·2	4·37	4·78	......	285	297	1681
Feb. 18, 1870.........	6·5	5·8	6·15	......	284	296	1680
Feb. 18, 1870.........	6·5	5·8	6·15	308	......	296	1680
Feb. 18, 1870.........	16·6	10·8	13·7	305	......	293	1677
Feb. 23, 1870.........	18·7	18·45	18·57	302	......	290	1674
April 6, 1870.........	......	......	12·6	309	......	297	1681

All the values obtained for this condenser at different temperatures are shown in the following Table arranged in the order of temperature:—

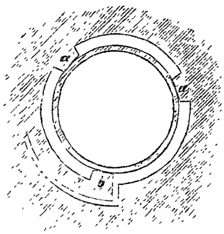
Date.	Temperature.	Value.
Feb. 15, 1870	−12°·15	1677
Feb. 10, 1870	−12·1	1678
Feb. 15, 1870	−11·375	1681
Feb. 15, 1870	−10·79	1682
Feb. 15, 1870	−10·025	1678
Feb. 11, 1870	− 0·2	1676
Feb. 16, 1870	0·46	1681
Dec. 29, 1869	2·7	1694
Feb. 17, 1870	4·78	1681
Feb. 15, 1870	5·125	1679
Dec. 30, 1869	5·7	1691
Feb. 18, 1870	6·15	1680
Feb. 18, 1870	6·15	1680
Jan. 11, 1870	7·1	1691
Dec. 22, 1869	7·5	1702
Jan. 8, 1870	8·2	1693
Jan. 6, 1870	10·35	1687
April 6, 1870	12·6	1681
Jan. 6, 1870	13	1693
Feb. 18. 1870	13·7	1677
Dec. 30, 1869	15·55	1689
Jan. 8, 1870	16·3	1690
Jan. 8, 1870	16·3	1690
Dec. 22, 1869	17·6	1702
Feb. 23, 1870	18·57	1674
Jan. 6, 1870	19·8	1685
Jan. 7, 1870	20·85	1689
Jan. 12, 1870	23·65	1688
Jan. 12, 1870	24·35	1687

The differences of these values show no alteration of specific inductive capacity of paraffine due to variations of temperature.

The absolute value of this condenser with air as the dielectric, obtained by measurement of it on the sliding condenser described in the Appendix, is 35·394 centimetres; and taking the mean of these values with paraffine, 1684 scale-divisions, or 69·552 centimetres, the specific inductive capacity of paraffine is found to be 1·965, that of air being taken as unity. The layer of paraffine under the insulated tube had been left in for the purpose of supporting it. The correction necessary on account of this increases this number to 1·977.

Appendix.

The sliding condenser described in this paper was found to be too limited in range, and liable to failure of insulation owing to the admission of shreds. Another sliding condenser (referred to at pages 576, 578, and 583) was therefore employed of such a construction as to give greater range and more perfect insulation. The tubes of this instrument were arranged vertically, and were supported by a wide brass cylinder surrounding the insulated tube at a considerable distance from it, and secured to a massive upright iron support. To a brass disk forming the cover of this cylinder the insulated tube was fastened by a vulcanite collar, into which it was screwed, so as to hang vertically in the middle of the surrounding cylinder. This cover also supported the uninsulated tube directly above the insulated tube so as to be coaxial with it. The core, instead of sliding upon feet as in the instrument formerly described, moved in two V supports cut in the brass plates which closed the ends of the uninsulated tube. The points of support and the springs which press the moving core against them are formed by cutting the brass plates in the manner shown in the annexed figure. Another tube, about 7 millimetres greater in diameter than the insulated tube, was supported by the plate closing the lower end of the wide brass cylinder so as to slide vertically round the insulated tube. By altering the position of the core and of this outer tube, which was uninsulated, the value of the condenser could be so varied that capacities ranging from 47 to 180 centimetres could be measured upon it. The position of the outer tube and that of the core were indicated upon scales of $\frac{1}{40}$ of an inch engraved upon the tubes themselves.



a, *a*. Points of support.
b. Spring.

XXI. *On Cyclides and Sphero-Quartics.*

By JOHN CASEY, *LL.D., M.R.I.A.* *Communicated by* A. CAYLEY, *F.R.S.*

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CHAPTER I.

ERRATA.

Page 627, foot note, line 8 from bottom, for *quadrinodal surface* read *quartic cyclide*.
" 628, art. 86, *before* ϱ' in equation (85), *omit* $\frac{1}{2}$.
" 635, art. 109, *after* the determinant (93), *add* $=0$.
" 638, art. 118, line 7, *for* $U^2=$ *read* $U^2\equiv$.
" 639, art. 121, *after* $\alpha^2+\beta^2+\gamma^2+\delta^2+\epsilon^2$, *add* $=0$.
" 643, art. 143, line 5 from bottom, *for* (407) *read* (107).
" 694, art. 278, *for* $=\left(\alpha\frac{d}{d\alpha''}+\beta'\frac{d}{d\beta'}+\gamma'\frac{d}{d\gamma''}+\delta'\frac{d}{d\delta}\right)W''$

read $=\left(\alpha'\frac{d}{d\alpha''}+\beta'\frac{d}{d\beta''}+\gamma'\frac{d}{d\gamma''}+\delta'\frac{d}{d\delta''}\right)W''$.

" 705, art. 319, in equation (193) *supply* $\sqrt{F'}$.
" 707, art 324, the numbers 3, 4, 5, 6, 7, 8, 9, 10 in the determinant (200) should be small suffix numbers.

Cor. If a sphere S cuts four spheres, S′, S″, S‴, S⁗, orthogonally, it also cuts orthogonally $\lambda S'+\mu S''+\nu S'''+\varrho S''''$ when λ, μ, ν, ϱ are any multiples.

3. The following method finds the equation of the orthogonal sphere in tetrahedral coordinates. Let S′, S″, S‴, S⁗ be the given spheres, then $\lambda S'+\mu S''+\nu S'''+\varrho S''''$ is coorthogonal with S′, S″, &c.; and if the radius of $\lambda S'+\mu S''+\nu S'''+\varrho S''''$ be evanescent, its centre must be a point on the required orthogonal sphere; but if its radius be zero, it represents an imaginary cone and the discriminant vanishes. It is easy to see that λ, μ, ν, ϱ are the tetrahedral coordinates of the centre of $\lambda S'+\mu S''+\nu S'''+\varrho S''''$, the tetrahedron of reference having its angular points at the centres of S′, S″, &c.

Now let the spheres S′, S″, &c. be given in the form

$$(x-a')^2+(y-b')^2+(z-c')^2-r^2=0 \text{ \&c.},$$

and then the required discriminant will be, after dividing by the factor $(\lambda+\mu+\nu+\varrho)^2$*,

$$\begin{aligned}(\lambda+\mu+\nu+\varrho)\{\lambda(a'^2+b'^2+c'^2-r'^2)+\mu(a''^2+b''^2+c''^2-r''^2)\\ +\nu(a'''^2+b'''^2+c'''^2-r'''^2)+\varrho(a''''^2+b''''^2+c''''^2-r''''^2)\}\\ =(a'\lambda+a''\mu+a'''\nu+a''''\varrho)^2+(b'\lambda+b''\mu+b'''\nu+b''''\varrho)^2\\ +(c'\lambda+c''\mu+c'''\nu+c''''\varrho)^2;\end{aligned}$$

and this is readily found to be equivalent to the following equation, in which (S′ S″) &c. denotes the angle of intersection of the spheres S′, S″, &c.:—

$$\left.\begin{aligned}(\lambda r')^2+(\mu r'')^2+(\nu r''')^2+(\varrho r'''')^2\\ -2\lambda\mu r'r''\cos(S'S'')-2\lambda\nu r'r'''\cos(S'S''')-2\lambda\varrho r'r''''\cos(S'S'''')\\ -2\mu\nu r''r'''\cos(S''S''')-2\nu\varrho r'''r''''\cos(S'''S'''')-2\varrho\mu r''''r''\cos(S''''S'')=0.\end{aligned}\right\} \quad . \quad . \quad (3)$$

See my paper "On the Equations of Circles &c.," in the Proceedings of the Royal Irish Academy, vol. ix. pt. iv. p. 410.

This equation is simplified by incorporating the radii r', r'', &c. with the variables $\lambda, \mu, \nu, \varrho$; thus put $\lambda r'=x$, $\mu r''=y$, &c., and we get the equation of the sphere orthogonal to four given spheres in the form

$$\left.\begin{aligned}x^2+y^2+z^2+w^2-2xy\cos(S'S'')-2xz\cos(S'S''')-2xw\cos(S'S'''')\\ -2yz\cos(S''S''')-2zw\cos(S'''S'''')-2wy\cos(S''''S'')=0.\end{aligned}\right\} \quad . \quad . \quad (4)$$

Cor. 1. Hence, if the four given spheres be mutually orthogonal, the equation of their orthogonal sphere in tetrahedral coordinates is

$$(\lambda r')^2+(\mu r'')^2+(\nu r''')^2+(\varrho r'''')^2=0 \text{ or } x^2+y^2+z^2+w^2=0. \quad . \quad . \quad . \quad . \quad (5)$$

Cor. 2. The sphere orthogonal to four given spheres is inscribed in each of the eight quadrics,

$$U^2=(\lambda r'\pm\mu r''\pm\nu r'''\pm\varrho r'''')^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

where U denotes the orthogonal sphere†.

* [The vanishing of the factor $(\lambda+\mu+\nu+\varrho)^2$ is the condition that the sphere $\lambda S+\mu S'+\nu S''+\varrho S'''$ may become a plane. Hence $\lambda+\mu+\nu+\varrho=0$ may be regarded as the tangential equation of the centre of the sphere which cuts orthogonally the four spheres S, S′, S″, S‴.—January 1872.]

† [Professor CAYLEY remarks as follows on this article:—"You give in passing what appears to me an interesting theorem, when you say 'it is easy to see that $\lambda, \mu, \nu, \varrho$ are the tetrahedral coordinates of the centre of the sphere $\lambda S+\mu S'+\nu S''+\varrho S'''=0$.' Take any four quadric surfaces $S=0$, $S'=0$, $S''=0$, $S'''=0$; and establish the relation $\lambda S+\mu S'+\nu S''+\varrho S'''=$ a cone. This establishes between $\lambda, \mu, \nu, \varrho$ and x, y, z, w four lineo-linear equations, so that, eliminating either set of variables, we have between the other set a quartic equation; moreover, the variables of each set are proportional to cubic functions of the other set (see my "Memoir on Quartic Surfaces," vol. iii. pp. 19–69 of the Proceedings of the London Mathematical Society). Then your theorem is, that when the four quadrics have a common conic, the x, y, z, w and $\lambda, \mu, \nu, \varrho$ are linear functions each of the other, so that the two quartic surfaces are homographically related, or, by a proper interpretation of the coordinates, may be regarded as being one and the same surface." My theorem, that $\lambda, \mu, \nu, \varrho$ are the tetrahedral coordinates of the centre of the sphere $\lambda S+\mu S'+\nu S''+\varrho S'''=0$, is easily proved as follows: the centre of the sphere $\lambda S+\mu S'+\nu S''+\varrho S'''$ is evidently the mean centre of the centres of S, S′, S″, S‴ for the system of multiples $\lambda, \mu, \nu, \varrho$; in other words, it is the centre of gravity of four masses proportional to $\lambda, \mu, \nu, \varrho$ placed at those points. Hence the proposition follows at once by a well-known theorem in Statics.—January 1872.]

4. If $W \equiv (a, b, c, d, l, m, n, p, q, r \between \alpha, \beta, \gamma, \delta)^2 = 0$, where $\alpha=0, \beta=0, \gamma=0, \delta=0$ are the equations of four given spheres, which I shall by analogy to known systems call the spheres of *reference*, then $W=0$ is evidently the most general equation of a surface of the fourth degree, having the imaginary circle at infinity as a double line. Such a surface has been called by MOUTARD an "anallagmatic surface," and by DARBOUX a "cyclide" (see 'Comptes Rendus,' June 7, 1869). I shall adopt the latter name.

5. The cyclide $W=0$ is by the usual theory the envelope of the sphere

$$x\alpha+y\beta+z\gamma+w\delta=0,$$

where x, y, z, w are variable multiples, provided the condition holds:

$$\begin{vmatrix} a, & n, & m, & p, & x, \\ n, & b, & l, & q, & y, \\ m, & l, & c, & r, & z, \\ p, & q, & r, & d, & w, \\ x, & y, & z, & w, & 0; \end{vmatrix} = 0. \quad \ldots \ldots \ldots \quad (7)$$

now the sphere $x\alpha+y\beta+z\gamma+w\delta=0$ cuts orthogonally the Jacobian of $\alpha, \beta, \gamma, \delta$, and the equation (7) is the equation of a quadric. Hence we have the following theorem:—

A quartic cyclide is the envelope of a variable sphere whose centre moves on a given quadric, and which cuts a given fixed sphere orthogonally.

6. If the equation of the cyclide be of the form

$$(a, b, c, f, g, h \between \alpha, \beta, \gamma)^2 = 0, \quad \ldots \ldots \ldots \quad (8)$$

it is shown, as in the last article, that it is the envelope of a variable sphere whose centre moves along a plane conic and which cuts a given fixed sphere orthogonally. Now from the form of equation (8) it is evident that this species of cyclide has two nodes, namely, the two points common to the three spheres of reference α, β, γ, and that these nodes are conic nodes, that is, nodes which have these points as vertices of tangent cones to the cyclide. I shall call this species of cyclide a *binodal* cyclide*.

SECTION II.—*Generalization of methods of Section* I.

7. The results of Section I. admit of important generalization, to the exposition of which I shall devote a few articles.

Let $S^{\frac{1}{2}}-A=0$, $S^{\frac{1}{2}}-B=0$ be two quadrics inscribed in the same quadric, $S \equiv x^2+y^2+z^2+w^2$, A and B being the planes $ax+a'y+a''z+a'''w=0$ and $bx+b'y+b''z+b'''w=0$ respectively; we see that $S^{\frac{1}{2}}-A+k(S^{\frac{1}{2}}-B)$ is the equation of a quadric inscribed in S, and passing through one of the conics of intersection of $S-A^2$ and $S-B^2$, namely, through the common intersection of these two quadrics with the plane $A-B=0$. But if we clear $(S^{\frac{1}{2}}-A)+k(S^{\frac{1}{2}}-B)=0$ from radicals, the discriminant

* [The cyclide (8) must, from the form of the equation, have two nodes; but in certain special cases which will be discussed in the sequel, it will have one or two additional nodes.—January 1872.]

of the result equated to zero gives, as is easily seen,

$$(1-S'')k^2+2(1-R)k+(1-S')=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

where S′, S″ are the results of substituting the coordinates of the poles of the planes A and B in S, and R the result of substituting the coordinates of the pole of A in B.

We should arrive at the same result if we had taken the equations of the two quadrics under the form $S^{\frac{1}{2}}+A$ and $S^{\frac{1}{2}}+B$. But if we had worked with $S^{\frac{1}{2}}\mp A$ and $S^{\frac{1}{2}}\pm B$, we should get

$$(1-S'')k^2+2(1+R)k+(1-S'). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

8. As the equations (9) and (10) are of the second degree in k, we see that, through each conic of intersection of $S-A^2$ and $S-B^2$, there pass two cones circumscribed to S. The equations of these cones are obtained by eliminating k between $S^{\frac{1}{2}}-A+k(S^{\frac{1}{2}}-B)$, and the two equations (9) and (10). They are:

$$(1-S'')(S^{\frac{1}{2}}-A)^2-2(1-R)(S^{\frac{1}{2}}-A)(S^{\frac{1}{2}}-B)+(1-S')(S^{\frac{1}{2}}-B)^2, \quad . \quad . \quad (11)$$

$$(1-S'')(S^{\frac{1}{2}}-A)^2-2(1+R)(S^{\frac{1}{2}}-A)(S^{\frac{1}{2}}+B)+(1-S')(S^{\frac{1}{2}}+B)^2. \quad . \quad . \quad (12)$$

These cones correspond to the limiting points of two spheres, as these latter are evidently imaginary cones passing through the circle of intersection of the two spheres and circumscribed to the imaginary circle at infinity.

9. If we put

$$1-R=\sqrt{(1-S')(1-S'')}\cos\theta,$$
$$1+R=\sqrt{(1-S')(1-S'')}\cos\phi,$$

the ratio of the roots of equation (9) is $e^{2\theta\sqrt{-1}}$, and of equation (10) $e^{2\phi\sqrt{-1}}$. Now if $\theta=\frac{\pi}{2}$ the ratio of the roots is negative unity, and we have an harmonic pencil of four planes, namely the planes A, B, and the planes passing through the intersection of the planes A, B, and which are also planes of contact of the cones of article (8) with the quadric (S); in other words, the poles of A and B and the vertices of the cones form an harmonic range of points. When two quadrics, then, are connected by the relation

$$1\pm R=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

I shall, by an extension of a known term, say that they cut *orthogonally* or *harmonically*.

10. It is easy to see that, being given by its general equation, a quadric S, and two planes $\lambda x+\mu y+\nu z+\rho w$, $\lambda' x+\mu' y+\nu' z+\rho' w$, the result of substituting the coordinates of the pole with respect to the quadric of one of the planes in the equation of the other, multiplied by the discriminant of the quadric, is equal to the determinant:

$$\begin{vmatrix} a, & n, & m, & p, & \lambda, \\ n, & b, & l, & q, & \mu, \\ m, & l, & c, & r, & \nu, \\ p, & q, & r, & d, & \rho, \\ \lambda, & \mu', & \nu', & \rho', & 0; \end{vmatrix}$$

and denoting this by Π, we infer from equation (13) that the condition that should cut orthogonally two quadrics, $S-(\lambda x+\mu y+\nu z+\varrho w)^2$, $S-(\lambda' x+\mu' y+\nu' z+\varrho' w)^2$, both inscribed in the same quadric S given by its general equation, is the invariant relation

$$\Delta \pm \Pi = 0. \qquad (14)$$

11. To find the equation of a quadric cutting four given quadrics orthogonally. Let

$$S^{\frac{1}{2}} \pm A, \quad S^{\frac{1}{2}} \pm B, \quad S^{\frac{1}{2}} \pm C, \quad S^{\frac{1}{2}} \pm D$$

be the four given quadrics. It follows from equation (13) that we must have

$$\begin{aligned} a\lambda + a'\mu + a''\nu + a'''\varrho \pm 1 &= 0, \\ b\lambda + b'\mu + b''\nu + b'''\varrho \pm 1 &= 0, \\ c\lambda + c'\mu + c''\nu + c'''\varrho \pm 1 &= 0, \\ d\lambda + d'\mu + d''\nu + d'''\varrho \pm 1 &= 0, \end{aligned}$$

λ, μ, ν, ϱ being the coordinates of the pole of the plane of contact of the sought quadric with respect to S, and that this quadric will be then $S^{\frac{1}{2}}-(\lambda x+\mu y+\nu z+\varrho w)=0$.

Hence, eliminating λ, μ, ν, ϱ from these five equations, we get

$$\begin{vmatrix} S^{\frac{1}{2}}, & x, & y, & z, & w, \\ \mp 1, & a, & a', & a'', & a''', \\ \mp 1, & b, & b', & b'', & b''', \\ \mp 1, & c, & c', & c'', & c''', \\ \mp 1, & d, & d', & d'', & d''', \end{vmatrix} = 0, \qquad (16)$$

where the double signs of the first column answer to those of the binomial, $S^{\frac{1}{2}} \pm A$. Hence if we denote for shortness by the notation $(S^{\frac{1}{2}}\ a\ b'\ c''\ d''')$ the determinant (16), in the case where all the units in the first column are positive, we shall have eight orthogonal quadrics, whose equations are as follows:—

$$(S^{\frac{1}{2}} \quad a \quad b' \quad c'' \quad d''')=0, \qquad (17)$$

$$(S^{\frac{1}{2}} \quad -a \quad b' \quad c'' \quad d''')=0, \qquad (18)$$

$$(S^{\frac{1}{2}} \quad a \quad -b' \quad c'' \quad d''')=0, \qquad (19)$$

$$(S^{\frac{1}{2}} \quad a \quad b' \quad -c'' \quad d''')=0, \qquad (20)$$

$$(S^{\frac{1}{2}} \quad a \quad b' \quad c'' \quad -d''')=0, \qquad (21)$$

$$(S^{\frac{1}{2}} \quad -a \quad -b' \quad c'' \quad d''')=0, \qquad (22)$$

$$(S^{\frac{1}{2}} \quad -a \quad b' \quad -c'' \quad d''')=0, \qquad (23)$$

$$(S^{\frac{1}{2}} \quad -a \quad b' \quad c'' \quad -d''')=0, \qquad (24)$$

Thus, for example, equation (20) developed is

$$\begin{vmatrix} S^{\frac{1}{2}}, & x, & y, & z, & w, \\ 1, & a, & a', & a'', & a''', \\ 1, & b, & b', & b'', & b''', \\ 1, & -c, & -c', & -c'', & -c''', \\ 1, & d, & d', & d'', & d''', \end{vmatrix} = 0.$$

12. Denoting by $J_1 \ldots J_8$ the eight orthogonal quadrics (17) ... (24), and remembering that λ, μ, ν, ϱ are the coordinates of the pole of the plane of contact of one of these surfaces with S, since these coordinates satisfy the first four equations of art. 11, we see easily that they belong to the point common to the system of six planes represented by the system of six equations,

$$\pm A = \pm B = \pm C = \pm D,$$

in which the arrangement of the signs correspond to the quadric which we consider. We have then the following theorem:—

The poles of contact of the eight orthogonal quadrics $J_1 \ldots J_8$ are the eight radical centres of the four quadrics, $S-A^2$, $S-B^2$, $S-C^2$, $S-D^2$.

13. The polar of the point λ, μ, ν, ϱ, with respect to $S-A^2$, is

$$\lambda(x-Aa)+\mu(y-Aa')+\nu(zAa'')+\varrho(w-Aa'''),$$

and this reduces, in virtue of the first equation of art. 11, to

$$\lambda x+\mu y+\nu z+\varrho w=\mp A;$$

and eliminating λ, μ, ν, ϱ from this and the four equations of the same article, we get

$$\begin{vmatrix} \pm A, & x, & y, & z, & w, \\ \pm 1, & a, & a', & a'', & a''', \\ \pm 1, & b, & b', & b'', & b''', \\ \pm 1, & c, & c', & c'', & c''', \\ \pm 1, & d, & d', & d', & d''', \end{vmatrix} = 0, \quad \ldots \ldots \quad (25)$$

where the choice of signs depends on the quadric J. This is evidently the plane of contact of one of the conics of intersection of J and $S-A^2$. We have then the following construction for the eight orthogonal quadrics:—

Let us imagine tangent cones whose vertices are the eight radical centres, and the required quadrics pass through the conics of contact.

14. The equations of J_1, J_2, &c. take a very simple form when referred to the tetrahedron which has for vertices the poles of the planes A, B, C, D with respect to S in tetrahedral coordinates.

Let x', y', z', w' be the new coordinates of the point (x, y, z, w), the poles being a, a', a'',

a''', b, b', b'', b''', &c., and we have then the following substitutions to make:

$$\begin{aligned} x &= ax' + by' + cz' + dw', \\ y &= a'x' + b'y' + c'z' + d'w', \\ z &= a''x' + b''y' + c''z' + d''w', \\ w &= a'''x' + b'''y' + c'''z' + d'''w'; \end{aligned}$$

and hence, by the equation of art. 11,

$$\lambda x + \mu y + \nu z + \varrho w = \pm x' \pm y' \pm z' + w'.$$

Consequently the transformed equation of J (corresponding to the choice of the double signs) is simply

$$\pm(x' \pm y' \pm z' \pm w') = S^{\frac{1}{2}}.$$

Hence

$$\begin{aligned} (\pm x' \pm y' \pm z' \pm w')^2 = &(ax' + by' + cz' + dw')^2 + (a'x' + b'y' + c'z' + d'w')^2 \\ &+ (a''x' + b''y' + c''z' + d''w')^2 + (a'''x' + b'''y' + c'''z' + d'''w')^2. \end{aligned}$$

This can be written in a more convenient manner by the following substitution, and by suppressing accents as being no longer necessary.

Let us denote the result of substituting the coordinates of the pole of

B in C by L and of A in D by P,
C „ A „ M „ B „ D „ Q,
A „ B „ N „ C „ D „ R,

and we shall have the equation of J_1 in the following form:

$$\left.\begin{aligned} J_1 \equiv &(1-S')x^2 + (1-S'')y^2 + (1-S''')z^2 + (1-S'''')w^2 \\ &+ 2(1-L)yz + 2(1-M)zx + 2(1-N)xy \\ &+ 2(1-P)xw + 2(1-Q)yw + 2(1-R)zw = 0. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (26)$$

15. The equation J_1 is the locus of all the double points of the quadric

$$\lambda(S^{\frac{1}{2}} - A) + \mu(S^{\frac{1}{2}} - B) + \nu(S^{\frac{1}{2}} - C) + \varrho(S^{\frac{1}{2}} - D).$$

In fact this is equivalent to the equation

$$(\lambda + \mu + \nu + \varrho)^2 S - (\lambda A + \mu B + \nu C + \varrho D)^2,$$

the discriminant of which is easily found to be

$$\begin{aligned} (\lambda + \mu + \nu + \varrho)^2 = &(\lambda a' + \mu b + \nu c + \varrho d)^2 + (\lambda a' + \mu b' + \nu c' + \varrho d') \\ &+ (\lambda a'' + \mu b'' + \nu c'' + \varrho d'')^2 + (\lambda a''' + \mu b''' + \nu c''' + \varrho d''')^2; \end{aligned}$$

and λ, μ, ν, ϱ being replaced by x, y, z, w, we have the equation of J_1.—Q.E.D.

It is instructive to compare the modes of investigation employed in this article and article 3 of the last section.

16. The equation (26) of J_1 can be written in a more suitable form by means of the anharmonic angles of art. 9; for this purpose let us denote

$$\frac{1-R}{\sqrt{(1-S')(1-S'')}} \text{ by } \cos L; \quad \frac{1+R}{\sqrt{(1-S')(1-S'')}} \text{ by } -\cos L', \text{ \&c.}$$

Let us denote also S′ by $\cos^2 \varrho'$, S″ by $\cos^2 \varrho''$, &c. It is evident that the angles L, L′, ϱ', ϱ'', &c. may be either real or imaginary; when the substitutes are made, we get

$$\left.\begin{aligned} J_1 \equiv & x^2 \sin^2 \varrho' + y^2 \sin^2 \varrho'' + z^2 \sin^2 \varrho''' + w^2 \sin^2 \varrho'''' \\ & + 2yz \sin \varrho'' \sin \varrho''' \cos L + 2zx \sin \varrho''' \sin \varrho' \cos M + 2xy \sin \varrho' \sin \varrho'' \cos N \\ & + 2xw \sin \varrho' \sin \varrho'''' \cos P + 2yw \sin \varrho'' \sin \varrho''' \cos Q + 2zw \sin \varrho''' \sin \varrho'''' \cos R = 0. \end{aligned}\right\} \quad (27)$$

Compare equation (3), art. 3.

Cor. If the four quadrics $S^{\frac{1}{2}}-A$, $S^{\frac{1}{2}}-B$, &c. be mutually orthogonal, the equation of their orthogonal quadric will be $J_1 \equiv x^2 \sin^2 \varrho' + y^2 \sin^2 \varrho'' + z^2 \sin^2 \varrho''' + w^2 \sin^2 \varrho''''$, or of the form $x^2+y^2+z^2+w^2=0$, and there will be only one orthogonal quadric instead of eight.

Observation. This section is abridged from a Memoir by me in TORTOLINI's 'Annali di Matematica,' serie ii. tomo ii. fasc. 4.

CHAPTER II.

SECTION I.—*Generation of Cyclides.*

17. We have seen that a cyclide is the envelope of a variable sphere whose centre moves along a given quadric, and which cuts a given sphere orthogonally (see art. 5). I shall call the variable sphere the *generating sphere* (a name which I find more convenient than *enveloped sphere* or enveloppée), and the quadric which is the locus of the centres of the generating sphere I shall call the *focal quadric*, a term expressive of an important property which it possesses. In DE LA GOURNERIE's Memoir "Sur les lignes Sphériques," he uses the name *déférente* in an analogous case (see LIOUVILLE's Journal, 1869). The sphere which is cut orthogonally we shall call the sphere of inversion, and it will be always denoted by the letter U, unless the contrary be stated, and the focal quadric by the letter F.

18. If we draw any tangent plane to F this will intersect F in two lines, the generating lines of F at the point of contact. Now let us conceive three spheres whose centres are at the intersections of these lines and at a consecutive point on each line respectively; then if they cut U orthogonally, the two points common to the three will evidently be their points of contact with their envelope. Now it is evident that these points are the limiting points of the system composed of U, and the tangent plane to F. Hence we have the following method of generating cyclides:—

Being given a quadric F *and a sphere* U, *draw any tangent plane* P *to* F, *the locus of the limiting points of* U *and* P *will be a cyclide.*

19. Let the two lines in which the tangent plane P of the last article cuts F be denoted by L, L′; now all the generating spheres whose centres are on L have a common circle of intersection; if this circle be called C, and the corresponding circle for L′ be called C′, then these circles are evidently homospheric, and the two points common to them are plainly points on the cyclide. I say moreover that the circles C, C′ lie altogether on the cyclide. For through L draw another tangent plane to F intersecting F in another line L″; then corresponding to this line we have another circle, C″, which is also homospheric with C, and their points of intersection are points on the cyclide; hence the circle C lies altogether in the cyclide, and so do the circles C′, C″, &c. Hence we infer the following method of generating cyclides analogous to the rectilinear generation of quadrics:—

Being given three circles, C, C′, C″, *cutting* U *orthogonally, the intersection of their planes being the centre of* U, *then the envelope of a variable circle whose motion is directed by cutting each of these circles twice is a cyclide.*

Cor. 1. Every generating sphere of a cyclide intersects it in the two generating circles passing through its points of contact with the cyclide.

Cor. 2. The generating spheres touch but do not intersect the cyclide if their focal quadric be not a ruled surface.

20. If the sphere U reduce to a point, which will happen when the four spheres of reference α, β, γ, δ (see art. 4) pass through a common point, the method of generating cyclides given in art. 18 becomes simplified as follows:—

Being given a quadric F *and a point* U, *then the locus of the reflection of* U *made by any tangent plane to* F *will be a cyclide.* This is plainly equivalent to the following:—*The pedal of a quadric is a cyclide;* or again, *the inverse of a quadric with respect to any arbitrary point is a cyclide.*

Or we may state the whole matter thus:—Being given a quadric F and a point U, from U draw a perpendicular UT on any tangent plane to F, and on UT take P, P′ in opposite directions such that $UT^2-TP^2=UT^2-TP'^2=k^2$, where k is a constant, then the locus of P, P′ is a cyclide.

There are three cases to be considered.

1°. If k^2 be positive the sphere U is real.

2°. If k^2 be negative the sphere U is imaginary; this will happen when the radical centre of the spheres of reference, α, β, γ, δ, is internal to these spheres.

3°. If k^2 vanish, U reduces to a point. The cyclide is in this case the inverse of a quadric.

The point U is a nodal point on the cyclide. The tangent planes to the cyclide at the node U form a cone, which is reciprocal to the cone whose vertex is at U and which circumscribes F. Hence the point U will be a conic node when F is either an ellipsoid or hyperboloid. We shall examine more minutely the species of the node in this case when we come to the Chapter on the Inversion of Cyclides.

21. If the focal quadric F be a cone the cyclide becomes modified in a remarkable way,

which it is necessary to examine, as this species of cyclide will occupy much of our space in the present memoir.

Since all the tangent planes of a cone pass through the same point, and since every tangent plane determines two points on the cyclide, it is plain that all the points lie on the surface of a sphere whose centre is at the vertex of the cone.

Again, since the cone is a ruled surface, each edge of it will determine, as in art. 19, a circle which will be a generating circle of the cyclide; but the circle will not in this case lie altogether in the envelope as in art. 19, because in art. 19 the points of contact of any line on the quadric are distinct for all the planes passing through it, whereas in the cone only one tangent plane, properly so called, can be drawn through any edge of it. But although the circles which answer to each edge of the cone do not lie altogether in the cyclide, yet the envelope of these circles is the cyclide, which in this case is evidently a twisted curve, which, as will be shown, is of the fourth degree. On this account I have called this species of cyclide, for the sake of distinction, a sphero-quartic.

22. Since the planes of the generating circles in the last article are perpendicular to the edges of the focal cone, the envelope of these planes is another cone; and as each plane passes through the centre of the sphere U, the vertex of the second cone is at the centre of U. Hence the sphero-quartic is the intersection of a sphere and a cone. Hence we have the following theorem:—*When a cyclide reduces to a sphero-quartic, it is the intersection of a sphere and a quadric.*

23. If we denote the sphere on which we have proved the sphero-quartic lies by Ω, then the generating circles are circles on Ω; and as Ω evidently cuts U orthogonally, the circle of intersection of U and Ω, which we denote by J, will be orthogonal to all the generating circles, and the focal cone pierces Ω in a sphero-conic. Hence we have the following method of generating sphero-quartics:—

Being given a circle J *on a sphere, and a sphero-conic on the same sphere. A sphero-quartic is the envelope of a variable circle whose centre moves along the sphero-conic, and which cuts the circle* J *orthogonally.*

24. From the last article we infer this other method of generating sphero-quartics.

Let F be a sphero-conic on a sphere Ω, U a point on the surface of Ω; from U draw an arc UT perpendicular to any great circle tangential to F, and take two points, P, P′, such that

$$\cos \mathrm{UT} : \cos \mathrm{TP} = \cos \mathrm{UT} : \cos \mathrm{TP'} = k,$$

where k is a constant.

The locus of the points P, P′ is a sphero-quartic.

Cor. If $k=1$ the point P coincides with U, and the point will in this case be a double point in the sphero-quartic, and the sphero-quartic itself will be the inverse of a plane conic from a point outside the plane of the conic. In fact if the sphere Ω be inverted into a plane from the point U, it is easy to see that the point P′ will be inverted into a point whose locus is a conic.

SECTION II.—*Generation of Quartic Surfaces having a Conic Nodal Line.*

Lemma. If $S\equiv x^2+y^2+z^2+w^2=0$, $A\equiv ax+by+cz+dw=0$, then the result of the operation $\lambda\frac{d}{dx}+\mu\frac{d}{dy}+\nu\frac{d}{dz}+\varrho\frac{d}{dw}$ performed on the quadric $S^{\frac{1}{2}}-A$ is

$$(a\lambda+b\mu+c\nu+d\varrho)S^{\frac{1}{2}}-(\lambda x+\mu y+\nu z+\varrho w),$$

which is evidently connected with $S^{\frac{1}{2}}-A$ by the invariant relation (13) of art 9. Hence we have the following theorem:—

The result of the operation $\lambda\frac{d}{dx}+\mu\frac{d}{dy}+\nu\frac{d}{dz}+\varrho\frac{d}{dw}$ *performed on a quadric of the form* $S^{\frac{1}{2}}\pm A$ *is a quadric orthogonal to* $S^{\frac{1}{2}}\pm A$.

25. Being given quadric J inscribed in another S and a point $(\alpha, \beta, \gamma, \delta)$, we can find, by the method of the preceding lemma, a quadric Q orthogonal to J, whose pole of contact with S is the point $(\alpha, \beta, \gamma, \delta)$. When $(\alpha, \beta, \gamma, \delta)$ varies, Q varies also; and I say if the locus of $(\alpha, \beta, \gamma, \delta)$ is a quadric F, that the envelope of Q is a quartic surface having a conic nodal line.

Demonstration. Let $Q_1=S^{\frac{1}{2}}-A$, $Q_2=S^{\frac{1}{2}}-B$, $Q_3=S^{\frac{1}{2}}-C$, $Q_4=S^{\frac{1}{2}}-D$ be four particular quadrics of the system cutting J orthogonally. Let us consider the quadric Q as having its pole of contact with S at the centre of mean distances of the poles of contact of Q_1, Q_2, Q_3, Q_4 for any suitable system of multiples x, y, z, w, and as we are only concerned with their mutual ratios, we can put $x+y+z+w=1$. Hence we get the following system of eqnations:—

$$\begin{aligned}
\alpha&=ax\;+by\;+cz\;+dw,\\
\beta&=a'x\;+b'y\;+c'z\;+d'w,\\
\gamma&=a''x+b''y+c''z+d''w,\\
\delta&=a'''x+b'''y+c'''z+d'''w;
\end{aligned}$$

and from these we have $Q=Q_1x+Q_2y+Q_3z+Q_4w$; and since the locus of $(\alpha, \beta, \gamma, \delta)$ is the quadric F, we see by substitution of the preceding values of $\alpha, \beta, \gamma, \delta$ in its equation, that x, y, z, w are connected by an equation of the second degree. If we denote this equation by

$$(a, b, c, d, l, m, n, p, q, r)(x, y, z, w)^2=0,$$

we have to find the envelope of $Q_1x+Q_2y+Q_3z+Q_4w$ subject to this condition. The theory of envelopes gives

$$\begin{vmatrix}
a, & n, & m, & p, & Q_1,\\
n, & b, & l, & q, & Q_2,\\
m, & l, & c, & r, & Q_3,\\
p, & q, & r, & d, & Q_4,\\
Q_4, & Q_2, & Q_3, & Q_4, & 0,
\end{vmatrix}=0.$$

Hence the required envelope is

$$\begin{vmatrix} a, & n, & m, & p, & S^{\frac{1}{2}}-A, \\ n, & b, & l, & q, & S^{\frac{1}{2}}-B, \\ m, & l, & c, & r, & S^{\frac{1}{2}}-C, \\ p, & q, & r, & d, & S^{\frac{1}{2}}-D, \\ S^{\frac{1}{2}}-A, & S^{\frac{1}{2}}-B, & S^{\frac{1}{2}}-C, & S^{\frac{1}{2}}-D, & 0, \end{vmatrix}=0. \quad . \quad . \quad . \quad (28)$$

The expansion of this determinant gives evidently a result of the form $U_2+U_1S^{\frac{1}{2}}=0$, where U_2 represents a function of the second degree, and U_1 a function of the first degree in the variables; and clearing off radicals we get $U_2^2-U_1^2S=0$; and this is the equation of a surface of the fourth degree, having the conic of intersection of U_2 and U_1 as a double line. Hence the proposition is proved.

26. The quantities x, y, z, w of the last article are evidently proportional to the tetrahedral coordinates of the point $\alpha, \beta, \gamma, \delta$, referred to the tetrahedron whose vertices are the poles of the planes A, B, C, D of the quadrics $S-A^2$, $S-B^2$, $S-C^2$, $S-D^2$, so that the equation of condition in x, y, z, w is only the equation of the surface F referred to this tetrahedron. Hence the method of generation of surfaces of the fourth degree having a conic for a nodal line is exactly the same as the method of generating cyclides given in art. 5; and in fact the two surfaces are identical, since the cyclide has the imaginary circle at infinity for a nodal line, so that by linear transformation we could get one surface from the other; and to every property of a cyclide there is a corresponding property of the more general surface here considered: but I thought that it would be useful to show that their equations, the equations of the surface cutting them orthogonally &c., are identical in form; so that for every theorem which I shall prove to hold for a cyclide the reader may if he chooses put in the more general surface here considered*.

* [Professor CAYLEY has remarked to me that, instead of the method of Chapter II., the immediate generalization would be to consider, instead of spheres, quadric surfaces of the form $S+LM$, $S+LN$, &c., and that it is a *further* generalization, or rather an extension, of $S-A^2$, $S-B^2$, &c. Professor CAYLEY remarks that it is a pity to omit the intermediate step. Before Professor CAYLEY had drawn my attention to it, the intermediate step had not occurred to me; however, any person who reads Chapter II. will find it easy to supply, by the assistance of the two following propositions, the omissions referred to:—

I. $S+LM=0$, $S+LN=0$ are two quadrics; it is required to find the condition that the pole of L with respect to $S+LM$ will be the pole of $M-N$ with respect to $S+LN$. Let

$$S+LM\equiv x^2+y^2+z^2+w^2+2(by+cz+dw)x,$$

$$S+LN\equiv x^2+y^2+z^2+w^2+2(b'y+c'z+d'w)x;$$

and let $\lambda, \mu, \nu, \varrho$ be the coordinates of the pole of the plane x with respect to the quadric $S+LM=0$; then we get the four equations:

$$\lambda+b\mu+c\nu+d\varrho=1, \qquad \lambda b+\mu=0,$$

$$\lambda c+\nu=0, \qquad \lambda d+\varrho=0.$$

CHAPTER III.

SECTION I.—*Different forms of the Equations of Cyclides.*

27. Let T be a tangent plane to the focal quadric F; P, P′ the corresponding points of the cyclide; then (art. 18) P, P′ are the limiting points of the sphere U and the plane T.

Hence

$$\lambda=\frac{1}{1-(b^2+c^2+d^2)},\quad \mu=\frac{-b}{1-(b^2+c^2+d^2)},$$

$$\nu=\frac{-c}{1-(b^2+c^2+d^2)},\quad \varrho=\frac{-d}{1-(b^2+c^2+d^2)}.$$

Hence forming the condition that the polar plane of the point $(\lambda, \mu, \nu, \varrho)$ with respect to the quadric S+LN is

$$(b-b')y+(c-c')z+(d-d')w=0,$$

we get

$$bb'+cc'+dd'=1.$$

This condition I propose to call the orthotomic invariant of the two quadrics.

If we take the more general forms,

$$x^2+y^2+z^2+w^2+2(ax+by+cz+dw)x,$$

$$x^2+y^2+z^2+w^2+2(a'x+b'y+c'z+d'w)x,$$

for S+LM, S+LN, these may, without loss of generality, be written in the more compact forms

$$ax^2+y^2+z^2+w^2+2(by+cz+dw)x,$$

$$a'x^2+y^2+z^2+w^2+2(b'y+c'z+d'w)x;$$

and we find, as before, the orthotomic invariant to be

$$2bb'+2cc'+2dd'=a+a'.$$

Compare equation (1), article 1.

The two quadrics related, as here considered, have many important properties. Thus *the poles of the plane* L *with respect to the quadrics, and the four points in which the line of connexion of these poles meets the quadrics, form a system of six points in involution.*

Def. Two quadrics related as in this proposition may be said to cut orthogonally.

II. Given five quadrics, S+LM, S+LN, &c., where

$$M=a'x+b'y+c'z+d'w=0,$$

$$N=a''x+b''y+c''z+d''w=0,$$

then the condition that the five quadrics should be coorthogonal is the determinant

$$\begin{vmatrix} a', & b', & c', & d', & 1, \\ a'', & b'', & c'', & d'', & 1, \\ a''', & b''', & c''', & d''', & 1, \\ a'''', & b'''', & c'''', & d'''', & 1, \\ a''''', & b''''', & c''''', & d''''', & 1, \end{vmatrix}=0.$$

Hence we infer the following theorem:—If α, β, γ, δ be any four quadrics of the form S+LM=0, S+LN=0, &c., then the quadric $\lambda\alpha+\mu\beta+\nu\gamma+\varrho\delta$ is coorthogonal with α, β, γ, δ, and the pole of the plane L=0 with respect to $\lambda\alpha+\mu\beta+\nu\gamma+\rho\delta$ will be a point whose tetrahedral coordinates are proportional to λ, μ, ν, ϱ, the angular points of the tetrahedron of reference being the poles of L with respect to α, β, γ, δ respectively.—January 1872.]

Let a, b, c, d be the centres of the spheres of reference, α, β, γ, δ; r', r'', r''', r'''' their radii; then, since U is the Jacobian of α, β, γ, δ, the tangents from a, b, c, d to U are equal to r', r'', r''', r'''' respectively. Again, let perpendiculars from a, b, c, d on the tangent plane T be denoted by λ, μ, ν, ρ. Now the result of substituting the coordinates of P in $\alpha = \mathrm{P}a^2 - r^2 =$ difference of squares of tangents from a to the limiting point P, and the orthogonal sphere $\mathrm{U} = 2\lambda . \mathrm{OP}$ (O being the centre of U). Hence the results of substituting the coordinates of any point P of the cyclide in the equations of the spheres of reference are proportional to the perpendiculars λ, μ, ν, ρ. Hence we have the following theorem :—

If $(a, b, c, d, l, m, n, p, q, r \between \alpha, \beta, \gamma, \delta)^2 = 0$ *be the equation of any cyclide,*

$$(a, b, c, d, l, m, n, p, q, r \between \lambda, \mu, \nu, \rho)^2 = 0$$

is the tangential equation of the corresponding focal quadric of the cyclide.

Cor. 1. Hence if we are given the equation of the focal quadric, we are given the equation of the cyclide, and *vice versâ.*

Cor. 2. Hence, when the sphere of inversion U and the focal quadric F of a cyclide are given, the cyclide is determined; but U is determined by four constants and F by nine. Hence a cyclide is determined by $4+9=13$ constants.

28. By means of the last article we are enabled to get a very important expression for the sphere U in terms of the four spheres of references. Thus, since a cyclide is the envelope of a variable sphere cutting U orthogonally, and whose centre moves along the surface of a given quadric F, now if the given quadric F be the sphere U itself, it is plain that the cyclide will in this case be the sphere U counted twice, that is U^2. But the equation of U in tetrahedral coordinates, x, y, z, w being the coordinates, is (see art. 3)

$$\begin{aligned}&(r'x)^2+(r''y)^2+(r'''z)^2+(r''''w)^2\\&-2r'r''xy\cos(\alpha\beta)-2r'r'''xz\cos(\alpha\gamma)-2r'r''''xw\cos(\alpha\delta)\\&-2r''r'''yz\cos(\beta\gamma)-2r''r''''yw\cos(\beta\delta)-2r'''r''''zw\cos(\gamma\delta)=0.\end{aligned}$$

Hence, forming the corresponding tangential equation, and substituting α, β, γ, δ for the variables, we get the following determinant for the square of U :—

$$\begin{vmatrix} -r'^2, & r'r''\cos(\alpha\beta), & r'r'''\cos(\alpha\gamma), & r'r''''\cos(\alpha\delta), & \alpha, \\ r''r'\cos(\beta\alpha), & -r''^2, & r''r'''\cos(\beta\gamma), & r''r''''\cos(\beta\delta), & \beta, \\ r'''r'\cos(\gamma\alpha), & r'''r''\cos(\gamma\beta), & -r'''^2, & r'''r''''\cos(\gamma\delta), & \gamma, \\ r''''r'\cos(\delta\alpha), & r''''r''\cos(\delta\beta), & r''''r'''\cos(\delta\gamma), & -r''''^2, & \delta, \\ \alpha, & \beta, & \gamma, & \delta, & 0. \end{vmatrix}$$

This determinant may be simplified as follows :—divide the first row by r', the second

by r'', &c. Again divide the first column by r', the second by r'' &c., and we get the following result:—

$$U^2= \begin{vmatrix} -1, & \cos(\alpha\beta), & \cos(\alpha\gamma), & \cos(\alpha\delta), & \alpha\div r', \\ \cos(\beta\alpha), & -1, & \cos(\beta\gamma), & \cos(\beta\delta), & \beta\div r'', \\ \cos(\gamma\alpha), & \cos(\gamma\beta), & -1, & \cos(\gamma\delta), & \gamma\div r''', \\ \cos(\delta\alpha), & \cos(\delta\beta), & \cos(\delta\gamma), & -1 & \delta\div r'''', \\ \alpha\div r' & \beta\div r'' & \gamma\div r''' & \delta\div r'''' & 0. \end{vmatrix} \quad . \quad . \quad (29)$$

Cor. Hence, if the four spheres of reference α, β, γ, δ be mutually orthogonal, the equation becomes

$$-U^2\equiv\left(\frac{\alpha}{r'}\right)^2+\left(\frac{\beta}{r''}\right)^2+\left(\frac{\gamma}{r'''}\right)^2+\left(\frac{\delta}{r}\right)^2=0;$$

and by incorporating constants with the variables it becomes $\alpha^2+\beta^2+\gamma^2+\delta^2=0$.

We shall find the value of U^2 in this latter form very important.

29. If the tetrahedron to which F is referred be inscribed in F, the coefficients a, b, c, d vanish; then forming the tangential equation and replacing λ, μ, ν, ϱ by α, β, γ, δ, we have the following theorem. If the equation of a cyclide be in the form

$$\begin{vmatrix} 0, & n, & m, & p, & \alpha, \\ n, & 0, & l, & q, & \beta, \\ m, & l, & 0, & r, & \gamma, \\ p, & q, & r, & 0, & \delta, \\ \alpha, & \beta, & \gamma, & \delta, & 0 \end{vmatrix}=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

that is, of a symmetrical determinant bordered with the variables whose diagonal terms are each zero, the spheres of reference have each double contact with the cyclide; in other words, they are generating spheres.

Cor. From this theorem, combined with article 3, we can easily get the equation of a sphere circumscribing a tetrahedron.

30. If the equation of a cyclide be given in the form

$$\left.\begin{aligned}\alpha^2+\beta^2+\gamma^2+\delta^2+2l(\beta\gamma+\alpha\delta)\\ +2m(\alpha\gamma+\beta\delta)+2n(\alpha\beta+\gamma\delta)=0\end{aligned}\right\}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

where $1+2\,lmn=l^2+m^2+n^2$, it can be proved, precisely as in SALMON'S 'Geometry of Three Dimensions,' p. 153, that each of the spheres of reference cuts the cyclide in two circles. Hence (see art. 19, *Cor.* 1) each of the spheres of references is a generating sphere.

31. If the coefficients of α^2, β^2, γ^2, δ^2 in the general equation of a cyclide vanish, then the coefficients of λ^2, μ^2, ν^2, ϱ^2 vanish in the tangential equation of the focal quadric; and hence *if the coefficients of the squares of the variables vanish in the equation of a*

cyclide, the focal quadric is inscribed in the tetrahedron formed by the centres of the spheres of reference.

32. Let $W\equiv(a, b, c, d, l, m, n, p, q, r\between\alpha, \beta, \gamma, \delta)^2=0$ be the equation of a cyclide, and we know that the square of the Jacobian of $\alpha, \beta, \gamma, \delta$ is given by the equation (29). Now, substituting $\lambda, \mu, \nu, \varrho$ for $\alpha, \beta, \gamma, \delta$ in these equations, we have the tangential equations of the focal quadric F of the cyclide and the sphere U; but if F and U be referred to their common self-conjugate tetrahedron, their equations will take the form

$$a\lambda^2+b\mu^2+c\nu^2+d\varrho^2=0, \quad \lambda^2+\mu^2+\nu^2+\varrho^2=0.$$

Hence we have W and U^2 given by the equations

$$W=a\alpha^2+b\beta^2+c\gamma^2+d\delta^2=0,$$
$$U^2\equiv-(\alpha^2+\beta^2+\gamma^2+\delta^2)=0 \text{ (see art. 28. } Cor.).$$

Now if, for the sake of uniformity, we represent U by ε, we have the following equation identically true,

$$\alpha^2+\beta^2+\gamma^2+\delta^2+\varepsilon^2=0; \quad \dots \dots \dots \quad (32)$$

and since the addition of any multiple of an expression which vanishes identically does not alter a function, we see that the equation of a cyclide may be written in the following form by adding $e(\alpha^2+\beta^2+\gamma^2+\delta^2+\varepsilon^2)$ to $a\alpha^2+b\beta^2+c\gamma^2+d\delta^2=0$, and afterwards putting a for $(a+e)$, b for $(b+e)$ &c.,

$$W\equiv a\alpha^2+b^2\beta+c\gamma^2+d\delta^2+e\varepsilon^2=0, \quad \dots \dots \dots \quad (33)$$

in which each of the spheres of reference is cut orthogonally by all the others.

We could show *à priori* that W can be expressed in either of the forms

$$a\alpha^2+b\beta^2+c\gamma^2+d\delta^2=0,$$

or

$$a\alpha^2+b\beta^2+c\gamma^2+d\delta^2+e\varepsilon^2=0;$$

for the first form contains explicitly three constants, and each sphere contains four constants, so that there are $3+4\times4=19$ constants at our disposal; but each pair of spheres being mutually orthogonal is equivalent to six conditions. Hence we have thirteen constants, which is the number required. Similarly, in the second form we have twenty-four constants; but these are subject to the equation of condition $\alpha^2+\beta^2+\gamma^2+\delta^2+\varepsilon^2$, which does not vanish identically except by the incorporation of certain constants, and the condition of orthogonality of each pair of spheres is equivalent to ten conditions. Hence, as before, we have thirteen conditions remaining.

33. By means of the identical relation $\alpha^2+\beta^2+\gamma^2+\delta^2+\varepsilon^2=0$, we can eliminate in succession each of the five spheres $\alpha, \beta, \gamma, \delta, \varepsilon$ from the equation (33) of the cyclide; and we see that the same cyclide can be written in five different ways, the letter eliminated representing the sphere used in the corresponding equation of the surface. Hence we have

$$\left.\begin{array}{ll} W=(a-b)\beta^2+(a-c)\gamma^2+(a-d)\delta^2+(a-e)\epsilon^2=0, & (\mathrm{I}) \\ W=(b-c)\gamma^2+(b-d)\delta^2+(b-e)\epsilon^2+(b-a)\alpha^2=0, & (\mathrm{II}) \\ W=(c-d)\delta^2+(c-e)\epsilon^2+(c-a)\alpha^2+(c-b)\beta^2=0, & (\mathrm{III}) \\ W=(d-e)\epsilon^2+(d-a)\alpha^2+(d-b)\beta^2+(d-c)\gamma^2=0, & (\mathrm{IV}) \\ W=(e-a)\alpha^2+(e-b)\beta^2+(e-c)\gamma^2+(e-d)\delta^2=0. & (\mathrm{V}) \end{array}\right\} \quad \ldots \quad (34)$$

If we denote the focal quadrics corresponding to these different forms of the equation of W by F F′ F″ F‴ $F^{(iv)}$, we get the following as the tangential equations of the five focal quadrics: —

$$\left.\begin{array}{l} F=(a-b)\mu^2+(a-c)\nu^2+(a-d)\varrho^2+(a-e)\sigma^2=0, \\ F'=(b-c)\nu^2+(b-d)\varrho^2+(b-e)\sigma^2+(b-a)\lambda^2=0, \\ F''=(c-d)\varrho^2+(c-e)\sigma^2+(c-a)\lambda^2+(c-b)\mu^2=0, \\ F'''=(d-e)\sigma^2+(d-a)\lambda^2+(d-b)\mu^2+(d-c)\nu^2=0, \\ F^{(iv)}=(e-a)\lambda^2+(e-b)\mu^2+(e-c)\nu^2+(e-d)\varrho^2=0. \end{array}\right\} \quad \ldots \quad (35)$$

So that the cyclide W can be generated in five different ways as the envelope of a variable sphere whose centre moves on a quadric and cuts a given sphere orthogonally; the corresponding spheres and quadrics for generating W being α, F; β, F′; γ, F″; δ, F‴; ϵ, $F^{(iv)}$ respectively.

34. Since the tangential equation of α is plainly $\mu^2+\nu^2+\varrho^2+\sigma^2=0$,

of β, $\nu^2+\varrho^2+\sigma^2+\lambda^2$, &c.,

we get the equations of the developables circumscribed to the pairs of quadrics α, F; β, F′, &c. by a known process; thus the developable circumscribed about α and F will be the envelope of the quadric, whose tangential equation

$$(a-b)\mu^2+(a-c)\nu^2+(a-d)\varrho^2+(a-e)\sigma^2+k(\mu^2+\nu^2+\varrho^2+\sigma^2)=0;$$

and by taking $k=(b-a)$, $(c-a)$, $(d-a)$, $(e-a)$ in succession, we see that the double lines of the developable are the plane conics, whose tangential equations are:

$$\left.\begin{array}{l} (b-c)\nu^2+(b-d)\varrho^2+(b-e)\sigma^2=0, \\ (c-b)\mu^2+(c-d)\varrho^2+(c-e)\sigma^2=0, \\ (d-b)\mu^2+(d-c)\nu^2+(d-e)\sigma^2=0, \\ (e-b)\mu^2+(e-c)\nu^2+(e-d)\varrho^2=0. \end{array}\right\} \quad \ldots \quad (36)$$

By comparing these with the system of equations (35), we see that the first conic is a plane section of the quadric F′, the second of F″, the third of F‴, and the fourth of F⁗.

Hence, if we call the spheres α, β, γ, δ, ϵ the spheres of inversion of the cyclide (we shall prove this in a future Chapter), and call Σ, Σ', Σ'', Σ''', Σ'''' the five developables circumscribed to the spheres of inversion and their corresponding focal quadrics, we have the following theorem:—

The double lines of Σ	*are plane*	*sections of*	F', F'', F''', F'''',
,,	Σ'	,,	F'', F''', F'''', F,
,,	Σ''	,,	F''', F'''', F, F',
,,	Σ'''	,,	F'''', F, F', F'',
,,	Σ''''	,,	F, F', F'', F'''.

35. If we take the first of the equations (34) to represent W, the corresponding sphere of inversion is α^2; but this, in virtue of the identical relation $\alpha^2+\beta^2+\gamma^2+\delta^2+\varepsilon^2=0$, is also given by the equation $\beta^2+\gamma^2+\delta^2+\varepsilon^2=0$; and eliminating β^2, γ^2, δ^2, ε^2 in succession between W and α^2, we see that each of the four binodal cyclides are inscribed in W,

$$\left.\begin{aligned}(b-c)\gamma^2+(b-d)\delta^2+(b-e)\varepsilon^2&=0,\\(c-b)\beta^2+(c-d)\delta^2+(c-e)\varepsilon^2&=0,\\(d-b)\beta^2+(d-c)\gamma^2+(d-e)\varepsilon^2&=0,\\(e-b)\beta^2+(e-c)\gamma^2+(e-d)\delta^2&=0,\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (37)$$

and these cyclides have the double lines of Σ as focal quadrics.

It is plain that we get corresponding results for each of the five forms (34) in which the equation of W may be written, so that *the cyclide* W *is circumscribed about ten binodal cyclides. The focal quadrics of these binodals are plane conics, and through each conic two developables pass.*

SECTION II.—*Sphero-quartics.*

36. Let P, a point on the surface of the sphere U, be the centre of the small circle S on the surface of the same sphere, O a fixed point, also on the surface of U, which we shall take as origin, OX a great circle of U corresponding to the initial line in plane geometry, and let $OP=n$ and the angle $POX=m$; then m and n are what I shall call the spherical coordinates of the point P; and whenever I shall use the term spherical coordinates it is in the sense here explained. Now let θ and ρ be the spherical coordinates of any point Q of the circle S, then (the reader can easily construct the figure) we have from the spherical triangle OPQ, r being the radius of S,

$$\cos r=\cos n\cos\rho+\sin n\sin\rho\cos(\theta-m). \quad . \quad . \quad . \quad . \quad . \quad . \quad (38)$$

This equation may be taken as the equation of the small circle S. Now if in the equation (38) we substitute the spherical coordinates of any point Q′ whose distance from P is the arc r', we plainly get $\cos r-\cos r'$; but $\cos r-\cos r'$ is equal to the perpendicular let fall from the point Q′ on the plane of the small circle S, hence we have the following theorem:—*The result of substituting the spherical coordinates of any point on the surface of a sphere radius unity in the equation of any small circle on the sphere is equal to the perpendicular distance of the point from the plane of the small circle.*

37. If any sphere Ω intersect a cyclide W, the curve of intersection is a sphero-quartic.

Demonstration. Let $W=a\alpha^2+b\beta^2+c\gamma^2+d\delta^2$, and let perpendiculars from any point P

of the curve $W\Omega$ on the planes through the intersection of Ω and α, Ω and β, &c. be denoted by x, y, z, w; then if the distances from the centre of Ω to the centres of $\alpha, \beta, \gamma, \delta$ be denoted by $\omega_1, \omega_2, \omega_3, \omega_4$ respectively, it is easy to see that the results of substituting the coordinates of P in $\alpha, \beta, \gamma, \delta$ are $2\omega_1 x$, $2\omega_2 y$, $2\omega_3 z$, $2\omega_4 w$, and therefore the quadric

$$a\omega_1^2x^2+b\omega_2^2y^2+c\omega_3^2z^2+d\omega_4^2w^2=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (39)$$

passes through the curve $W\Omega$. Hence the curve $W\Omega$ is also the intersection of the sphere Ω and the quadric (39), and therefore it is a sphero-quartic.

38. If in the last article we suppose the sphere Ω to coincide with U, the sphere orthogonal to $\alpha, \beta, \gamma, \delta$, and if we denote the circles in which the spheres $\alpha, \beta, \gamma, \delta$ intersect U_1 by the same notation, that is, by $\alpha, \beta, \gamma, \delta$, then if the radii of the circles $\alpha, \beta, \gamma, \delta$ be r', r'', r''', r'''', it is plain that $\omega_1, \omega_2, \omega_3, \omega_4$ of the last article become $\sec r'$, $\sec r''$, $\sec r'''$, $\sec r''''$, the radius of U being denoted by unity. Hence, by articles 36 and 37, we have the following theorem:—*If* $W=(a, b, c, d, l, m, n, p, q, r\between\alpha, \beta, \gamma, \delta)^2=0$ *be the equation of a cyclide, the equation of the sphero-quartic* WU *will be*

$$(a, b, c, d, l, m, n, p, q, r)\left(\frac{\alpha}{\cos r'}, \frac{\beta}{\cos r''}, \frac{\gamma}{\cos r'''}, \frac{\delta}{\cos r''''}\right)^2=0, \quad . \quad . \quad . \quad (40)$$

where $\alpha, \beta, \gamma, \delta$ *are the small circles of intersection of the spheres* $\alpha, \beta, \gamma, \delta$ *with* U.

39. From the three last articles, combined with art. 28, we have at once the following theorem, which is a very important one in the theory of sphero-quartics:—*If* $\alpha, \beta, \gamma, \delta$ *be any four small circles on the sphere* U, *the following relation will be true for any point on the surface of* U, *and will therefore be an identical relation*:

$$\begin{vmatrix} -1, & \cos(\alpha\beta), & \cos(\alpha\gamma), & \cos(\alpha\delta), & \alpha\div\sin r', \\ \cos(\beta\alpha), & -1, & \cos(\beta\gamma), & \cos(\beta\delta), & \beta\div\sin r'', \\ \cos(\gamma\alpha), & \cos(\gamma\beta), & -1, & \cos(\gamma\delta), & \gamma\div\sin r''', \\ \cos(\delta\alpha), & \cos(\delta\beta), & \cos(\delta\gamma), & -1, & \delta\div\sin r'''', \\ \alpha\div\sin r', & \beta\div\sin r'', & \gamma\div\sin r''', & \delta\div\sin r'''', & 0, \end{vmatrix}=0. \quad . \quad . \quad (41)$$

Cor. If the circles $\alpha, \beta, \gamma, \delta$ on the surface of U be mutually orthogonal, the relation is identically true for any point on U,

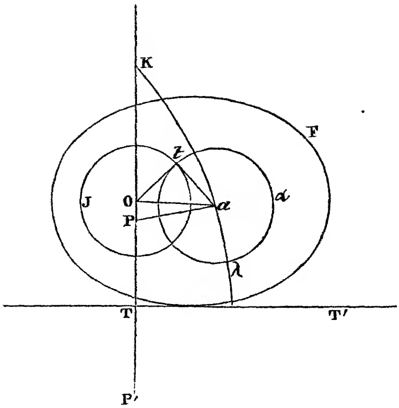
$$\frac{\alpha^2}{\sin^2 r'}+\frac{\beta^2}{\sin^2 r''}+\frac{\gamma^2}{\sin^2 r'''}+\frac{\delta^2}{\sin^2 r''''}=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (42)$$

If we incorporate the constants with the variables this becomes $\alpha^2+\beta^2+\gamma^2+\delta^2=0$.

40. If α, β, γ be three small circles on the sphere U, and if a sphero-quartic be given by an equation of the second degree $(a, b, c, f, g, h\between\alpha, \beta, \gamma)^2=0$, I say the tangential equation of the corresponding focal sphero-conic is

$$(a, b, c, f, g, h\between\lambda, \mu, \nu)^2=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (43)$$

Fig. 1.



Demonstration. Let F be the focal sphero-conic, a the centre of α, one of the circles of reference; then if P, P′ be limiting points of the system, composed of J and any tangent TT′ to F; P, P′ are points on the sphero-quartic. Let K be the pole of the great circle TT′. Now if at be the tangent from a to J, by art. 36, the result of substituting the coordinates of P in the small circle $\alpha = \cos at - \cos a\mathrm{P}$, this may be written

$$\alpha = \cos at - \cos a\mathrm{P};$$

but

$$\cos at = \frac{\cos a\mathrm{O}}{\cos \mathrm{O}t} = \frac{\sin\lambda \sin \mathrm{OT} + \cos\lambda \cos \mathrm{OT} \cos \mathrm{OK}a}{\cos \mathrm{O}t},$$

and

$$\cos a\mathrm{P} = \sin\lambda \sin \mathrm{PT} + \cos\lambda \cos \mathrm{PT} \cos \mathrm{OK}a$$

$$= \sin\lambda \sin \mathrm{PT} + \frac{\cos\lambda \cos \mathrm{OT} \cos \mathrm{OK}a}{\cos \mathrm{O}t}.$$

Hence, by substitution, we get

$$\alpha = \sin\lambda \left\{ \frac{\sin \mathrm{OT}}{\cos \mathrm{O}t} - \sin \mathrm{PT} \right\};$$

and putting for $\cos \mathrm{O}t$ its value $\cos \mathrm{OT} \div \cos \mathrm{PT}$ (see art. 24), we get

$$\alpha = \frac{\sin\lambda \sin \mathrm{OP}}{\cos \mathrm{OT}}.$$

Hence the results of substituting the coordinates of any point P of the sphero-quartic in the equation of the small circles α, β, γ are proportional to the sines of the arcs from the centres of α, β, γ to a great circle tangential to the sphero-conic F, and hence the proposition is proved.

41. If the cyclide W be expressed in terms of four spheres α, β, γ, δ which are mutually orthogonal, then the sphero-quartic WU will be expressed in terms of four circles which

are mutually orthogonal, and its equation will be of the form $a\alpha^2+b\beta^2+c\gamma^2+d\delta^2=0$; but $\alpha^2+\beta^2+\gamma^2+\delta^2=0$ is an identical relation. Hence, eliminating each of the variables α^2, β^2, γ^2, δ^2 in succession, we see that the same sphero-quartic may be expressed by either of the four equations:

$$\left.\begin{aligned}(a-b)\beta^2+(a-c)\gamma^2+(a-d)\delta^2=0,\\(b-a)\alpha^2+(b-c)\gamma^2+(b-d)\delta^2=0,\\(c-a)\alpha^2+(c-b)\beta^2+(c-d)\delta^2=0,\\(d-a)\alpha^2+(d-b)\beta^2+(d-c)\gamma^2=0,\end{aligned}\right\} \quad \ldots \ldots \quad (44)$$

and by the last article we see that the sphero-quartic has four focal sphero-conics, whose tangential equations are:

$$\left.\begin{aligned}(a-b)\mu^2+(a-c)\nu^2+(a-d)\varrho^2=0,\\(b-a)\lambda^2+(b-c)\nu^2+(b-d)\varrho^2=0,\\(c-a)\lambda^2+(c-b)\mu^2+(c-d)\varrho^2=0,\\(d-a)\lambda^2+(d-b)\mu^2+(d-c)\nu^2=0.\end{aligned}\right\} \quad \ldots \ldots \quad (45)$$

Cor. *Sphero-quartics may be generated in four different ways as the envelope of a variable circle* which cuts a given circle orthogonally, and whose *centre moves along a given sphero-conic.*

42. If $W=(a, b, c, d, l, m, n, p, q, r)(\alpha, \beta, \gamma, \delta)^2=0$ be the general equation of a cyclide, and U the sphere orthogonal to $\alpha, \beta, \gamma, \delta$, then it is easy to see that the results of substituting the coordinates of any point P of the sphero-quartic (WU) in the equations of $\alpha, \beta, \gamma, \delta$ are proportional to the perpendiculars from the centres of $\alpha, \beta, \gamma, \delta$ on the tangent plane to U at the point P; but if these be perpendiculars to $\lambda, \mu, \nu, \varrho$, we see that the surface whose tangential equation is

$$(a, b, c, d, l, m, n, p, q, r)(\lambda, \mu, \nu, \varrho)^2$$

is inscribed in the developable formed by the tangent planes to U along the sphero-quartic WU, but this tangential equation is that of the focal quadric of W. Hence we have the following theorem:—*The developable circumscribed about the focal quadric of a cyclide and the corresponding sphere of inversion* U *touches the sphere along the sphero-quartic* (WU), *and the cones whose vertices are at the centre of* U, *and which stand on the nodal conics of the developable, intersect* U *in the focal sphero-conics of the sphero-quartic* WU. The latter part of the theorem is evident by writing the equation of the cyclide in terms of four spheres mutually orthogonal, and from the equations (45) of the last article*.

* [We have given in art. 33 the equations in tangential coordinates of the five focal quadrics of a cyclide; the following investigation gives, being given the equations of a focal quadric and the corresponding sphere of inversion in Cartesian coordinates, the equations in Cartesian coordinates of the four remaining focal quadrics.

I. Let $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1=0$ be the focal quadric F of a cyclide W, and $(x-f)^2+(y-g)^2+(z-h)^2-r^2=0$ be the corresponding sphere of inversion; then if from the centre O of the sphere we let fall a perpendicular OT

CHAPTER IV.

Sphero-quartics (continued).

43. In discussing the properties of sphero-quartics, we have hitherto considered a sphero-quartic as the intersection of a sphere and a cyclide. There is another mode of considering sphero-quartics, which offers many advantages for the investigation of these curves, namely, to consider a sphero-quartic as the curve of intersection of a sphere and a quartic cone tangential to the cyclide, the vertex of the cone being at the centre of the sphere, which we shall take as one of the spheres of inversion of the cyclide. This method of studying the sphero-quartic will give us an opportunity of showing the connexion which exists between the invariants and covariants of plane conics and of circles

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on any tangent plane to F, and take two points P, P′ in opposite directions from T on OT so that

$$OT^2-TP^2=OT^2-TP'^2=r^2,$$

the locus of the points P, P′ is the cyclide W; but denoting OT by p, and OP by ς, this gives us $2p\varsigma=r^2+\varsigma^2$, or

$$2\sqrt{a^2\cos^2\alpha+b^2\cos^2\beta+c^2\cos^2\gamma}-2(f\cos\alpha+g\cos\beta+h\cos\gamma)=r^2+\varsigma^2,$$

$\cos\alpha$, $\cos\beta$, $\cos\gamma$ being the direction cosines of OT. Hence, if the centre of the sphere be now taken as origin, we have the equation of the cyclide

$$4(a^2x^2+b^2y^2+c^2z^2)=(x^2+y^2+z^2+2fx+2gy+2hz+r^2)^2.$$

II. The equation of the cyclide given in I. is the envelope of the quadric $S+\mu C+\mu^2=0$, where S represents the cone $a^2x^2+b^2y^2+c^2z^2$, and C the sphere $x^2+y^2+z^2+2fx+2gy+2hz+r^2$; and the condition that this should represent a cone is the discriminant

$$(a^2+\mu)(b^2+\mu)(c^2+\mu)(\mu r^2+\mu^2)-\mu^2f^2(b^2+\mu)(c^2+\mu)-\mu^2g^2(c^2+\mu)(a^2+\mu)-\mu^2h^2(a^2+\mu)(b^2+\mu)=0,$$

or, as it may be written,

$$\frac{\mu^2f^2}{a^2+\mu}+\frac{\mu^2g^2}{b^2+\mu}+\frac{\mu^2h^2}{c^2+\mu}=\mu r^2+\mu^2.$$

If the five values of μ in this equation be denoted by μ_1, μ_2, μ_3, μ_4, μ_5, we have the equations of the five cones which have double contact with the cyclide (see art. 187), $S+\mu_1C+\mu_1^2$, $S+\mu_2C+\mu_2^2$, &c.: and the vertices of these five cones are, by the same article, the five centres of inversion of the cyclide. Since one value is obviously $=0$ in the foregoing equation, we see that the cone whose vertex is the centre of the sphere of inversion

$$(x-f)^2+(y-g)^2+(z-h)^2-r^2=0$$

will be, when that centre is taken as origin,

$$a^2x^2+b^2y^2+c^2z^2=0.$$

Hence, if the other centres be taken respectively as origin, the equations of the other cones will be

$$(a^2+\mu_2)x^2+(b^2+\mu_2)y^2+(c^2+\mu_2)z^2=0 \quad \ldots \ldots \quad (\alpha')$$

$$(a^2+\mu_3)x^2+(b^2+\mu_3)y^2+(c^2+\mu_3)z^2=0 \quad \ldots \ldots \quad (\beta')$$

$$(a^2+\mu_4)x^2+(b^2+\mu_4)y^2+(c^2+\mu_4)z^2=0 \quad \ldots \ldots \quad (\gamma')$$

$$(a^2+\mu_5)x^2+(b^2+\mu_5)y^2+(c^2+\mu_5)z^2=0 \quad \ldots \ldots \quad (\delta')$$

Now, since the cone $a^2x^2+b^2y^2+c^2z^2=0$ is the reciprocal of the asymptotic cone of the focal quadric $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1$, we infer that the cones (α'), (β'), (γ'), (δ') are the reciprocals of the asymptotic cones of the

on the sphere, and to show that sphero-quartics may be generated in the same way as I have given in the Fifth Chapter of my 'Bicirculars' for generating plane quartics having two finite double points.

44. The equation of a right cone whose semivertical angle is ϱ is

$$L^2-\cos^2\varrho(x^2+y^2+z^2)=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (46)$$

where L is a plane through the vertex of the cone: now this cone intersects a sphere of radius unity whose centre is at its vertex in two small circles; and I say that the two factors of the equation (46), namely $L\pm\cos\varrho(x^2+y^2+z^2)^{\frac{1}{2}}$, may be taken to represent these two circles; for the equation (38), which represents a small circle on the sphere, will become by transformation to three rectangular planes $L=\cos\varrho(x^2+y^2+z^2)^{\frac{1}{2}}$, and its twin circle will be the other factor, $L+\cos\varrho(x^2+y^2+z^2)^{\frac{1}{2}}$.

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other four focal quadrics; and hence we have the following system of Cartesian equations of these focal quadrics:—

$$\frac{x^2}{a^2+\mu_2}+\frac{y^2}{b^2+\mu_2}+\frac{z^2}{c^2+\mu_2}=1,$$

$$\frac{x^2}{a^2+\mu_3}+\frac{y^2}{b^2+\mu_3}+\frac{z^2}{c^2+\mu_3}=1,$$

$$\frac{x^2}{a^2+\mu_4}+\frac{y^2}{b^2+\mu_4}+\frac{z^2}{c^2+\mu_4}=1,$$

$$\frac{x^2}{a^2+\mu_5}+\frac{y^2}{b^2+\mu_5}+\frac{z^2}{c^2+\mu_5}=1,$$

so that the five focal quadrics are confocal, as we know otherwise.

III. Since the equation

$$\frac{\mu^2f^2}{a^2+\mu}+\frac{\mu^2g^2}{b^2+\mu}+\frac{\mu^2h^2}{c^2+\mu}=\mu^2+\mu r^2$$

may be written in the form

$$\frac{f^2}{a^2+\mu}+\frac{g^2}{b^2+\mu}+\frac{h^2}{c^2+\mu}=1+\frac{r^2}{\mu},$$

and this is the discriminant of $\mu F+J$, where

$$F\equiv\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1=0, \quad J\equiv(x-f)^2+(y-g)^2+(z-h)^2-r^2=0$$

(see SALMON's 'Geometry of Three Dimensions,' p. 146), we infer that the same values which will make $\mu F+J$ a cone will also make $S+\mu C+\mu^2$ (see II.) a cone. The two cones will have a common vertex, their equations referred to that vertex as origin being

$$\frac{x^2(a^2+\mu)}{a^2}+\frac{y^2(b^2+\mu)}{b^2}+\frac{z^2(c^2+\mu)}{c^2}=0,$$

$$x^2(a^2+\mu)+y^2(b^2+\mu)+z^2(c^2+\mu)=0.$$

Hence we have the following remarkable theorem:—If F and J be a corresponding focal quadric and sphere of inversion of a cyclide, and if $\mu_1, \mu_2, \mu_3, \mu_4$ be the four roots of the biquadratic which is the discriminant of $\mu F+J$, then if F be given in its canonical form,

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1=0,$$

the equations of the four other focal quadrics are got from this by changing a^2, b^2, c^2 respectively into $(a^2+\mu_1)$, $(b^2+\mu_1)$, $(c^2+\mu_1)$, &c.

Now if we put S for the point sphere $x^2+y^2+z^2$, the equations of the two circles may be written

$$S^{\frac{1}{2}}\pm L\sec\rho=0. \quad (47)$$

It is clear that these equations (47) may also be interpreted as denoting separately the two sheets of the cone (46)—that is $S^{\frac{1}{2}}+L\sec\rho$ represents one sheet of it, and $S^{\frac{1}{2}}-L\sec\rho$ the other. Hence we infer from this article and from article 38 *that the equation*

$$(a, b, c, d, l, m, n, p, q, r\between\alpha, \beta, \gamma, \delta)^2=0$$

will represent a cyclide, a sphero-quartic, or a tangent cone to the cyclide, whose vertex is

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IV. When $S+\mu C+\mu^2$ (see II.) represents a cone, the coordinates of the vertex are, by the usual process,

$$\frac{-\mu f}{a^2+\mu}, \quad \frac{-\mu g}{b^2+\mu}, \quad \frac{-\mu h}{c^2+\mu},$$

if referred to the centre of J as origin, or

$$\frac{a^2 f}{a^2+\mu}, \quad \frac{b^2 g}{b^2+\mu}, \quad \frac{c^2 h}{c^2+\mu},$$

if referred to the centre of F as origin. Hence we have the following theorem:—If $F\equiv\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1=0$, and $J\equiv(x-f)^2+(y-g)^2+(z-h)^2-r^2=0$ be a corresponding focal quadric and sphere of inversion of a cyclide, and if $\mu_1, \mu_2, \mu_3, \mu_4$ be the four roots of the biquadratic in μ, which is the discriminant of $\mu F+J$, then the coordinates of the centres of the other four spheres of inversion are:

$$\frac{a^2 f}{a^2+\mu_1}, \quad \frac{b^2 g}{b^2+\mu_1}, \quad \frac{c^2 h}{c^2+\mu_1},$$
$$\frac{a^2 f}{a^2+\mu_2}, \quad \frac{b^2 g}{b^2+\mu_2}, \quad \frac{c^2 h}{c^2+\mu_2},$$
$$\frac{a^2 f}{a^2+\mu_3}, \quad \frac{b^2 g}{b^2+\mu_3}, \quad \frac{c^2 h}{c^2+\mu_3},$$
$$\frac{a^2 f}{a^2+\mu_4}, \quad \frac{b^2 g}{b^2+\mu_4}, \quad \frac{c^2 h}{c^2+\mu_4}.$$

Cor. These values satisfy the system of determinants

$$\left|\begin{matrix} \frac{x}{a^2}, & \frac{y}{b^2}, & \frac{z}{c^2}, \\ (x-f), & (y-g), & (z-h), \end{matrix}\right|=0,$$

or

$$\left|\begin{matrix} \partial_x & \partial_y & \partial_z \\ , & , & , \end{matrix}\right| \begin{matrix} F, \\ J, \end{matrix} = 0.$$

Hence we have the following theorem:—If F and J be a corresponding focal quadric and sphere of inversion of a cyclide, then the five centres of inversion of the cyclide lie on the Jacobian curve of J and F (see CAYLEY, "Memoir on Quartic Surfaces," Proceedings of the London Mathematical Society).

V. Being given

$$F\equiv\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1=0, \quad J\equiv(x-f)^2+(y-g)^2+(z-h)^2-r^2=0,$$

the equation of the cyclide is

$$4(a^2x^2+b^2y^2+c^2z^2)-(x^2+y^2+z^2+2fx+2gy+2hz+r^2)^2=0. \quad (A)$$

at the centre of the sphere of inversion U *of the cyclide, according as we regard the variables* $\alpha, \beta, \gamma, \delta$ *as spheres, as circles on the sphere* U, *the Jacobian of the spheres* $\alpha, \beta, \gamma, \delta$, *or as single sheets of cones having their common vertex at the centre of* U.

45. From the double interpretation of the equation $S^{\frac{1}{2}}+L\sec\rho=0$ as denoting a small circle on the sphere, or as denoting a single sheet of the cone (46), all the results which we shall prove in the following articles are twofold in their application; for simplicity, however, I shall consider it as denoting a circle unless the contrary is expressed. If the equation of the plane L be $ax+by+cz=0$, it is clear that a, b, c may be regarded either as the direction cosines of L, or the coordinates of its pole on the sphere U, for

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Again, being given

$$F'=\frac{x^2}{a^2+\mu_1}+\frac{y^2}{b^2+\mu_1}+\frac{z^2}{c^2+\mu_1}-1=0,$$

$$J'=\left(x-\frac{a^2f}{a^2+\mu_1}\right)^2+\left(y-\frac{b^2g}{b^2+\mu_1}\right)^2+\left(z-\frac{c^2h}{c^2+\mu_1}\right)^2-r'^2=0,$$

the equation of the cyclide is

$$\left.\begin{aligned}&4\{(a^2+\mu_1)x^2+(b^2+\mu_1)y^2+(c^2+\mu_1)z^2\}\\&=x^2+y^2+z^2+\frac{2a^2f}{a^2+\mu_1}x+\frac{2b^2g}{b^2+\mu_1}y+\frac{2c^2h}{c^2+\mu_1}z+r'^2.\end{aligned}\right\} \quad \ldots \ldots \quad \text{(B)}$$

The origin in equation (A) is the centre of J, and the origin in equation (B) is the centre of J′, that is, the point whose coordinates with respect to the centre of J are

$$\frac{-\mu_1 f}{a^2+\mu_1},\ \frac{-\mu_1 g}{b^2+\mu_1},\ \frac{-\mu_1 h}{c^2+\mu_1}.$$

In order to compare the equations (A) and (B), which represent the same surface, we must transform (B) to the same origin as (A), or (A) to the same origin as (B): we will adopt the latter transformation, and we get the following result:—

$$\left.\begin{aligned}&4\left\{a^2\left(x+\frac{\mu_1 f}{a^2+\mu_1}\right)^2+b^2\left(y+\frac{\mu_1 g}{b^2+\mu_1}\right)^2+c^2\left(z+\frac{\mu_1 h}{c^2+\mu_1}\right)^2\right\}\\&=\left\{\left(x+\frac{\mu_1 f}{a^2+\mu_1}\right)^2+\left(y+\frac{\mu_1 g}{b^2+\mu_1}\right)^2+\left(z+\frac{\mu_1 h}{c^2+\mu_1}\right)^2\right.\\&\quad\left.+2f\left(x+\frac{\mu_1 f}{a^2+\mu_1}\right)+2g\left(y+\frac{\mu_1 g}{b^2+\mu_1}\right)+2h\left(z+\frac{\mu_1 h}{c^2+\mu_1}\right)+r^2\right\}^2.\end{aligned}\right\} \quad \ldots \quad \text{(C)}$$

Since the equations (B) and (C) represent the same cyclide and are referred to the same origin, by comparing the absolute terms, we shall get the value of r^2 in terms of r'^2, μ, and known constants. The absolute term in equation (C) is

$$\left\{\left(\frac{\mu_1 f}{a^2+\mu_1}+f\right)^2+\left(\frac{\mu_1 g}{b^2+\mu_1}+g\right)^2+\left(\frac{\mu_1 h}{c^2+\mu_1}+h\right)^2-(f^2+g^2+h^2)+r^2\right\}^2$$
$$-4\left\{\frac{a^2\mu_1^2f^2}{(a^2\mu_1)^2}+\frac{b^2\mu_1^2g^2}{(b^2+\mu_1)^2}+\frac{c^2\mu_1^2h^2}{(c^2+\mu_1)^2}\right\},$$

which, being reduced by means of the relation

$$\frac{f^2}{a^2+\mu_1}+\frac{g^2}{b^2+\mu_1}+\frac{h^2}{c^2+\mu_1}=1+\frac{r^2}{\mu_1},$$

becomes

$$\left\{\frac{\mu_1^2f^2}{(a^2+\mu_1)^2}+\frac{\mu_1^2g^2}{(b^2+\mu_1)^2}+\frac{\mu_1^2h^2}{(c^2+\mu_1)^2}-r^2\right\}^2;$$

these are both the same thing. Hence it follows, when we represent a circle on U by the equation $S^{\frac{1}{2}}-L=0$, where $L=ax+by+cz=0$, that a, b, c are equal to the direction cosines of L multiplied respectively by the secant of the spherical radius of the circle.

46. Now let us take, as in conics, the small circles

$$S^{\frac{1}{2}}-L=0, \quad S^{\frac{1}{2}}=M=0,$$

and form the invariants of this system; thus

$$S^{\frac{1}{2}}-L+k(S^{\frac{1}{2}}-M)=0$$

is a small circle coaxal with $S^{\frac{1}{2}}-L$ and $S^{\frac{1}{2}}-M$, $L-M=0$ being the great circle through

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and the absolute term in equation (B) is r'^4. Hence we get

$$r^2+r'^2=\frac{\mu_1^2f^2}{(a^2+\mu_1)^2}+\frac{(\mu_1^2g^2}{(b^2+\mu_1)^2}+\frac{\mu_1^2h^2}{(c^2+\mu_1)^2}.$$

That is the sum of the squares of the radii of J and J'= square of the distance between their centres, and hence J and J' cut orthogonally.

VI. The cyclide got from J and F is the envelope of the quadric $S+\mu C+\mu^2$, where

$$S\equiv a^2x^2+b^2y^2+c^2z^2, \quad C\equiv x^2+y^2+z^2+2fx+2gy+2hz+r^2.$$

The same cyclide, got from J' and F', is the envelope of the quadric $S'+\lambda C'+\lambda^2$, where

$$S'\equiv(a^2+\mu_1)x^2+(b^2+\mu_1)y^2+(c^2+\mu_1)z^2,$$

and

$$C'\equiv(x^2+y^2+z^2)+\frac{2a^2f}{a^2+\mu_1}x+\frac{2b^2g}{b^2+\mu_1}y+\frac{2c^2h}{c^2+\mu_1}z+r'^2.$$

Now, to show that $S+\mu C+\mu^2$ and $S'+\lambda C'+\lambda^2$ represent different quadrics, we are to observe that the first is referred to the centre of J as origin, and the second to the centre of J' as origin. Now let us transform the first to the same origin as the second; we must change x into $x-\frac{\mu_1 f}{a^2+\mu_1}$, y into $y-\frac{\mu_1 g}{b^2+\mu_1}$, z in to $z-\frac{\mu_1 h}{c^2+\mu_1}$; and, in order that they may be identical, we must have $\mu=\mu_1+\lambda$; this will make the coefficients of x^2, y^2, z^2 the same in both, but the coefficients of x, y, z will be different. Hence $S+\mu C+\mu^2$ and $S'+\lambda C'+\lambda^2$ cannot represent the same quadric. Hence we have the following theorem:—*A cyclide which has no node may be generated in five different ways as the envelope of a variable quadric.*

VII. If it be required to find how many double tangents can be drawn from a given point to a cyclide, let us substitute the coordinates of the given point in the quadric $S+\mu C+\mu^2$, and we shall have a quadratic in μ; hence two quadrics of each of the five systems of generating quadrics pass through the given point, and two rectilinear generators of each quadric pass through the given point; now each rectilinear generator of the generating quadric is a double tangent of the cyclide. Hence we have the following theorem:—*The tangent cone from an arbitrary point to a cyclide which has no node has twenty double edges.*

VIII. If $F\equiv(a, b, c, d, l, m, n, p, q, r)(x, y, z, 1)^2=0$,

$$J\equiv x^2+y^2+z^2-r^2=0,$$

the cyclide is given by the determinant

$$\begin{vmatrix} a, & n, & m, & p, & -2x, \\ n, & b, & l, & q, & -2y, \\ m, & l, & c, & r, & -2z, \\ p, & q, & r, & d, & (x^2+y^2+z^2+r^2), \\ -2x & -2y & -2z, & (x^2+y^2+z^2+r^2), & 0, \end{vmatrix}=0.\text{—January 1872.]}$$

their points of intersection. Forming the discriminant, we get

$$(1-S'')k^2+2(1-R)k+(1-S')=0, \quad \ldots \ldots \ldots \quad (48)$$

where S′, S″ denote the results of substituting the coefficients of x, y, z from the planes L and M in the point sphere S $(x^2+y^2+z^2=0)$, and R the result of substituting the coefficients from one of these planes in the equation of the other. Hence if ϱ', ϱ'' be the spherical radii of the circle $S^{\frac{1}{2}}-L$ and $S^{\frac{1}{2}}-M$, we have

$$\begin{aligned} 1-S' &= -\tan^2\varrho', \\ 1-S'' &= -\tan^2\varrho'', \\ 1-R &= -\tan\varrho'\tan\varrho''\cos C, \end{aligned}$$

where C is the angle of intersection of the circles. Hence the quadratic (48) becomes

$$\tan^2\varrho''\,k^2+2(\tan\varrho'\tan\varrho''\cos C)k+\tan^2\varrho'=0, \quad \ldots \ldots \quad (49)$$

and the discriminant is

$$\tan^2\varrho\tan^2\varrho''\sin^2 C\,; \quad \ldots \ldots \ldots \ldots \quad (50)$$

and this is what corresponds, in the geometry of two small circles on the sphere, to the invariant of two conics,

$$(1-S')(1-S'')-(1-R)^2.$$

See SALMON's 'Conics,' page 343, or 'Bicircular Quartics,' art. 127.

47. If D be the spherical distance between the poles of the planes L, M, we have

$$1-R=1-\frac{\cos D}{\cos\varrho'\cos\varrho''}.$$

Hence if $1-R=0$, $\cos D=\cos\varrho'\cos\varrho''$, or the triangle is right-angled which is formed by D, ϱ', ϱ'', that is the circles $S^{\frac{1}{2}}-L$, $S^{\frac{1}{2}}-M$ cut orthogonally (compare art. 9).

48. The two factors $1-R\pm\sqrt{(1-S')(1-S'')}$ of the invariant

$$(1-R)^2-(1-S')(1-S'')$$

are plainly

$$\left.\begin{aligned} &\tan\varrho'\tan\varrho''\sin\tfrac{1}{2}C, \\ &\tan\varrho'\tan\varrho''\cos\tfrac{1}{2}C, \end{aligned}\right\} \quad \ldots \ldots \ldots \ldots \quad (51)$$

where C is the angle of intersection of the circles $S^{\frac{1}{2}}-L$, $S^{\frac{1}{2}}-M$; and these are respectively the sine squared of half the direct common tangent, and the sine squared of half the transverse common tangent of the two circles. We have therefore, from the extension of PTOLEMY's theorem in my memoir "On the Equations of Circles," this further extension to conics inscribed in the same conic, namely, the condition that four conics, $S^{\frac{1}{2}}-L$, $S^{\frac{1}{2}}-M$, $S^{\frac{1}{2}}-N$, $S^{\frac{1}{2}}-P$, should be all touched by a fifth conic of the same form is

$$\sqrt{(12)(34)}\pm\sqrt{(13)(24)}\pm\sqrt{(14)(23)}=0, \quad \ldots \ldots \quad (52)$$

where (12) stands for the invariant

$$(1-R)-\sqrt{(1-S')(1-S'')} \text{ of two conics.}$$

49. If we eliminate k between the equation

$$S^{\frac{1}{2}}-L+k(S^{\frac{1}{2}}-M),$$

and the discriminant,

$$k^2\tan^2\varrho''+2k\tan\varrho'\tan\varrho''\cos C+\tan^2\varrho'=0,$$

we get

$$(S^{\frac{1}{2}}-L)^2\tan^2\varrho''-2(S^{\frac{1}{2}}-L)(S^{\frac{1}{2}}-M)\tan\varrho'\tan\varrho''\cos C+(S^{\frac{1}{2}}-M)\tan^2\varrho'=0. \quad . \quad . \quad (53)$$

This is the equation of the limiting points of the two circles $S^{\frac{1}{2}}-L=0$ and $S^{\frac{1}{2}}-M=0$; and they evidently correspond to the vertices or points of intersection of the two pairs of lines which can be drawn touching the conic S through the points of intersection of the conics $S^{\frac{1}{2}}-L$ and $S^{\frac{1}{2}}-M$, with their common chord $L-M$. Compare art. 8, equations (11) and (12).

Cor. The equations of the pair of points diametrically opposite is got by changing the signs of L and M in the circles $S^{\frac{1}{2}}-L$ and $S^{\frac{1}{2}}-M$.

50. We may get the equations of the limiting points otherwise. Thus, if $\cos\alpha'$, $\cos\beta'$, $\cos\gamma'$; $\cos\alpha''$, $\cos\beta''$, $\cos\gamma''$ be the direction cosines of the planes L and M, then, when we write the equations of the small circles in the form $S^{\frac{1}{2}}-L=0$, $S^{\frac{1}{2}}-M=0$, we must have

$$L\equiv\sec\varrho'(x\cos\alpha'+y\cos\beta'+z\cos\gamma'),$$
$$M\equiv\sec\varrho''(x\cos\alpha''+y\cos\beta''+z\cos\gamma'').$$

Let, then, the circle $S^{\frac{1}{2}}-L+k(S^{\frac{1}{2}}-M)=0$ be denoted by

$$S^{\frac{1}{2}}-\sec r(x\cos\lambda+y\cos\mu+z\cos\nu)=0;$$

and if this reduce to a point, we must have $\sec r=1$.

Hence, comparing coefficients, we get

$$(1+k)\cos\lambda=\sec\varrho'\cos\alpha'+k\sec\varrho''\cos\alpha'',$$
$$(1+k)\cos\mu=\sec\varrho'\cos\beta'+k\sec\varrho''\cos\beta'',$$
$$(1+k)\cos\nu=\sec\varrho'\cos\gamma'+k\sec\varrho''\cos\gamma'';$$

square and add, and we get, after a slight reduction,

$$k^2\tan^2\varrho''+2k\tan\varrho'\tan\varrho''\cos C+\tan^2\varrho'=0,$$

the same as before.

51. We can now, from the results proved in 'Bicircular Quartics,' write out at once corresponding ones for three small circles on the sphere. Thus, from the equations of the four conics J, J$'$, J$''$, J$'''$ orthogonal to three given conics, $S-L^2$, $S-M^2$, $S-N^2=0$, we can write out the equations of the circles cutting three circles orthogonally. Thus if the circles be $S^{\frac{1}{2}}-L$, $S^{\frac{1}{2}}-M$, $S^{\frac{1}{2}}-N$, their spherical radii ϱ', ϱ'', ϱ'''; direction angles of the planes L, M, N be α', β', γ'; α'', β'', γ''; α''', β''', γ''', the orthogonal circle J is the

determinant

$$\begin{vmatrix} S^{\frac{1}{2}}, & x, & y, & z, \\ \cos\varrho', & \cos\alpha', & \cos\beta', & \cos\gamma', \\ \cos\varrho'', & \cos\alpha'', & \cos\beta'', & \cos\gamma'', \\ \cos\varrho''', & \cos\alpha''', & \cos\beta''', & \cos\gamma''', \end{vmatrix} = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (54)$$

Compare art 11, equation (16).

52. The foregoing equation can be got directly as follows, my object in giving the above method being to show the identity of the methods of spherical geometry, and the method of conics given in the 'Bicirculars;' and in fact it was geometrically, that is, from consideration of the sphere, that I first discovered the method given in the 'Bicirculars.'

Let π be the radius and λ, μ, ν the direction-angles of the axis of the orthogonal circle; then, from the condition $1-R=0$ (see art. 47), we get three equations,

$$\cos\varrho' \cos\pi - \cos\lambda\cos\alpha' - \cos\mu\cos\beta' - \cos\nu\cos\gamma' = 0,$$
$$\cos\varrho'' \cos\pi - \cos\lambda\cos\alpha'' - \cos\mu\cos\beta'' - \cos\nu\cos\gamma'' = 0,$$
$$\cos\varrho''' \cos\pi - \cos\lambda\cos\alpha''' - \cos\mu\cos\beta''' - \cos\nu\cos\gamma''' = 0;$$

and the required circles give us a fourth equation,

$$S^{\frac{1}{2}}\cos\pi - \cos\lambda(x) - \cos\mu(y) - \cos\nu(z) = 0.$$

Hence, eliminating linearly, we get the same determinant as before.

Cor. The equations of the three other J's are got from the equation (54) by putting negative signs to the direction cosines of the axes of the circles.

53. The equation (54) expanded is

$$S^{\frac{1}{2}}\begin{vmatrix} \cos\alpha', & \cos\beta', & \cos\gamma', \\ \cos\alpha'', & \cos\beta'', & \cos\gamma'', \\ \cos\alpha''', & \cos\beta''', & \cos\gamma''', \end{vmatrix} = \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (55)$$

$$x\begin{vmatrix} \cos\beta', & \cos\gamma', & \cos\varrho', \\ \cos\beta'', & \cos\gamma'', & \cos\varrho'', \\ \cos\beta''', & \cos\gamma''', & \cos\varrho''', \end{vmatrix} + y\begin{vmatrix} \cos\gamma', & \cos\alpha', & \cos\varrho', \\ \cos\gamma'', & \cos\alpha'', & \cos\varrho'', \\ \cos\gamma''', & \cos\alpha''', & \cos\varrho''', \end{vmatrix} + z\begin{vmatrix} \cos\alpha', & \cos\beta', & \cos\varrho', \\ \cos\alpha'', & \cos\beta'', & \cos\varrho'', \\ \cos\alpha''', & \cos\beta''', & \cos\varrho'''. \end{vmatrix}$$

Let this be written $GS^{\frac{1}{2}}=Ix+Hy+Kz$, and comparing it with the equation

$$S^{\frac{1}{2}}=\sec r(x\cos\lambda+y\cos\mu+z\cos\nu),$$

we get

$$\sec^2 r=\frac{I^2+H^2+K^2}{G^2}.$$

Hence the coordinates of the pole of the plane of the orthogonal circle, with respect to the sphere U, are

$$\frac{I}{G}, \frac{H}{G}, \frac{K}{G}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (56)$$

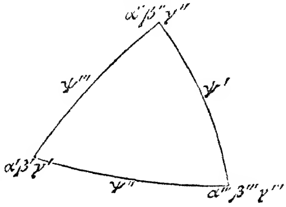
Now if this point be within the surface of the sphere U on which the circles $S^{\frac{1}{2}}-L$, $S^{\frac{1}{2}}-M$, $S^{\frac{1}{2}}-N$ are described, the orthogonal circle will be imaginary. But if the circles

$S^{\frac{1}{2}}-L$, $S^{\frac{1}{2}}-M$, $S^{\frac{1}{2}}-N$ be great circles, we have $\varrho'=\varrho''=\varrho'''=\frac{\pi}{2}$, and I, H, K each$=$zero, but G is finite. Hence the orthogonal circle becomes the imaginary circle at infinity. Hence we have the important theorem:—*That the imaginary circle at infinity is the circle which cuts three great circles of the sphere orthogonally, and is therefore a limiting case of the circle cutting any three circles orthogonally.*

Cor. In the geometry of a plane the two circular points at infinity are represented by the circle which cuts the three sides of any triangle orthogonally, and is therefore a limiting case of the circle which cuts any three circles on the plane orthogonally.

54. The following transformation of equation (55) will be useful in a subsequent article. Let the sides of the spherical triangle formed by joining the spherical centres of the three small circles $S^{\frac{1}{2}}-L$, $S^{\frac{1}{2}}-M$, $S^{\frac{1}{2}}-N$ be denoted by ψ', ψ'', ψ''' respectively, and the direction cosines of the planes of these sides, or, which is the same thing, of the lines from the centre of the sphere U to the angle points of the supplemental triangle be $\cos a'$, $\cos b'$, $\cos c'$; $\cos a''$, $\cos b''$, $\cos c''$; $\cos a'''$, $\cos b'''$, $\cos c'''$ respectively, then the equation (55) becomes transformed into the following:

Fig. 2.



$$S^{\frac{1}{2}}\begin{vmatrix} \cos\alpha', & \cos\beta', & \cos\gamma' \\ \cos\alpha'', & \cos\beta'', & \cos\gamma'' \\ \cos\alpha''', & \cos\beta''', & \cos\gamma''' \end{vmatrix} = \left\{\begin{array}{l} +\cos\varrho' \ \sin\psi' \ (x\cos a'+y\cos b'+z\cos c' \\ +\cos\varrho'' \ \sin\psi'' (x\cos a''+y\cos b''+z\cos c'' \\ +\cos\varrho''' \sin\psi'''(x\cos a'''+y\cos b'''+z\cos c'''. \end{array}\right\} \quad (57)$$

55. Let us seek the locus of all the point circles of the system

$$l(S^{\frac{1}{2}}-L)+m(S^{\frac{1}{2}}-M)+n(S^{\frac{1}{2}}-N).$$

The equation $S^{\frac{1}{2}}=x\cos\lambda+y\cos\mu+z\cos\nu$ denotes a point circle (see art. 50). Hence, comparing coefficients, we get

$$\cos\lambda=l\sec\varrho'\cos\alpha'+m\sec\varrho''\cos\alpha''+n\sec\varrho'''\cos\alpha'''$$
$$\cos\mu=l\sec\varrho'\cos\beta'+m\sec\varrho''\cos\beta''+n\sec\varrho'''\cos\beta'''$$
$$\cos\nu=l\sec\varrho'\cos\gamma'+m\sec\varrho''\cos\gamma''+n\sec\varrho'''\cos\gamma''' ;$$

square and add, and we get

$$\left.\begin{array}{l} 1=l^2\sec^2\varrho'+m\sec^2\varrho''+n^2\sec^2\varrho'''+2lm\sec\varrho'\sec\varrho''\sin\psi''' \\ \quad +2mn\sec\varrho''\sec\varrho'''\sin\psi'+2nl\sec\varrho'''\sec\varrho'\sin\psi''. \end{array}\right\} \quad . \quad . \quad . \quad . \quad (58)$$

Now, since all that we are concerned with is the mutual ratios of the multiples l, m, n, let us suppose $l+m+n=1$; square and subtract from equation (58), and then replace l, m, n by x, y, z, and we have the equation of J (see art. 51) in the form

$$\left.\begin{array}{l} (\tan^2\varrho', \quad \tan^2\varrho'', \quad \tan^2\varrho''', \ -\tan\varrho'\tan\varrho''\cos C, \\ -\tan\varrho''\tan\varrho'''\cos A, \ -\tan\varrho'''\tan\varrho'\cos B)(x, y, z)^2=0. \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad (59)$$

Compare 'Bicircular Quartics,' art. 139.

56. The condition that four small circles

$$S^{\frac{1}{2}}-L,\ S^{\frac{1}{2}}-M,\ S^{\frac{1}{2}}-N,\ S^{\frac{1}{2}}-P$$

should be coorthogonal is easily seen to be the determinant

$$\begin{vmatrix} \cos\varrho', & \cos\alpha', & \cos\beta', & \cos\gamma', \\ \cos\varrho'', & \cos\alpha'', & \cos\beta'', & \cos\gamma'', \\ \cos\varrho''', & \cos\alpha''', & \cos\beta''', & \cos\gamma''', \\ \cos\varrho'''', & \cos\alpha'''', & \cos\beta'''', & \cos\gamma'''', \end{vmatrix}=0. \quad . \quad . \quad . \quad . \quad . \quad (60)$$

Now, if we form the equations of J', J'', J''' (see art. 52, *Cor.*), and using transformations similar to those of art. 54, we see that the four J's (J, J', J'', J''') fulfil the condition of being coorthogonal. Hence we have the following theorem:—*The four circles are coorthogonal which are orthogonal to the four triads of circles*, $S^{\frac{1}{2}}\pm L$, $S^{\frac{1}{2}}\pm M$, $S^{\frac{1}{2}}\pm N$.

57. The plane of the great circle coaxal with J and the small circle $S^{\frac{1}{2}}-L=0$ is the polar plane of the point whose coordinates are $\frac{I}{G}, \frac{H}{G}, \frac{K}{G}$ (see art. 53) with respect to the cone $S-L^2=0$; and this is easily found to be the determinant

$$\begin{vmatrix} L, & x, & y, & z, \\ \cos\varrho', & \cos\alpha', & \cos\beta', & \cos\gamma', \\ \cos\varrho'', & \cos\alpha'', & \cos\beta'', & \cos\gamma'', \\ \cos\varrho''', & \cos\alpha''', & \cos\beta''', & \cos\gamma''', \end{vmatrix}=0. \quad . \quad . \quad . \quad . \quad . \quad (61)$$

Compare art. 13.

58. From the condition (art. 56) that four small circles on the sphere should be coorthogonal, it is easily inferred that the poles with respect to the sphere of the planes of these circles are complanar. Hence the planes of these small circles pass through a common point. Conversely, *any four small circles on the sphere are coorthogonal whose planes pass through a common point.*

Cor. The common point through which the planes of four coorthogonal circles pass is the pole with respect to the sphere of their common Jacobian.

59. The orthogonal circle J, as will appear evident from the form of its equation (see art. 55, equation 59), has double contact with each of the four sphero-conics:

$$\frac{\cos^2\frac{1}{2}A}{x\tan\varrho'}+\frac{\cos^2\frac{1}{2}B}{y\tan\varrho''}+\frac{\cos^2\ C}{z\tan\varrho'''}=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (62)$$

$$\frac{\cos^2\frac{1}{2}A}{x\tan\varrho'}-\frac{\sin^2\frac{1}{2}B}{y\tan\varrho''}-\frac{\sin^2\frac{1}{2}C}{z\tan\varrho'''}=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (63)$$

$$\frac{-\sin^2\frac{1}{2}A}{x\tan\varrho'}+\frac{\cos^2\frac{1}{2}B}{y\tan\varrho''}-\frac{\sin^2\frac{1}{2}C}{z\tan\varrho'''}=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (64)$$

$$\frac{-\sin^2\frac{1}{2}A}{x\tan\varrho'}-\frac{\sin^2\ B}{y\tan\varrho''}+\frac{\cos^2\frac{1}{2}C}{z\tan\varrho'''}=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (65)$$

the four chords of contact being the four great circles

$$x \tan \varrho' \pm y \tan \varrho'' \pm z \tan \varrho''' = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (66)$$

These great circles are the four axes of similitude of the three circles $S^{\frac{1}{2}}-L$, $S^{\frac{1}{2}}-M$, $S^{\frac{1}{2}}-N$, and the four sphero-conics are the four director or focal conics of the four pairs of circles (regarded as sphero-quartics) which are tangential to the three circles $S^{\frac{1}{2}}-L$, $S^{\frac{1}{2}}-M$, $S^{\frac{1}{2}}-N$. (See my memoir "On the Equations of Circles.")

60. Each of the four sphero-conics (62) to (65) inscribed in J touches four other sphero-conics inscribed in J; their equations are

$$J = \{x \tan \varrho' \cos (B \pm C) + y \tan^4 \varrho'' \cos (C \pm A) + z \tan \varrho''' \cos (A \pm B)^2\}. \quad . \quad (67)$$

By means of these sphero-conics can be proved Dr. HART'S extension of FEUERBACH'S theorem, and the equations of Dr. HART'S circles can be found. (See my memoir "On the Equations of Circles.")

61. We now return, after this long digression, to the sphero-quartic.

Let us consider the function of the second degree,

$$(a, b, c, f, g, h \between S^{\frac{1}{2}}-L, S^{\frac{1}{2}}-M, S^{\frac{1}{2}}-N)^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (68)$$

This will represent a sphero-quartic on the surface of the sphere U, or a quartic cone having its vertex at the centre of U, according as we interpret $S^{\frac{1}{2}}-L$ &c. as circles, or as single sheets of cones (see art. 44). Now the equation (68) is the envelope of $\lambda(S^{\frac{1}{2}}-L)+\mu(S^{\frac{1}{2}}-M)+\nu(S^{\frac{1}{2}}-N)$. If the condition be fulfilled,

$$(A, B, C, F, G, H \between \lambda\ \mu\ \nu)^2 = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (69)$$

where A, B, &c. denote as usual $bc-f^2$, $ca-g^2$, &c.; but the circle

$$\lambda(S^{\frac{1}{2}}-L)+\mu(S^{\frac{1}{2}}-M)+\nu(S^{\frac{1}{2}}-N)=0$$

has the plane $\lambda L+\mu M+\nu N=0$ perpendicular to its axis, that is, perpendicular to the radius of U passing through its centre, and in virtue of the condition (64) the envelope of the plane is the cone

$$(a, b, c, f, g, h \between L, M, N)^2 = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (70)$$

and therefore the locus of the centres of the generating circles of the sphero-quartic (68) is the sphero-conic, in which the cone reciprocal to (70) intersects the sphere U.

62. If we regard the equation (68) as representing a quartic cone, we see that the locus of the axis of its generating right cone is a cone of the second degree, namely, the reciprocal of the cone (70), and its generating right cone cuts orthogonally another right cone, namely, the right cone which is orthogonal to the three cones $S^{\frac{1}{2}}-L$, $S^{\frac{1}{2}}-M$, $S^{\frac{1}{2}}-N$; the equation of the orthogonal cone is given, art. 51, equation (54).

Cor. If we suppose any plane to cut the quartic cone of this article, it will intersect it in a quartic curve having two double points; it will also intersect the generating right cone in a conic, which will be the generating conic of the quartic curve; and finally it will intersect the directing cone of the quartic cone in a conic, which will be the directing conic of the quartic curve. It was in this manner I was first led to the discovery of the

method of generating quartics having two finite double points, and the transition from that was easy to the discovery of the method of generating surfaces of the fourth degree having a conic for a double line (see Chapter II., Section II. of this paper).

63. Since the plane of each of the circles $S^{\frac{1}{2}}-L$, $S^{\frac{1}{2}}-M$, $S^{\frac{1}{2}}-N$ passes through the pole of the plane of the orthogonal circle J with respect to U, that is through the point $\left(\frac{I}{G}, \frac{H}{G}, \frac{K}{G}\right)$ (see art. 53), it follows that the plane of the generating circle

$$\lambda(S^{\frac{1}{2}}-L)+\mu(S^{\frac{1}{2}}-M)+\nu(S^{\frac{1}{2}}-N)$$

passes through the same point, therefore the pole of the plane of the generating circle with respect to U lies in the plane $Ix+Hy+Kz=G$; in other words, the pole of the generating circle is complanar with the poles of the planes of the circles of reference, and therefore it describes a conic in space, namely, the conic in which the plane $Ix+Hy+Kz=G$ intersects the cone reciprocal to

$$(a, b, c, f, g, h\between L, M, N)^2=0.$$

64. Since the planes of the circles $S^{\frac{1}{2}}-L$, $S^{\frac{1}{2}}-M$, $S^{\frac{1}{2}}-N$ are respectively parallel to the planes L, M, N, the envelope of the plane of the generating circle is a cone similar and similarly placed with the cone $(a, b, c, f, g, h\between L, M, N)^2$, and its vertex is at the point $\frac{I}{G}, \frac{H}{G}, \frac{K}{G}$. Hence we are led to the known proposition, *that a sphero-quartic is the intersection of a sphere and a cone of the second degree, and therefore that it is the intersection of a sphere and a quadric.*

65. If the poles of the planes of the generating circles $S^{\frac{1}{2}}-L$, &c. with respect to U be the centres of three spheres α, β, γ which cut the sphere U orthogonally, then α, β, γ will intersect U in the circles $S^{\frac{1}{2}}-L$, $S^{\frac{1}{2}}-M$, $S^{\frac{1}{2}}-N$ respectively, and the cyclide

$$(a, b, c, f, g, h\between \alpha, \beta, \gamma)^2$$

will intersect U in the sphero-quartic

$$(a, b, c, f, g, h\between S^{\frac{1}{2}}-L, S^{\frac{1}{2}}-M, S^{\frac{1}{2}}-N)^2.$$

Hence we are led to the known theorem, *that a sphero-quartic is the intersection of a sphere and a cyclide.*

66. The results we have arrived at in this Chapter may be projectively extended to curves described on quadrics, in other words, the analytical proof is the same for this more general case as for the particular one we have examined. Thus, instead of a sphere U of radius unity, let us take a quadric $S-K^2$ inscribed in S as the surface on which the curve is described. Now the quadric $S-L^2$ intersects $S-K^2$ in two plane conics, and we may take the equations of these two conics to $S^{\frac{1}{2}}-L=0$, and $S^{\frac{1}{2}}+L=0$, precisely similar to the method we have given of representing small circles on the surface of a sphere. It is plain that the equations $S^{\frac{1}{2}}-L$, $S^{\frac{1}{2}}+L$ have another interpretation, namely, they represent respectively the two parts into which the surface $S-L^2$ is divided by the plane $L=0$; so that on the surface of the quadric $S-K^2=0$ a plane conic is represented by an equation of the same form as that of a circle on the surface of a sphere. Again, if

we have two circles on the surface of a sphere, the condition that the pole with respect to the sphere of the plane of one may lie on the plane of the other is expressed by the invariant $1-R=0$. Hence it is evident that the same invariant relation will express for two conics on a quadric that the pole of the plane of each with respect to the quadric lies on the plane of the other; and as circles so related cut orthogonally, we shall extend the term as in art. 9, and say that two conics so related cut orthogonally. In fact the relation $1-R=0$ may be called (see art. 9) the *harmonic invariant* of the quadrics or conics whose equations are connected by it.

67. It is evident from the last article that, being given four plane conics on a quadric, the condition that the four conics should be tangential to a fifth is the determinant

$$\begin{vmatrix} 0\,, & (12), & (13), & (14), \\ (12), & 0\,, & (23), & (24), \\ (13), & (23), & 0\,, & (34), \\ (14), & (24), & (34), & 0, \end{vmatrix} = 0; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (71)$$

and in fact Dr. SALMON's direct proof of this theorem in case of conics on a plane will apply *verbatim* to the more general case here considered (see SALMON's 'Conics,' 5th edition, page 366).

Cor. From the equation (71) we can find, as Dr. SALMON has done for conics on a plane, the equations of the pairs of conics which touch three conics on a quadric; the equation is

$$\sqrt{(23)(S^{\frac{1}{2}}-L)} \pm \sqrt{(31)(S^{\frac{1}{2}}-N)} \pm \sqrt{(12)(S^{\frac{1}{2}}-N)} = 0; \quad . \quad . \quad . \quad . \qquad (72)$$

or this equation may be inferred also from art. 59.

68. If we are given any three conics, $S^{\frac{1}{2}}-L$, $S^{\frac{1}{2}}-M$, $S^{\frac{1}{2}}-N$, on the surface of a quadric, we get, precisely the same as in art. 51, the equation of the conic J which cuts them orthogonally. And so in general, being given any homogeneous function of the second degree $(a, b, c, f, g, h \between S^{\frac{1}{2}}-L, S^{\frac{1}{2}}-M, S^{\frac{1}{2}}-N)$, we see that it represents a twisted quartic of the first family, and that all the properties of sphéro-quartics may be applied projectively to it (see observations, art. 26).

CHAPTER V.—INVERSION AND CENTRES OF INVERSION.

SECTION I.—*Cyclides.*

69. If α, β, P be any three spheres, δ_1, δ_2, δ_3 their diameters, DE, FG common tangents to α, P_1; β, P respectively; then if the system be inverted from any arbitrary point, and denoting the inverse system by the same letters accented, we have (see SALMON's 'Conics,' fifth edition, page 114),

$$\frac{DE^2}{\delta_1} : \frac{FG^2}{\delta_2} = \frac{D'E'^2}{\delta_1'} : \frac{F'G'^2}{\delta_2'}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (73)$$

Now this result holds whatever be the magnitude of P; it will be true in the limit when P *reduces to a point, in which case* DE^2, FG^2 become the result of substituting the

coordinates of P in the equations of the spheres α, β respectively. Hence we have the following theorem:—

The results of substituting the coordinates of any point P *in the equations of two spheres, divided respectively by the diameters of those spheres, have a ratio which is unaltered by inversion.*

70. The general equation of any cyclide,

$$W=(a, b, c, d, l, m, n, p, q, r,\between \alpha, \beta, \gamma, \delta,)^2=0,$$

may be written in the form

$$W=(a\delta_1^2, b\delta_2^2, c\delta_3^2, d\delta_4^2, l\delta_2\delta_3, m\delta_3\delta_1, n\delta_1\delta_2, p\delta_1\delta_4, q\delta_2\delta_4, r\delta_3\delta_4 \between \frac{\alpha}{\delta_1}, \frac{\beta}{\delta_2}, \frac{\gamma}{\delta_3}, \frac{\delta}{\delta_4})^2=0;$$

and by the last article the six ratios $\frac{\alpha}{\delta_1}:\frac{\beta}{\delta_2}:\frac{\gamma}{\delta_3}:\frac{\delta}{\delta_4}$ remain unaltered by inversion. Hence, denoting the inverses by the same letters accented, the cyclide W will be inverted into a cyclide W′ given by the equation

$$\left.\begin{array}{l} W=\left(a\left(\frac{\delta_1'}{\delta_1}\right)^2, b\left(\frac{\delta_2'}{\delta_2}\right)^2, c\left(\frac{\delta_3'}{\delta_3}\right)^2, d\left(\frac{\delta_4'}{\delta_4}\right)^2, l\left(\frac{\delta_2'\delta_3'}{\delta_2\delta_3}\right), m\left(\frac{\delta_3'\delta_1'}{\delta_3\delta_1}\right), n\left(\frac{\delta_1'\delta_2'}{\delta_1\delta_2}\right),\right. \\ \qquad p\left(\frac{\delta_1'\delta_4'}{\delta_1\delta_4}\right), q\left(\frac{\delta_2'\delta_4'}{\delta_2\delta_4}\right), r\left(\frac{\delta_3'\delta_4'}{\delta_3\delta_4}\right)\between \alpha', \beta', \gamma', \delta',)^2=0. \end{array}\right\} \quad . \quad . \quad . \quad . \quad (74)$$

71. We have shown that the equation of any cyclide may be written in the form

$$a\alpha^2+b\beta^2+c\gamma^2+d\delta^2+e\epsilon^2=0 \text{ (see art. 32)},$$

where $\alpha^2+\beta^2+\gamma^2+\delta^2+\epsilon^2=0$ is an identical relation. I shall call this form of the equation of a cyclide the canonical form; and we see by the last article that the equation of the inverse of a cyclide given by its canonical form is also in its canonical form.

Cor. If the cyclide be of the form $a\alpha^2+b\beta^2+c\gamma^2=0$, the inverse cyclide will be of the same form, that is, the inverse of a binodal cyclide is a binodal cyclide.

72. The five spheres α, β, γ, δ, ϵ of the canonical form are mutually orthogonal; and if we take the centre of α for a centre of inversion, and α for the sphere of inversion, each of the five spheres will be inverted into itself. Hence the centre of α is a centre of self-inversion of the cyclide. Similarly the centres of the spheres β, γ, δ, ϵ are centres of self-inversion. Hence we have the following theorem:—*A cyclide is an anallagmatic surface, and the centres of the five spheres of the canonical form are its five centres of inversion.*

73. We can confirm some of the foregoing results by Cartesian methods. Thus let the spheres of reference α, β, γ, δ be expressed in rectangular coordinates, and putting s for the sum of the coefficients a, b, c, &c., we get

$$W=s(x^2+y^2+z^2)^2+U_1(x^2+y^2+z^2)+U_2=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad (75)$$

where U_1 and U_2 are the general equations of the first and the second degree. Now transforming into polar coordinates by putting

$$x=\rho\cos\theta, \quad y=\rho\cos\phi, \quad z=\rho\cos\psi,$$

where

$$\cos^2\theta+\cos^2\phi+\cos^2\psi=1,$$

and then inverting by putting $\rho=\frac{k^2}{\rho'}$, and changing back again to Cartesian coordinates, we evidently get an equation of the same form, which proves that the inverse of a cyclide is a cyclide.

74. If the absolute term in the equation (75) vanishes, it is evident that the coefficient of $(x^2+y^2+z^2)^2$ in the inverse surface vanishes; in other words, the inverse surface will be a cubic cyclide, that is, a cubic surface passing through the imaginary circle at infinity. The section of this cyclide by any plane will be a circular cubic, and its focal quadric will be a paraboloid. Hence *the inverse of a cyclide from any point on the cyclide will be a cubic cyclide.* This corresponds to the theorem that the inverse of a bicircular quartic from any point on the quartic itself will be a circular cubic.

75. If $s=0$ in the general equation (75), that coefficient vanishes in the inverse surface, that is, the inverse surface will want the absolute term. Hence *if we invert a cubic cyclide from any point not on the cubic itself, the inverse surface will be a quartic cyclide passing through the origin.*

76. If in the general equation (75) not only s vanish, but also the coefficients of x, y, z in U_1 each separately vanish, that equation will represent a quadric, and the centre we invert from will be a node on the inverse surface. *Hence the inverse of a quadric will be a cyclide having the origin or centre of inversion as a node.*

The species of the node will depend on that of the quadric which is inverted.

1°. If the quadric inverted be central, let its equation be

$$\frac{(x-a)^2}{L}+\frac{(y-b)^2}{M}+\frac{(z-c)^2}{N}-1=0.$$

If this be inverted by the process of art. 73, we get

$$\left(\frac{a^2}{L}+\frac{b^2}{M}+\frac{c^2}{N}-1\right)(x^2+y^2+z^2)^2-2k^2\left(\frac{ax}{L}+\frac{by}{M}+\frac{cz}{N}\right)(x^2+y^2+z^2)+k^4\left(\frac{x^2}{L}+\frac{y^2}{M}+\frac{z^2}{N}\right)=0. \quad (76)$$

Hence *the node has the cone* $\frac{x^2}{L}+\frac{y^2}{M}+\frac{z^2}{N}$ *for a tangent cone, that is, the node is a conic node; the cone will evidently be real or imaginary according as the quadric inverted is an hyperboloid or ellipsoid.*

2°. Let the quadric inverted be non-central; let its equation be (see SALMON'S 'Geometry of Three Dimensions')

$$ax^2\pm by^2+2px+2qy+2rz+d=0.$$

The process of art. 73 gives for the inverse surface

$$k^4(ax^2\pm by^2)+2k^2(px+qy+rz)(x^2+y^2+z^2)+d(x^2+y^2+z^2)^2=0. \quad . \quad . \quad (77)$$

Hence the tangent cone at the node reduces to the pair of planes $ax^2\pm by^2=0$, *and the pair of planes will be imaginary or real according as the quadric inverted represents an elliptic or a hyperbolic paraboloid.*

This species of node is called by Professor CAYLEY a *binode* (see his "Cubic Surfaces" in the Philosophical Transactions for 1869, p. 231).

3°. If $b=0$ in the equation of a non-central surface the two planes become coincident, that is, the quadric cone becomes a coincident plane pair. Professor CAYLEY calls this species of node a *unode*. Hence *the inverse of a parabolic cylinder is a unodal cyclide.*

77. If we invert a binodal cyclide, $a\alpha^2+b\beta^2+c\gamma^2$, from one of the points common to the spheres of reference α, β, γ, the spheres α, β, γ will be inverted into three planes, and *therefore the inverse of a binodal cyclide from one of the nodes will be a cone of the second degree.* Conversely, *the inverse of a cone of the second degree will be a binodal cyclide.*

This conclusion may also be inferred otherwise; for as in 1°, art. 76, the inverse of the cone

$$\frac{(x-a)^2}{L}+\frac{(y-b)^2}{M}+\frac{(z-c)^2}{N}=0$$

is the cyclide

$$\left(\frac{a^2}{L}+\frac{b^2}{M}+\frac{c^2}{N}\right)(x^2+y^2+z^2)^2-2k^2\left(\frac{ax}{L}+\frac{by}{M}+\frac{cz}{N}\right)(x^2+y^2+z^2)+k^4\left(\frac{x^2}{L}+\frac{y^2}{M}+\frac{z^2}{N}\right)=0; \quad (78)$$

and it is evident from the form of this equation that the origin is a conic node. Again, if we transfer the origin to the point a, b, c, it will be seen that the new origin is a conic node. *Hence the inverse of a cone has two conic nodes.*

Cor. If we invert a quadric from a point on the quadric we get a cubic cyclide.

78. If one of the spheres of inversion touch one of the focal quadrics of the cyclide at one point, the focal quadric and the sphere of inversion can be expressed by the two tangential equations

$$\left.\begin{aligned} a\lambda^2+b\mu^2+c\nu^2+2n\nu\varrho=0,\\ a'\lambda^2+b'\mu^2+c'\nu^2+2n'\nu\varrho=0.\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (79)$$

See CAYLEY "On Developable Surfaces of the Second Order," Cambridge and Dublin Mathematical Journal, vol. v. p. 51.

Hence it follows that the equation of the cyclide and the square of its sphere of inversion U are given by the equations

$$\left.\begin{aligned} a\alpha^2+b\beta^2+c\gamma^2+2n\gamma\delta=0,\\ a'\alpha^2+b'\beta^2+c'\gamma^2+2n'\gamma\delta=0,\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (80)$$

where α, β are spheres of inversion of the cyclide, γ a point sphere, which is a centre of inversion of the cyclide in the sense that it is the centre of a circle which inverts the cyclide into a quadric. Hence in this case there are *four* centres of inversion, namely, the centre of U and the centres of α, β, and the point sphere γ.

The spheres of inversion are, 1°, *the sphere which inverts the quadric into a cyclide*; 2°, *the inverses of the principal planes of the quadric.*

The quadric must be either an ellipsoid or a hyperboloid; and the cyclide which is its

inverse, will have the point of contact of the two surfaces (79) for a node. It is plain this cyclide is the pedal of a quadric. FRESNEL'S surface of elasticity is an example.

The generating spheres will be, 1°, the inverses of the three systems of spheres whose centres are in the principal planes of the quadric, and which have double contact with it; 2°, the inverses of the tangent planes of the quadric.

There will be four focal quadrics, namely, the loci of the four systems of generating spheres.

79. If one of the spheres of inversion osculate the focal quadric, the tangential equation of the focal quadric and sphere of inversion can be expressed by the system

$$\left.\begin{array}{l} l(\lambda^2-2\mu\nu)+m(\mu^2-2\nu\rho)=0, \\ l'(\lambda^2-2\mu\nu)+m'(\mu^2-2\nu\rho)=0 \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (81)$$

(see CAYLEY, *suprà*).

Hence the cyclide and the square of the sphere U may be expressed by the system of equations

$$\left.\begin{array}{l} l(\alpha^2-2\beta\gamma)+m(\beta^2-2\gamma\delta)=0, \\ l'(\alpha^2-2\beta\gamma)+m'(\beta^2-2\gamma\delta)=0, \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (82)$$

where β is a sphere of inversion, γ a point sphere, which is a centre of inversion in the sense that it is the centre of a sphere which inverts the cyclide into a paraboloid. Hence in this case there are only three centres of inversion, namely, the centre of U, the centre of β, and the point sphere $\gamma=0$.

The three spheres of inversion are U, β, and the sphere whose centre is γ, which inverts the paraboloid into a cyclide. The spheres U and β are the inverses of the two planes of reflection of the paraboloid.

The generating spheres will be, 1°, the inverses of the two systems of spheres whose centres are in the two planes of reflection, and which have double contact with the paraboloid; 2°, the inverses of the tangent planes of the paraboloid.

There will be three focal quadrics, namely, the loci of the centres of the three systems of generating spheres.

80. The binodal cyclide $a\alpha^2+b\beta^2+c\gamma^2=0$ will have, besides the centres of the spheres α, β, γ, which are centres of inversion, an infinite number of centres of inversion lying on the line joining the two nodes.

This is the species of cyclide that is generated when the sphere of inversion U has double contact with the focal quadric F; each point of contact will plainly be a node of the cyclide; and since the surface is the inverse of a cone, there will be, as in the case of cyclides which are the inverses of hyperboloids, four systems of generating spheres*.

* [In writing this memoir I omitted trinodal and quadrinodal cyclides. My attention was directed to this omission by Professor CAYLEY. This omission was the less excusable, inasmuch as quadrinodal cyclides were the only surfaces to which the name cyclide was applied, until M. DARBOUX extended the term (see 'Comptes Rendus,' June 7, 1869). The existence of trinodal cyclides was first proved by Professor CAYLEY in the Quar-

SECTION II.—*Sphero-quartics.*

81. The theory of the centres of inversion of sphero-quartics is in a great measure identical with that of cyclides. Thus the analogue of the fundamental theorem (art. 69) is the following. If α, β, P be any three small circles on a sphere U, r_1, r_2, r_3 their radii, t, t_1 the common tangents of α, P; β, P respectively; then, if this system be inverted from any arbitrary origin on U, by the formula

$$\tan\tfrac{1}{2}\varrho\tan\tfrac{1}{2}\varrho'=\text{constant}, \quad \dots\dots\dots \quad (83)$$

ϱ, ϱ' being arcs drawn from the origin to a point and its inverse, we get, as in art. 69,

$$\frac{\sin^2\frac{1}{2}t}{\tan r_1}:\frac{\sin^2\frac{1}{2}t_1}{\tan r_2}=\frac{\sin^2\frac{1}{2}t'}{\tan r_1'}:\frac{\sin^2\frac{1}{2}t_1'}{\tan r_2'}. \quad \dots\dots\dots \quad (84)$$

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terly Journal, vol. x. p. 34, vol. xi. p. 15. The properties of quadrinodal cyclides were first studied by DUPIN. The principal ones will be contained in the following propositions, considered from my point of view.

I. If a quadric of revolution be inverted with respect to any point, the inverse surface will be a trinodal cyclide.

Demonstration. Let S be the quadric, L a plane of circular section, P the centre of inversion, then the sphere W, which passes through P and through the circle of intersection of S and L, will intersect S in another circle M, and will pass through a circle having OP as radius (O being the point in which a plane through P parallel to L and M is intersected by the axis of revolution of S). Now if we invert from P, the sphere W will invert into a plane V, the circles of intersection of L and M with S will invert into circles Q, R in the plane V, and the fixed circle having OP as radius will invert into the radical axis of Q and R, and the points in which Q and R meet this radical axis will be nodes of the cyclide into which S inverts; the point P will be a third node. Hence the proposition is proved.

Cor. 1. If the quadric of revolution be a cone, the inverse of its vertex will be a node of the cyclide. Hence *the inverse of a cone of revolution will be a quadrinodal cyclide.*

Cor. 2. Since a cone of revolution is the envelope of a variable sphere which touches three planes, we infer by inversion *that a quadrinodal cyclide is the envelope of a variable sphere which touches three fixed spheres.*

If in this mode of generation the points common to the three spheres be real, two of the four nodes of the cyclide will be real and two imaginary.

If the points common to the three spheres be imaginary, the four nodes of the cyclide will be imaginary. In this case it is evident the three spheres may be inverted into three spheres whose centres are collinear; now the envelope of a variable sphere which touches three spheres whose centres are collinear is evidently a ring formed by the revolution of a circle round an axis in its plane. Hence *the inverse of such a ring is a quadrinodal cyclide whose four nodes are imaginary*: in fact, a ring formed by the revolution of a circle round an axis in its plane is a quadrinodal cyclide which has two imaginary nodes at infinity.

II. The envelope of the sphere

$$(x-\alpha)^2+(y-\beta)^2+(z-a)^2=m^2\{(\alpha-c)^2+\beta^2\},$$

which cuts orthogonally the plane $z=a$, will, if α, β vary, be a cone of revolution, that is the envelope of

$$x^2+y^2+z^2-2(\alpha x+\beta y+az)+(1-m^2)(\alpha^2+\beta^2)+2m^2c\alpha+a^2-m^2c^2=0;$$

or, putting for a moment

$$(1-m^2)(\alpha^2+\beta^2)+2m^2c\alpha+a^2-m^2c^2=\Omega,$$

the envelope of

$$x^2+y^2+z^2-2(\alpha x+\beta y+az)+\Omega=0$$

is a cone of revolution. Now the sphere inverse to

$$x^2+y^2+z^2-2(\alpha x+\beta y+az)+\Omega$$

Now, reasoning as in art. 69, if P become a point, since $\sin^2 \frac{1}{2}t = \frac{a}{2\cos r_1}$ (see art. 36), we have the following theorem:—

The results of substituting the spherical coordinates of any point P *on the surface of a sphere* U *in the equations of two small circles on* U, *divided respectively by the sines of the radii of these circles, have a ratio which is unaltered by inversion.*

From this theorem it follows, as in art. 70, if $W = (a, b, c, d, l, m, n, p, q, r)(\alpha, \beta, \gamma, \delta)^2 = 0$, where α, β, γ, δ are small circles on U, be the equation of any sphero-quartic, that the

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will evidently cut orthogonally the sphere inverse to the plane $z=a$, that is, the sphere

$$x^2+y^2+z^2-\frac{2}{\Omega}(\alpha x+\beta y+az)+\frac{1}{\Omega}$$

will cut orthogonally the sphere $x^2+y^2+z^2=\frac{z}{a}$,

and its envelope will be the inverse of the before-mentioned cone of revolution.

Let the coordinates of the centre of

$$x^2+y^2+z^2-\frac{2}{\Omega}(\alpha x+\beta y+az)+\frac{1}{\Omega} \text{ be X, Y, Z,}$$

hence

$$\alpha=\Omega X, \quad \beta=\Omega Y, \quad a=\Omega Z;$$

$$\therefore \quad \alpha=\frac{aX}{Z}, \quad \beta=\frac{aY}{Z}.$$

Hence, restoring the value of Ω, we get

$$a=\{(1-m^2)(\alpha^2+\beta^2)+2m^2c\alpha+a^2-m^2c^2\}X;$$

and substituting the values of α and β we get

$$(1-m^2)a^2(X^2+Y^2)+(a^2-m^2c^2)Z^2+2m^2caXZ-aZ=0,$$

or, say, $\nabla=0$; and therefore *the quadrinodal cyclide, which is the inverse of the before-mentioned cone of revolution, is the envelope of a variable sphere whose centre moves on the quadric* $\nabla=0$, *and which cuts orthogonally the sphere* $x^2+y^2+z^2=\frac{z}{a}$.

III. The plane $Z=0$ touches $\nabla=0$ at an umbilic; and it is also a tangent plane to the sphere $x^2+y^2+z^2=\frac{z}{a}$; and the centre of the sphere is on one of the principal axes of ∇. Hence it touches ∇ at another umbilic; but if a sphere touch a quadric at two umbilics, it touches at two others. Hence we have the following theorem:— *A quadrinodal cyclide is the envelope of a variable sphere whose centre moves on a given quadric, and which cuts orthogonally a sphere which touches the quadric at four umbilics.*

Cor. We can get from the canonical form of the equation of the cyclide in terms of its five spheres of inversion the condition for four nodes. Thus, let the cyclide be

$$a\alpha^2+b\beta^2+c\gamma^2+d\delta^2+e\epsilon^2=0,$$

we have

$$\alpha^2+\beta^2+\gamma^2+\delta^2+\epsilon^2=0 \text{ identically.}$$

Now if two of the coefficients a, b, c, d, e be equal to one another, for instance d and e, we get for the focal quadrics three quadrics and one conic, which must be a focal conic of the three focal quadrics. The cyclide will therefore in this case have two nodes.

If two distinct pairs of the five quantities a, b, c, d, e be equal, such as $b=c$, $d=e$, then we get for the focal quadrics one quadric and two conics, and the cyclide will have four nodes.

IV. If $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1=0$ be an ellipsoid, then the sphere $\left(x-\frac{hk}{a}\right)^2+y^2+z^2=\left(\frac{bc}{a}\right)^2$ touches it at four um-

equation of the inverse sphero-quartic will be of the same form; and in particular if the sphero-quartic be of the form $a\alpha^2+b\beta^2+c\gamma^2+d\delta^2=0$, the inverse sphero-quartic will be of the same form.

82. We have seen that the general equation of a sphero-quartic can be written in the form

$$a\alpha^2+b\beta^2+c\gamma^2+d\delta^2=0,$$

where $\alpha^2+\beta^2+\gamma^2+\delta^2=0$ is an identical relation, and the circles are mutually orthogonal.

This is also evident otherwise; for the given form contains fifteen constants, namely,

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bilics if $h^2=a^2-b^2$, $k^2=a^2-c^2$; two of the umbilics of contact are in the plane of (xz) and are real, and two in the plane (xy) and are imaginary. Now to find the equation of the cyclide that will have the ellipsoid and sphere for focal quadric and sphere of inversion, that is, for F and J; the perpendicular let fall from the centre of J on any tangent plane to F is evidently equal to $\sqrt{a^2\cos^2\alpha+b^2\cos^2\beta+c^2\cos^2\gamma}-\frac{hk\cos\alpha}{a}$; but if O be the centre of J and T the foot of the perpendicular on the tangent plane, and if the points P, P′ be taken so that $OT^2-TP^2=OT^2-TP'^2=\left(\frac{bc}{a}\right)^2$, then P, P′ are points on the cyclide; and denoting OP by ρ and OT by p, we get

$$2p\rho=\frac{b^2c^2}{a^2}+\rho^2,$$

or,

$$2\left\{\sqrt{a^2\cos^2\alpha+b^2\cos^2\beta+c^2\cos^2\gamma}-\frac{hk\cos\alpha}{a^2}\right\}\rho=\frac{b^2c^2}{a^2}+\rho^2.$$

Hence, since $\cos\alpha$. $\cos\beta$, $\cos\gamma$ are the direction cosines of OP or ρ, we get the equation of the cyclide

$$2\sqrt{a^2x^2+b^2y^2+c^2z^2}=x^2+y^2+z^2+\frac{2hkx}{a}+\frac{b^2c^2}{a^2},$$

or, denoting the second side of the equation by C, $4(a^2x^2+b^2y^2+c^2z^2)=C^2$.

V. If in the equation $4(a^2x^2+b^2y^2+c^2z^2)=C^2$, we put $y=0$, $z=0$, we get

$$\left(x^2+\frac{2hkx}{a}+\frac{b^2c^2}{a^2}+2ax\right)\left(x^2+\frac{2hkx}{a}+\frac{b^2c^2}{a^2}-2ax\right)=0;$$

and the four values of x in this equation being denoted by x_1, x_2, x_3, x_4, we get $x_1+x_2+x_3+x_4=-\frac{4hk}{a}$. Hence it follows that the centre of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ is the centre of the cyclide. The values of x_1, x_2, x_3, x_4 are easily seen to be given by the equations

$$x_1=\ \ a+h-k-\frac{hk}{a};\qquad x_2=\ \ a-h+k-\frac{hk}{a};$$

$$x_3=-a+h+k-\frac{hk}{a};\qquad x_4=-a-h-k-\frac{hk}{a}.$$

Hence if we had taken the centre of F for origin of coordinates in the equation of the cyclide, the four values of x would be given by the system of four equations, $\pm a\pm h\pm k=0$.

VI. The equation of the cyclide referred to the centre of F as origin is given by the following symmetrical equation:

$$(x^2+y^2+z^2)^2-2x^2(a^2+h^2+k^2)-2y^2(a^2-h^2+k^2)-2z^2(a^2+h^2-k^2)$$
$$+8ahkx+(a^4+h^4+k^4-2a^2h^2-2h^2k^2-2k^2a^2)=0.$$

The sections by the planes $y=0$, $z=0$ are evidently the two pairs of circles

$$(x+k)^2+z^2=(a+h)^2,\qquad (x-k)^2+z^2=(a-h)^2,$$
$$(x+h)^2+y^2=(a+k)^2,\qquad (x-h)^2+y^2=(a-k)^2;$$

three explicitly and twelve implicitly, each of the circles α, β, γ, δ containing three; but six equations of condition are implied by the fact of the four circles being mutually orthogonal, and the identical relation $\alpha^2+\beta^2+\gamma^2+\delta^2=0$ (for constants are incorporated) is equivalent to one condition. Hence the given form contains eight independent constants; and this is the number which determines a sphero-quartic.

83. Since the circle α is orthogonal to the circles β, γ, δ, if we take either of the points in which a diameter of U perpendicular to the plane of α, called the polar line of α (see SALMON'S 'Geometry of Three Dimensions,' art. 358), cuts U, and tan² of half the cor-

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and the centres of these two pairs of circles are the foci respectively of the sections of the ellipsoid in whose planes they lie; also the radical axis of each pair is the line joining the pair of umbilics which lies in its plane.

VII. In the equation of the cyclide given in the last article, if we put $y=0$, $z=0$, we get

$$x^4-2x^2(a^2+h^2+k^2)+8ahkx+(a^4+h^4+k^4-2a^2h^2-2h^2k^2-2k^2a^2)=0\,;$$

and the four roots being denoted as before by x_1, x_2, x_3, x_4, we see, by the system of values given in article V., that a^2, h^2, k^2 are the roots of EULER's reducing cubic for this quartic in x; and if we denote by λ, μ, ν the roots of SIMPSON's reducing cubic for the same quartic, we get

$$\lambda=-(a^2-h^2-k^2),\quad \mu=-(a^2+k^2-h^2),\quad \nu=-(a^2+h^2-k^2).$$

Hence by these values we can write the equation of the quadrinodal cyclide in the following form, due to Professor CAYLEY, who arrived at it by a mode of investigation altogether different from that used here:

$$(y^2+z^2)^2+2x^2(y^2+z^2)+\mu y^2+\nu z^2(x-x_1)(x-x_2)(x-x_3)(x-x_4)=0.$$

VIII. The centre of similitude of the first pair of circles of article VI., that is, of the pair of circles

$$(x+k)^2+z^2=(a+h)^2,\qquad (x-k)^2+z^2=(a-h)^2,$$

is easily seen to be the point whose coordinates are $\left(\frac{ak}{h}, 0\right)$, that is, the middle point of the line joining two umbilics of F; or, in other words, the middle point of the line joining two nodes of the cyclide, and the centre of similitude of the other pair of circles, is the middle point of the line joining the other pair of nodes.

IX. The equation of the cyclide of article IV., that is, $4(a^2x^2+b^2y^2+c^2z^2)-C^2=0$, is the envelope of the quadric $a^2x^2+b^2y^2+c^2z^2+\mu C+\mu^2=0$, or, by restoring the value of C, of the quadric

$$(a^2+\mu)x^2+(b^2+\mu)y^2+(c^2+\mu)z^2+\frac{2\mu hk}{a}x+\mu^2+\frac{\mu b^2c^2}{a^2}\,;$$

and the condition that this should represent a cone is given by the equation

$$(a^2+\mu)(b^2+\mu)(c^2+\mu^2)\left(\mu+\frac{b^2c^2}{a^2}\right)-(b^2+\mu)(c^2+\mu)\mu\left(\frac{k^2h^2}{a^2}\right)=0.$$

This equation is satisfied by four values of μ, showing there are four cones, each having double contact with the cyclide; and the vertex of each cone is a centre of inversion. Now one value of μ is evidently $-b^2$, and the corresponding cone is

$$(a^2-b^2)x^2+(c^2-b^2)y^2-\frac{2b^2hkx}{a}+b^4\left(\frac{a^2-c^2}{a^2}\right)=0\,;$$

and when this is transferred to the centre of F as origin, it becomes

$$(hx-ak)^2-(k^2-h^2)z^2=0,$$

which represents a pair of planes whose line of intersection passes through a pair of umbilics of F; any point on this line may therefore be called a centre of inversion of the cyclide, that is, any point on the line joining a corresponding pair of nodes of the cyclide is a centre of inversion of the cyclide. In like manner, from the root

responding radius as the constant of inversion (see equation 83), then in this case each of the four circles α, β, γ, δ will be inverted into itself. Hence we have the following theorem:—

Every sphero-quartic has in general four circles and eight centres or poles of inversion.

84. If we are given a focal sphero-conic F of the sphero-quartic and its corresponding circle of inversion α, the remaining circles and centres of inversion may be constructed as follows. Through the four points in which α cuts F draw three pairs of great circles, each pair will intersect in two points diametrically opposite; the six points thus deter-

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$\mu=-c^2$, it can be shown that the line joining the other pair of nodes is such that any point of it is a centre of inversion; and from the two remaining roots the two other centres of inversion can be found.

X. If $F\equiv\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{2z}{c}=0$ be an elliptic paraboloid, and J the sphere which touches it at the umbilics, then we find, as in article IV., the equation of the cubic cyclide with four nodes to be

$$z\left(z+\frac{a^2}{c}\right)\left(z+\frac{b^2}{c}\right)+x^2\left(z+\frac{a^2}{c}\right)+y^2\left(z+\frac{b^2}{c}\right)=0.$$

The section of this surface by the plane xz consists of a line passing through the two nodes, and a circle whose centre is the focus of the section of the paraboloid by the same plane; and the section in like manner by the plane yz consists of the line joining the two other nodes, and a circle whose centre is the focus of the parabola, which is the section of the paraboloid by the plane yz.

XI. The sections of the surface by the planes xz and yz are lines of curvature of the surface.

Demonstration. The section by the plane xz is such that the centres of the generating spheres which touch the cyclide along it lie on the section of the paraboloid made by the plane of xz; hence, taking any two consecutive points on the section, it is evident that the normals at these points will lie in the plane of the section, since they pass through the centres of the generating spheres. Hence they intersect, and the proposition is proved.

We can verify this analytically; for the differential equation of the lines of curvature of any surface (see SALMON's 'Geometry of Three Dimensions,' p. 234) is

$$\left|\begin{matrix} dx, & dy, & dz, \\ L, & M, & N, \\ dL, & dM, & dN, \end{matrix}\right|=0;$$

and it is easy to see that this is satisfied by the equations $x=C\left(z+\frac{b^2}{c}\right)$ and $y=C'\left(z+\frac{a^2}{c}\right)$ combined with the equation of the surface, C, C′ being any constants. Hence *any plane passing through a line joining either of the pairs of nodes will be a line of curvature.*

XII. *The inverse of a line of curvature on any surface is a line of curvature on the inverse surface.* This is easily inferred from SALMON's 'Geometry of Three Dimensions,' articles 294, 479. Now, since trinodal and quadrinodal cyclides are the inverses of quadrics of revolution, we easily infer the following theorems:—

1°. *Any section of a quadrinodal surface through two nodes, either both real or both imaginary, will consist of two circles, which will be lines of curvature.*

2°. *A section of a cubic cyclide through two nodes will consist of a line and a circle, which will be lines of curvature.*

3°. *If through any point* P *on a trinodal or quadrinodal cyclide we draw two planes passing through the nodal axes, the circles which are the sections at* P *intersect orthogonally.*

4°. *Every line of curvature on a trinodal or quadrinodal cyclide consists of two circles, or a line and a circle.*

5°. *The locus of the generating spheres which touch a trinodal or quadrinodal cyclide at all the points of a line of curvature is a plane conic on the corresponding focal quadric.*—January 1872.]

mined, together with the poles of α, will be the eight centres of inversion, and the three circles cutting α orthogonally, and having the first three pairs of points as poles, will be the circles of inversion.

85. If two of the points coincide in which α cuts F, there will be then only six centres and three circles of inversion; but since the coincident circles whose centres are the point of contact of α and F reduce to a point, that point will not be a centre of inversion in the ordinary sense, that is, the centre of a circle which inverts the sphero-quartic into itself, but it will be the centre of a circle which inverts the sphero-quartic into a sphero-conic. Hence in this case there are only four ordinary centres of inversion.

If three of the points coincide, there will be then only two ordinary centres of inversion, namely, the poles of α; but the point of osculation and the point diametrically opposite to it will be the poles of a circle, which inverts the quartic into a sphero-conic, and α will be the ordinary circle of inversion.

86. The following proposition is not difficult to be proved.

If ρ be the radius vector from any origin on the surface of U to a sphero-conic and, measured in the same direction from the same origin, an arc ρ' be determined by the condition that

$$2 \tan \tfrac{1}{2} \rho' = \tan \tfrac{1}{2} \rho, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (85)$$

the locus of the extremity of ρ' is a sphero-quartic.

87. Besides the method of inversion hitherto used in this section, namely by the equation (83), by which any curve on the surface of a sphere is inverted into another curve on the surface of the same sphere, there is another, the more ordinary method, by which it can be inverted into a curve on the surface of another sphere or on a plane. Thus a sphero-quartic being the intersection of a cyclide W and a sphere U, and since if both be inverted from any point in space they invert into surfaces of the same kind, we see that a sphero-quartic inverts into another sphero-quartic, or into a bicircular quartic, if the arbitrary point be taken on the surface of U.

88. Since a sphero-quartic is the intersection of a sphere and a quadric, four cones can be described through it; then it is plain that the vertices of these four cones are four points in space, which are centres of self-inversion of the quartic, that is, the quartic is an anallagmatic curve, and it has four centres of self-inversion.

89. Let us consider the sphero-quartic WU, where W is the cyclide $a\alpha^2+b\beta^2+c\gamma^2+d\delta^2$, and U its sphere of inversion given by the equation $U^2=\alpha^2+\beta^2+\gamma^2+\delta^2=0$, then the centres of α, β, γ, δ are the four vertices of the four cones through WU, that is, they are the four centres of inversion. Hence we have the following theorem:—*If*

$$W=a\alpha^2+b\beta^2+c\gamma^2+d\delta^2+e\epsilon^2=0$$

be a cyclide given in its canonical form, then the sphero-quartic, which is the intersection of W and any of its spheres of inversion, has the centres of the four remaining spheres as centres of inversion.

90. If the arbitrary point we invert from be any point on the sphero-quartic WU

itself, then W and U invert respectively into a cubic cyclide and a plane, and hence the sphero-quartic inverts into a circular cubic. *Hence if a sphero-quartic be inverted from any point on itself it inverts into a circular cubic; and conversely, if a circular cubic be inverted from any arbitrary point in space, we get a sphero-quartic passing through the centre of inversion.*

91. If $W=a\alpha^2+b\beta^2+c\gamma^2+d\delta^2=0$, and $U^2=\alpha^2+\beta^2+\gamma^2+\delta^2=0$, and we eliminate α^2 between W and U^2, we get the cyclide $(b-a)\beta^2+(c-a)\gamma^2+(d-a)\delta^2=0$ passing through the sphero-quartic WU. But the cyclide $(b-a)\beta^2+(c-a)\gamma^2+(d-a)\delta^2=0$ is the envelope of a variable sphere whose centre moves on a conic; and inverting the curve WU from any arbitrary point on U, we get a bicircular quartic, whose generating circles will be the inverses of the circles in which the generating spheres of

$$(b-a)\beta^2+(c-a)\gamma^2+(d-a)\delta^2=0$$

intersect U, and the centres of the generating circles of the bicircular will be the points in which the lines from the origin to the centres of the generating spheres of

$$(b-a)\beta^2+(c-a)\gamma^2+(d-a)\delta^2=0$$

pierce the plane into which the sphere U inverts. Now the locus of the centres of the generating spheres of $(b-a)\beta^2+(c-a)\gamma^2+(d-a)\delta^2=0$ is one of the double lines of the developable formed by tangent planes to U along the sphero-quartic WU (see art. 42). Hence we infer the following theorem:—

If a sphero-quartic WU be inverted into a bicircular quartic, the four cones having the point we invert from as a common vertex, and whose bases are the double lines of the developable Σ formed by tangent planes to U along the sphero-quartic WU, will pierce the plane of the bicircular in four conics, which will be the focal conics of the bicircular.

92. If Z be a circle on U which osculates WU, then it is evident that the pole of the plane of Z with respect to U is a point on the cuspidal edge of Σ (see last article). Again, when we invert (WU) into a bicircular, Z will invert into an osculating circle of the bicircular. Hence we have the following theorem:—

If we invert a sphero-quartic into a bicircular, the evolute of the bicircular is the curve in which its plane is intersected by the cone whose vertex is the origin of inversion, and whose base is the cuspidal edge of the developable formed by tangent planes to U along WU.

93. We give in this and the following article some important properties of bicircular quartics, which follow at once from the properties we have demonstrated for sphero-quartics.

Since a sphero-quartic WU is the intersection of W and U, and WU is given by the general equation

$$WU=(a, b, c, d, l, m, n, p, q, r\between\alpha, \beta, \gamma, \delta)^2=0,$$

where α, β, γ, δ are circles on U, and U^2 is given by the equation (29), art. 28, hence, *when we invert U into a plane, the curve WU will be inverted into a bicircular whose equation is* $(a, b, c, d, l, m, n, p, q, r\between\alpha, \beta, \gamma, \delta)^2=0$, *and the following relation will be an identical one on the plane into which U inverts:—*

$$\left|\begin{array}{ccccc} -1, & \cos(\alpha\beta), & \cos(\alpha\gamma), & \cos(\alpha\delta), & \alpha\div r', \\ \cos(\beta\alpha), & -1, & \cos(\beta\gamma), & \cos(\beta\delta), & \beta\div r'', \\ \cos(\gamma\alpha), & \cos(\gamma\beta), & -1, & \cos(\gamma\delta), & \gamma\div r''', \\ \cos(\delta\alpha), & \cos(\delta\beta), & \cos(\delta\gamma), & -1, & \delta\div r'''', \\ \alpha\div r', & \beta\div r'', & \gamma\div{}''', & \delta\div r'''', & 0, \end{array}\right| = 0. \quad . \quad . \quad . \quad (86)$$

It hence follows that every bicircular can be expressed in the form $a\alpha^2+b\beta^2+c\gamma^2+d\delta^2=0$, where α, β, γ, δ are four circles mutually orthogonal, and that for this system of circles the relation is an identical one, $\alpha^2+\beta^2+\gamma^2+\delta^2=0$.

94. From the last article we see that the same bicircular can be written in the four following forms:

$$\left.\begin{array}{l} (d-a)\alpha^2+(d-b)\beta^2+(d-c)\gamma^2=0, \\ (a-b)\beta^2+(a-c)\gamma^2+(a-d)\delta^2=0, \\ (b-c)\gamma^2+(b-d)\delta^2+(b-a)\alpha^2, \\ (c-d)\delta^2+(c-a)\alpha^2+(c-b)\beta^2, \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (87)$$

and that consequently the tangential equations of the four focal conics of the quartic are given by the equations

$$\left.\begin{array}{l} (d-a)\lambda^2+(d-b)\mu^2+(d-c)\nu^2=0, \\ (a-b)\mu^2+(a-c)\nu^2+(a-d)\varrho^2=0, \\ (b-c)\nu^2+(b-d)\varrho^2+(d-a)\lambda^2=0, \\ (c-d)\varrho^2+(c-a)\lambda^2+(c-b)\mu^2=0. \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (88)$$

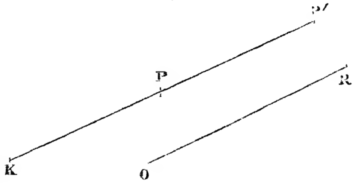
CHAPTER VI.

Projection of Sphero-quartics.

95. *If a sphero-quartic be projected on one of the planes of a circular section of any quadric passing through it by lines parallel to the greatest or least axis of the quadric, the projection will be a bicircular quartic whose centres of inversion will be the projections of the centres of inversion of the sphero-quartic.*

Demonstration. Let U be the sphere given by the equation $U^2=(\alpha^2+\beta^2+\gamma^2+\delta^2)=0$, and V the quadric which intersects U in the sphero-quartic, then the centres of α, β, γ, δ will be the vertices of the four cones which can be drawn through the sphero-quartic, that is they will be its centres of inversion. Let KPP′ be an edge of the cone passing through two points of the quartic, K being the centre of α; then, since α is a sphere of inversion of the quartic, the rectangle KP KP′ is constant. Hence if O be the centre of the quadric V, the radius vector OR of the quadric parallel to KP′ is constant; therefore the locus of R is a sphero-conic: and if the point R_1

Fig. 3.



be the projection of R on the plane of circular section, the locus of R_1 is a circle; therefore OR bears a constant ratio to its projection. Now let the projections of K, P, P′ by lines parallel to the greatest or least axis be K_1, P_1, P'_1, and it is evident that we have the proportion

$$KP \,.\, KP' : K_1P_1 \,.\, K_1P'_1 :: OR^2 : O_1R_1^2. \qquad (89)$$

Hence the rectangle $K_1P_1 \,.\, K_1P'_1$ is constant, and K_1 is a centre of inversion of the projection. The projection is therefore an anallagmatic curve, and being evidently of the fourth degree is a bicircular quartic. Hence the proposition is proved.

96. On account of the importance of the proposition of the preceding article, we give another proof by forming the equation of the projection in Cartesian coordinates.

Let U and V be given by the Cartesian equations

$$(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2-r^2=0,$$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1=0;$$

and changing the planes of reference to xy, xz, and one of the planes of circular section, we get, if θ denote the angle made by the plane of xy with the plane of circular section,

$$U\equiv(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2+2(x-\alpha)(z-\gamma)\cos\theta-r^2=0,$$

$$V\equiv\left(\frac{x+z\cos\theta}{a}\right)^2+\left(\frac{y}{b}\right)^2+\left(\frac{z\sin\theta}{c}\right)^2-1=0.$$

If we eliminate x between these equations, we get the projection of the curve UV on the plane of circular section; this elimination is most easily performed by the following substitution, namely,

$$a^2\left(1-\frac{y^2}{b^2}-\frac{z^2\sin^2\theta}{c^2}\right)=S,$$

$$r^2-(y-\beta)^2-(z-\gamma)^2\sin^2\theta=S',$$

$$(\alpha+\gamma\cos\theta)^2=S'',$$

and we have at once the equation

$$\sqrt{S}-\sqrt{S'}+\sqrt{S''}=0,$$

or, cleared of radicals,

$$(S-S')^2+S''(S''-2S-2S')=0 \qquad (90)$$

as the required projection; and substituting the value of $\sin^2\theta$, viz. $\left(\frac{a^2-b^2}{a^2-c^2}\right)\left(\frac{c^2}{b^2}\right)$ in $S-S'$, we get

$$-\left(\frac{a^2-b^2}{b^2}\right)\left\{y^2+z^2+\frac{2b^2\beta}{a^2-b^2}\,y+\frac{2c^2\gamma}{a^2-c^2}\,z-\left[\frac{a^2-\gamma^2}{a^2-c^2}\,b^2+\frac{b^2\beta^2}{a^2-b^2}+\frac{c^2\gamma^2}{a^2-c^2}\right]\right\},$$

which, equated to zero, represents a circle. Hence the proposition is proved.

Cor. $S+S'=-\left(\frac{a^2-b^2}{b^2}\right)\left\{\frac{a^2+b^2}{a^2-b^2}y^2+\frac{a^2+c^2}{a^2-c^2}z^2-\frac{2b^2\beta}{a^2-b^2}y-\frac{2c^2\gamma}{a^2-c^2}z+\frac{(\beta^2-a^2-\gamma^2)b^2}{a^2-b^2}+\frac{c^2\gamma^2}{a^2-c^2}\right\}.$

Hence the equation of the projection written in full, after replacing z by x, is

$$\left.\begin{array}{c}\left\{x^2+y^2+\frac{2c^2\gamma}{a^2-c^2}x+\frac{2b^2\beta}{a^2-b^2}y-\left(\frac{a^2-\gamma^2}{a^2-c^2}b^2+\frac{b^2\beta^2}{a^2-b^2}+\frac{c^2\gamma^2}{a^2-c^2}\right)\right\}^2 \\ -2\left\{\frac{\alpha b\sqrt{a^2-c^2}+\gamma a\sqrt{b^2-c^2}}{\sqrt{(a^2-b^2)(a^2-c^2)}}\right\}^2\times \\ \left\{\frac{a^2+c^2}{a^2-c^2}x^2+\frac{a^2+b^2}{a^2-b^2}y^2-\frac{2c^2\gamma}{a^2-c^2}x-\frac{2b^2\beta}{a^2-b^2}y+\frac{(\beta^2-a^2-\gamma^2)}{a^2-b^2}b^2+\frac{c^2\gamma^2}{a^2-c^2}\right\} \\ +\left\{\frac{\alpha b\sqrt{a^2-c^2}+\gamma a\sqrt{b^2-c^2}}{\sqrt{(a^2-b^2)(a^2-c^2)}}\right\}^4=0.\end{array}\right\} \quad . \quad . \quad (91)$$

97. *The elliptic projection of a sphero-quartic is a bicircular quartic.*

Definition. If through any point P on a quadric we describe two confocals, and if P′ be the point where the line of curvature common to the two confocals drawn through P intersects the plane of xy, P′ is what I call the elliptic projection of P.

If X, Y be the coordinates of the elliptic projection of a point on the sphero-quartic, x, y the coordinates of the projection of the same on either plane of circular section by lines parallel to the greatest axis, then, by SALMON's 'Geometry of Three Dimensions,' art. 180,

$$x^2 : \mathrm{X}^2 :: y^2 : \mathrm{Y}^2 :: b^2-c^2 : b^2.$$

Hence the locus of the point whose coordinates are X, Y is similar to the locus of the point whose coordinates are x, y. Hence the proposition is proved.

98. The four spheres $\alpha, \beta, \gamma, \delta$ intersect respectively the four cones through the sphero-quartic (that is, each sphere intersects the cone whose vertex is at its own centre) in four sphero-conics, and the projection of these sphero-conics on the planes of circular sections by lines parallel to the greatest or least axis will be four circles, and these will be the circles of inversion of the bicircular which results from projecting the sphero-quartic.

For if the sphere α intersect the line KPP′ in Q (see art. 95), $\mathrm{KQ}^2=\mathrm{KP}\ \mathrm{KP}'$. We can therefore account for the four circles of inversion of the bicircular.

99. The projecting lines of the four sphero-conics of the last article intersect the quadric in four curves, whose elliptic projections will be the circles of inversion of the bicircular which results from the elliptic projection of a sphero-quartic.

This is proved in the same way exactly as art. 97.

100. *Lemma.* If a sphere concentric with a quadric intersect it in a sphero-conic, and tangent planes to the quadric be parallel to the tangent planes to the cone whose vertex is at the centre of the sphere, and which stands on the sphero-conic, the locus of their points of contact is a line of curvature of the quadric.

This proposition is plainly the converse of art. 158 of SALMON's 'Geometry of Three Dimensions;' but we can give a direct proof of it as follows. Let the sphere and quadric be given by the equations

$$x^2+y^2+z^2=r^2,$$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,$$

then the equation of the cone is

$$\left(\frac{a^2-r^2}{a^2}\right)x^2+\left(\frac{b^2-r^2}{b^2}\right)y^2+\left(\frac{c^2-r^2}{c^2}\right)z^2=0.$$

Now the equation of a tangent plane to the quadric is

$$\frac{xx'}{a^2}+\frac{yy'}{b^2}+\frac{zz'}{c^2}=1.$$

Hence the equation of a parallel tangent plane through the centre is

$$\frac{xx'}{a^2}+\frac{yy'}{b^2}+\frac{zz'}{c^2}=0\,;$$

and the condition that this should be a tangent plane to the cone is

$$\frac{x'^2}{a^2(a^2-r^2)}+\frac{y'^2}{b^2(b^2-r^2)}+\frac{z'^2}{c^2(c^2-r^2)}=0.$$

Hence $x'\,y'\,z'$ is a point on the intersection of the confocals $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ and $\frac{x^2}{a^2-r^2}+\frac{y^2}{b^2-r^2}+\frac{z^2}{c^2-r^2}=1$. Hence the proposition is proved.

101. *If tangent planes be drawn to the quadric parallel to the tangent planes of the four cones through the sphero-quartic* UV, *the locus of their points of contact are four lines of curvature on* V.

Demonstration. Let OR be a central vector of V parallel to an edge of one of the cones, then OR is constant, and the proposition is evident from the last article.

Cor. If a developable be described about V *by drawing tangent planes to it along the sphero-quartic* UV, *the four cones whose common vertex is at the centre of* V, *and which stand on the double lines of the developable, intersect the quadric in the lines of curvature stated in the proposition.*

102. *If tangent planes be drawn to* U *parallel to the tangent planes to the cones, the loci of the points of contact are sphero-conics; these sphero-conics are the focal sphero-conics of the sphero-quartic.* This proposition is evident.

103. *If through any tangent line of a sphero-quartic four planes be drawn passing through its four centres of inversion, the anharmonic ratio of these four planes is constant.*

Demonstration. Let λ_1, λ_2, λ_3, λ_4 be the four values of λ, for which $U+\lambda V$ represents a cone; then if we represent by U_1 and V_1 the tangent planes to U and V through the given line, the four planes in question are evidently tangent planes to the four cones, and their equations are $U_1+\lambda_1V_1$, $U_1+\lambda_2V_1$, $U_1+\lambda_3V_1$, $U_1+\lambda_4V_1$, and the anharmonic ratio is

$$\frac{(\lambda_1-\lambda_2)(\lambda_3-\lambda_4)}{(\lambda_1-\lambda_3)(\lambda_2-\lambda_4)}. \qquad (92)$$

Cor. It is easy to see that the theorem, "that the anharmonic ratio is constant of the pencil formed by the four tangents which may be drawn from any point of a plane curve

of the third degree," follows as an immediate inference from the theorem of this article; for the point in the sphero-quartic will be the vertex of a cone of the third degree which stands on the quartic, and then, from what we have proved, it follows that if through any edge of a cone of the third degree four tangent planes be drawn to the cone their anharmonic ratio is constant.

CHAPTER VII.—FOCI AND FOCAL CURVES.

SECTION I.—*Foci of Cyclides.*

104. The conception of a focus which I shall use in this memoir is, in the case of surfaces, an infinitely small sphere having imaginary double contact with the surface; and for curves, that of an infinitely small circle, having imaginary double contact with the curve. This being premised, let us take the cyclide given by its canonical form,

$$W \equiv a\alpha^2+b\beta^2+c\gamma^2+d\delta^2+e\varepsilon^2=0.$$

We see from art. 33 that this cyclide is the envelope in five different ways of a variable sphere whose centre moves on a given quadric, and which cuts a given fixed sphere orthogonally. Thus, taking the quadric F of art. 33, the tangential equation to F is $(a-b)\mu^2+(a-c)\nu^2+(a-d)\varrho^2+(a-e)\sigma^2=0$; and corresponding to this we have the sphere α, which is the one which the variable sphere cuts orthogonally while its centre moves on F. Now let a developable be circumscribed to α and F, then the curve of taction of the developable and F, and the curve of intersection of α and F, divide the surface of F into three regions, which possess the following properties:—In the first region every point is such that any sphere having it as centre and cutting α orthogonally is real, and moreover such that this sphere meets the consecutive one in a real curve; in the second region every point is such that the spheres are real, but do not intersect the consecutive ones in real points; while in the third region the orthogonal spheres are altogether imaginary. Hence it follows that every point in the sphero-quartic (αF) is an infinitely small sphere having imaginary double contact with W, or, in other words, every point on (αF) is a focus of W.

In like manner every point on each of the four sphero-quartics ($\beta F'$), ($\gamma F''$), ($\delta F'''$), (εF^{iv}) is a focus of W, so that a cyclide has in general five focal sphero-quartics.

From this proposition it is evident that the name *focal quadric* which I have employed for the directing quadrics F, F′, &c. is appropriate as suggestive of an important property of these surfaces; had I followed M. DE LA GOURNERIE I should have called them *déférentes*. In the next proposition we shall see an additional reason in favour of the name I have given.

105. *Definition.* When the points of contact of a focus with the cyclide are points on the imaginary circle at infinity, I shall, following Dr. SALMON, call the focus a "double focus" (see SALMON's 'Higher Curves'). Professor CAYLEY, in his memoir on "Polyzonal Curves," uses the term "*nodo-focus,*" to express the same idea; and M. DE LA

GOURNERIE, in his memoir "Sur les Lignes Sphériques," "singular focus" (see LIOUVILLE'S Journal for 1869).

106. Let us suppose that we have a system of generating spheres passing through the same point P; from P let there be drawn a tangent cone to the focal quadric F, then any edge of this tangent cone meets the quadric F in two consecutive points, and the generating spheres whose centres are at these points touch each other at P, consequently each edge of the cone is a normal to the cyclide at P as well as being a tangent to F: now let us suppose the point P to be on the imaginary circle at infinity, and the normals to the cyclide at P will be also tangents to it at P, and we see that the tangent lines to the cyclide at the imaginary circle at infinity are also tangent lines to the focal quadric F. Hence we have this remarkable theorem:—

The three focal conics of the focal quadric F *of the cyclide are double or nodo-focal curves of the cyclide.* Compare the corresponding theorem, art. 28 in 'Bicircular Quartics.'

107. Since the nodo-focal curves of the cyclide W are the three focal conics of F, they are in like manner the three focal conics of F′, F″, F‴, F$^{\text{iv}}$. Hence the five F's are confocal.

That is, *the five focal quadrics of a cyclide are confocal, and their three focal conics are such that each point of any of them is a double focus of the cyclide.*

108. If one of the focal quadrics of a cyclide be a sphere, then the focal conics of this sphere reduce to the centre, and the cyclide must consequently have the imaginary circle at infinity as a cuspidal edge. Hence all the focal quadrics must be spheres, and these spheres must be concentric.

From the analogy of the corresponding case in 'Bicircular Quartics' I shall call this species of cyclide a *Cartesian cyclide.*

The ordinary foci of a Cartesian cyclide being the intersection of the spheres of inversion with the focal spheres are circles, and it has but one singular or nodo-focus, which in this case is a triple focus, namely, the common centre of the focal spheres.

109. If we take four foci for spheres of reference of a cyclide, since these foci are point spheres, the result of substituting the coordinates of any point in one of them will be the square of the radius vector to the focus. Hence if we denote the vectors to the four foci by ρ, ρ', ρ'', ρ''', the vector equation of a cyclide will be in the form (see art. 29)

$$\begin{vmatrix} 0, & n, & m, & p, & \rho^2, \\ n, & 0, & l, & q, & \rho'^2, \\ m, & l, & 0, & r, & \rho''^2, \\ p, & q, & r, & 0, & \rho'''^2, \\ \rho^2, & \rho'^2, & \rho''^2, & \rho'''^2, & 0. \end{vmatrix} \quad \ldots\ldots\ldots\ldots \quad (93)$$

110. If the focal quadric of a cyclide be a paraboloid, then the cyclide becomes a cubic surface together with the plane at infinity. Hence *the focal quadrics of cubic cyclides are five confocal paraboloids.*

111. If we are given a sphere of inversion and the corresponding focal quadric of a cyclide, we can construct the remaining focal quadrics. For by art. 34 let α be the sphere of inversion and F the focal quadric, and circumscribing a developable Σ to α and F, if the double lines of Σ, which are conics, be C, C', C'', C''', then through C, C', C'', C''' let confocals F', F'', F''', F'''' to F be described, and these will be the other focal quadrics of the cyclide.

Now if α touches F the developable Σ will have but three double lines, C, C', C'', and hence in this case there will be only four focal quadrics. If α touches F the cyclide will have the point of contact for a node, and moreover the cyclide will be the inverse of a central quadric. Hence it follows *that the cyclide which results from inverting a central quadric has but four focal quadrics, and that three of these confocal quadrics pass through the node* (see art. 78).

112. If the sphere of inversion be an osculating sphere of the focal quadric F, the developable Σ will have but two nodal lines, C, C', and therefore there will be but two additional focal quadrics, F', F''. Hence in all there will be but three focal quadrics. This is the species of cyclide which results from inverting a non-central quadric (see art. 79).

113. If the sphere of inversion has double contact with F the cyclide will be binodal; there will be, besides the focal quadric F, the two confocals to F, which can be drawn through the two points where α touches F. A cyclide of this form being given by the equation $(a, b, c, f, g, h \between \beta, \gamma, \delta)^2$, and the locus of the centre of the generating sphere being a conic, *it must be a focal conic of the three confocal quadrics which the cyclide must have, that is, every point of this conic must be a double focus of the cyclide, and moreover the four points in which it intersects the corresponding sphere of inversion must be single foci.*

When the sphere of inversion α has double contact with F, the curve of intersection of α and F breaks up into two circles; these circles are the inverses of the two focal lines of the cone, of which this species of cyclide is the inverse.

114. Since three of the spheres of inversion of a cyclide which has only four spheres of inversion, and which is consequently the inverse of a central quadric, are the inverses of the three principal planes of the quadric, and since the inverse of a focus is a focus, it follows *that in this case the inverses of the three focal conics of the quadric inverted will be the focal sphero-quartics of the cyclide.* In this case also we have the following theorem, which is an extension of one given in my 'Bircirculars,' art. 55:—

If we invert a quadric Q *from any point* P, *the principal planes of the focal quadrics of the resulting cyclide are parallel to the tangent planes at* P *drawn to three confocals of* Q *passing through* P.

115. In like manner two of the spheres of inversion of a cyclide which has only three spheres of inversion, and which is the inverse of a non-central quadric, are the inverses of the two planes of symmetry of the quadric; and since the focal conics of a paraboloid are either an ellipse and parabola or hyperbola and parabola, *we see that one of*

the focal sphero-quartics of such a cyclide must have a cusp, namely, the inverse of the point at infinity on the focal parabola.

Conversely, *if the sphere of inversion* α *of a cyclide be an osculating sphere of the focal quadric* F, *and if the whole system be inverted from the point of osculation, the sphere* α *will invert into a principal plane of the quadric into which the cyclide inverts, and the sphero-quartic in which* α *intersects* F *will invert into a parabola.*

116. Since the intersection of a focal quadric of a cyclide with the corresponding sphere of inversion gives a line of foci of the cyclide, then, if the cyclide be $a\alpha^2+b\beta^2+c\gamma^2+d\delta^2$, the focal quadric will be $a\lambda^2+b\mu^2+c\nu^2+d\varrho^2$; and if the sphere of inversion be given by the equation $U^2=\alpha^2+\beta^2+\gamma^2+\delta^2$, then the line of foci will be given as the intersector of the two surfaces in tetrahedral coordinates,

$$\frac{x^2}{a}+\frac{y^2}{b}+\frac{z^2}{c}+\frac{w^2}{d}=0,$$

$$x^2+y^2+z^2+w^2=0,$$

and therefore the quadric in tangential coordinates

$$\frac{a\lambda^2}{a+k}+\frac{b\mu^2}{b+k}+\frac{c\nu^2}{c+k}+\frac{d\varrho^2}{d+k}=0$$

passes through the line of foci of $a\alpha^2+b\beta^2+c\gamma^2+d\delta^2$. Hence it follows that the cyclide

$$\frac{a\alpha^2}{a+k}+\frac{b\beta^2}{b+k}+\frac{c\gamma^2}{c+k}+\frac{d\delta^2}{d+k}=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (94)$$

denotes in general a cyclide having one focal sphero-quartic in common with $a\alpha^2+b\beta^2+c\gamma^2+d\delta^2=0$.

117. In art. 33 we have seen that a cyclide given by the canonical form $a\alpha^2+b\beta^2+c\gamma^2+d\delta^2+e\varepsilon^2=0$ may be written in five different forms; and by the last article we see that to this cyclide correspond five different systems of cyclides, each system having one sphero-quartic of foci common with it. These systems are given by the equations:

$$\frac{(a-b)\beta^2}{a-b+k}+\frac{(a-c)\gamma^2}{a-c+k}+\frac{(a-d)\delta^2}{a-d+k}+\frac{(a-e)\varepsilon^2}{a-e+k}=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (95)$$

$$\frac{(b-c)\gamma^2}{b-c+k'}+\frac{(b-d)\delta^2}{b-d+k'}+\frac{(b-e)\varepsilon^2}{b-e+k'}+\frac{(b-a)\alpha^2}{(b-a)+k'}=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (96)$$

$$\frac{(c-d)\delta^2}{c-d+k''}+\frac{(c-e)\varepsilon^2}{c-e+k''}+\frac{(c-a)\alpha^2}{c-a+k''}+\frac{(c-b)\beta^2}{c-b+k''}=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (97)$$

$$\frac{(d-e)\varepsilon^2}{d-e+k'''}+\frac{(d-a)\alpha^2}{d-a+k'''}+\frac{(d-b)\beta^2}{d-b+k'''}+\frac{(d-c)\gamma^2}{d-c+k'''}=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (98)$$

$$\frac{(e-a)\alpha^2}{e-a+k''''}+\frac{(e-b)\beta^2}{e-b+k''''}+\frac{(e-c)\gamma^2}{e-c+k''''}+\frac{(e-d)\delta^2}{e-d+k''''}=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (99)$$

In these equations the k's may have any value.

118. The method of forming the reciprocal of one cyclide with respect to another will be given in a subsequent Chapter; in this we shall anticipate so much of the results as to say that it is identical with the method of quadrics. This being premised, if we form the reciprocal of the cyclides

$$a\alpha^2+b\beta^2+c\gamma^2+d\delta^2=0,$$

$$\frac{a\alpha^2}{a+k}+\frac{b\beta^2}{b+k}+\frac{c\gamma^2}{c+k}+\frac{d\delta^2}{d+k}=0$$

with respect to $U^2=\alpha^2+\beta^2+\gamma^2+\delta^2=0$, we get

$$\frac{\alpha^2}{a}+\frac{\beta^2}{b}+\frac{\gamma^2}{c}+\frac{\delta^2}{d}=0,$$

$$\frac{(a+k)\alpha^2}{a}+\frac{(b+k)\beta^2}{b}+\frac{(c+k)\gamma^2}{c}+\frac{(d+k)\delta^2}{d}=0;$$

and from the forms of these reciprocals it is plain that they have double contact along the whole sphero-quartic, in which each is intersected by the common sphere of inversion U.

119. *The three confocals to a given cyclide which can be drawn through any given point are mutually orthogonal.*

Definition. Confocal cyclides are cyclides having a common sphero-quartic of foci.

Demonstration. The focal quadrics of a confocal system of cyclides pass through a common curve of intersection; this is the sphero-quartic, which is their common line of foci. Now let P be the point through which the cyclides pass, and taking P′ the inverse of P with respect to U, then the plane which bisects P P′ perpendicularly forms with P, P′, and U a coaxial system, and the three quadrics touching this plane and passing through the common line of foci will be the focal quadrics of three confocal cyclides passing through P, P′ and cutting each other orthogonally. For if X, Y, Z be the points of contact of the quadrics with the plane, it is evident that the spheres whose centres are X, Y, Z, and which cut U orthogonally, are themselves mutually orthogonal. Hence the proposition is evident.

120. The cyclides in the last article are not only orthogonal at P, P′, but each pair of them are orthogonal throughout their whole intersection*.

Demonstration. Let us consider the two cyclides whose focal quadrics touch the plane at X, Y, and let us consider any edge of the developable which circumscribes the focal quadrics. This edge will be divided in involution by the system of quadrics passing through the common focal sphero-quartic. The double points of the involution will be the points of contact of the edge with the two quadrics which the developable circumscribes. Hence the spheres having these points as centres and cutting U orthogonally will themselves cut orthogonally, and hence it follows that the cyclides of which they are generators will cut orthogonally in a curve which has double contact with the circle of intersection of their generating spheres.

* Hence it follows, by DUPIN's theorem, that these cyclides intersect each other in lines of curvature.

Cor. Each cyclide of the three orthogonal cyclides being a surface of two sheets, hence there will be two systems, each consisting of three sheets, and each system will have eight points common to all. Hence the three orthogonal cyclides will have sixteen points common to all; these will be eight pairs of inverse points.

121. The two cyclides

$$a\alpha^2+b\beta^2+c\gamma^2+d\delta^2+e\epsilon^2,$$

$$\frac{\alpha^2}{a}+\frac{\beta^2}{b}+\frac{\gamma^2}{c}+\frac{\delta^2}{d}+\frac{\epsilon^2}{c}$$

have in common their five focal sphero-quartics.

Demonstration. For eliminate α^2 from these cyclides by means of the identical relation

$$\alpha^2+\beta^2+\gamma^2+\delta^2+\epsilon^2,$$

and we see, by making $k=a$ in the equation (95), that the cyclides have one focal sphero-quartic in common. Hence the proposition is proved.

122. *If two cyclides having the same spheres of inversion be reciprocals with respect to the square of any of these spheres, then each intersects this sphere in a sphero-quartic of foci of the other.*

For it is evident the cyclides

$$a\alpha^2+b\beta^2+c\gamma^2+d\delta^2$$

and

$$\frac{\alpha^2}{a}+\frac{\beta^2}{b}+\frac{\gamma^2}{c}+\frac{\delta^2}{d}$$

possess this property.

SECTION II.—*Foci of Sphero-quartics.*

123. We have seen that every sphero-quartic can be generated in four different ways as the envelope of a variable circle on the surface of a sphere U, the centre of the variable sphere moving along a sphero-conic while it cuts a fixed circle on U orthogonally. Now, if one of these sphero-conics be F, and α the corresponding circle, it can be seen, in the same way as in art. 104, that each point in which α intersects F is a focus of the sphero-quartic. Again, the four cones which stand on the sphero-conics (see equations (45), art. 41), and whose common vertex is at the centre of U, are plainly the reciprocals of the four cones which can be drawn through the sphero-quartic; but these latter cones have the same planes of circular section, therefore the former system have the same focal lines. Hence we have the following theorem, analogous to one in 'Bicirculars':—

Every sphero-quartic has sixteen single foci, and these lie four by four on four confocal sphero-conics, these sphero-conics being the déférentes or focal conics of the sphero-quartic.

124. Being given one circle of inversion and the corresponding focal sphero-conic of a sphero-quartic, the three remaining focal sphero-conics can be constructed. For *circumscribe a spherical quadrilateral to the circle and conic and through the three quartets of opposite intersections describe confocals, and we have the thing done.*

125. Let us consider one of the double lines of the developable which circumscribes

U along the sphero-quartic, then the cone whose vertex is at the centre of U, and which stands on this double line, intersects U in the corresponding focal sphero-conic; and it is plain that the four foci on this sphero-conic are the four points where the double line of the developable intersects U. Hence *the sixteen foci of a sphero-quartic are the sixteen points in which the double lines of the developable which circumscribes the sphere along the sphero-quartic intersect the sphere.*

126. In Chapter IV. we have shown that the equation of a sphero-quartic may be so interpreted as to represent a quartic cone, namely, by regarding the circles α, β, γ, δ, which enter into the equation of a sphero-quartic, as single sheets of a cone, whose vertex is at the centre of the sphere. Again, the equation of a sphero-quartic may be interpreted so as to represent a cyclide, that is, by regarding α, β, γ, δ as spheres cutting U orthogonally, and the quartic cone given by the former interpretation will be a tangent cone to the cyclide given by the latter. Hence we have, from article 123, the following theorem:—

The quartic cone which circumscribes a cyclide, and whose vertex is at the centre of a sphere of inversion of the cyclide, has sixteen focal lines, which are four by four the edges of four confocal cones.

127. The four confocal cones of the last article possess another important property; to demonstrate it we must prove some properties of binodal cyclides.

Let us consider the sphero-quartic WU, W being the cyclide $a\alpha^2+b\beta^2+c\gamma^2+d\delta^2$, and $U^2\equiv\alpha^2+\beta^2+\gamma^2+\delta^2$, then WU is the intersection of U with any of the four binodals got by eliminating α, β, γ, δ successively between W and U^2. Now each of these binodals has three focal quadrics and one focal conic, which focal conic is also a focal conic of the three confocal quadrics of the cyclide to which it belongs. Eliminating α, we get the binodal $(a-b)\beta^2+(a-c)\gamma^2+(a-d)\delta^2$, and the focal conic of this is one of the double lines of the developable Σ circumscribed about U along WU. Let the four double lines of Σ be the conics C, C$'$, C$''$, C$'''$; if C be the focal conic of $(a-b)\beta^2+(a-c)\gamma^2+(a-d)\delta^2$, then Σ is the developable circumscribed about C and U. Hence, by art. 34, the three focal quadrics will be the three quadrics described through the conics C$'$, C$''$, C$'''$ respectively, and having C for a focal conic.

128. Since the four cones standing on C, C$'$, C$''$, C$'''$ are confocal, whose vertex is at the centre of U, and the four cones are confocal which have the same vertex and of which one stands on C and the remaining three are circumscribed to the three quadrics of the last article, hence we have the theorem, *that the cones which stand on* C$'$, C$''$, C$'''$ *are circumscribed to the focal quadrics of the binodal which has* C *for a focal conic.*

129. From the theorems of the two last articles we infer at once the following, which is the one referred to at the commencement of art. 127:—*If* WU *be a sphero-quartic, the four cones having the centre of* U *for a common vertex, and standing on the confocal sphero-conics of* WU, *pass respectively through the focal conics of the four binodals of* WU, *and each is circumscribed to a focal quadric of each of three of these binodals.*

130. Let us denote the planes of circular section of the cones through WU by P, P′, then P passes through two of the four circular points at infinity on WU and P′ through the other two; and if Π, Π′ denote the focal lines of the four cones of recent articles, we see that the tangent planes to the quartic cone Q of art. 126, which touch it at the circular points at infinity, intersect two by two in the lines Π, Π′. Hence we have the following theorem:—*The focal lines of the four confocal cones of* Q *are the double focal lines of* Q *itself.*

Cor. 1. Since every quadric has six planes of circular section, including real and imaginary, *we infer that the cone* Q *has six double focal lines.*

Cor. 2. Since the points in which the focal lines of Q intersect the sphere U are double foci of WU, *it follows that every sphero-quartic has six double foci.*

131. The theorem of the last article may be established as follows. Any plane I will cut the sphero-quartic in four points; these points are common to the four cones passing through the sphero-quartic. Hence, by reciprocation with respect to U, through any point i can be drawn four planes to intersect the planes of the nodal conics of Σ, each in four lines, forming four tetragrams described about the nodal conics. And since each tetragram has six angular points, from any point i can be drawn six lines piercing the planes of the nodal conics each in six points, which will be the angular points of tetragrams described about the nodal conics; and by supposing the plane I to be at infinity, the point i will be the centre of U, and the six lines will be the six focal lines of the four confocal cones.—Q.E.D.

132. When the circle of inversion α touches the focal sphero-conic F, the sphero-quartic has a double point, and it is the spheric inversion of a sphero-conic (see art. 81), or the ordinary inversion of a plane conic from a point outside the plane of the conic; and the cyclide WU, which will be got from the equation of the sphero-quartic by putting spheres for circles, as previously explained, will be the inversion of a central quadric. Again, the quartic cone Q, got by substituting single sheets of cones for the circle, will have a double line and three focal cones.

133. When α is an osculating circle of F, the sphero-quartic has a cusp. This species of sphero-quartic is the spheric inversion of a spherical parabola, that is, a sphero-conic whose major axis is a quadrant, or the ordinary inversion of a plane parabola from a point outside the plane of the parabola. The cyclide WU will be the inverse of a non-central quadric, and the cone Q will have a cuspidal edge, and but two confocal cones.

134. It is shown in art. 28 of 'Bicircular Quartics' that the equation of every bicircular can be written in the form $\Sigma\Sigma'=k^2C$, where Σ, Σ′ are circles whose centres are the double foci of the quartic; and it is easy to see that this is equivalent to the equation in elliptic coordinates,

$$\mu^2-\nu^2=k\sqrt{C}.$$

Hence by inversion we see that any sphero-quartic can be written in the form

$$\Sigma\Sigma'=k^2C, \qquad (100)$$

where Σ, Σ' are small circles, whose centres are the double foci of the sphero-quartics, and that this is again equivalent to the equation in elliptic coordinates,

$$\mu^2-\nu^2=k\sqrt{C}. \qquad (101)$$

135. If in the equation $\Sigma\Sigma'=k^2C$ we put

$$\Sigma'=\Sigma''\pm h^2,$$

we get

$$\Sigma(\Sigma''\pm h^2)=k^2C,$$

and the intersections of the circle Σ'' with the sphero-quartic are also the points of intersection of the circle $k^2C\pm h^2\Sigma$ with the sphero-quartic. Hence the sphero-quartic meets the circle Σ'' only in two points. Hence we have the following theorem:—*Any circle whose plane is perpendicular to a double focal line of a sphero-quartic meets the sphero-quartic only in two points.*

136. Let P, P′ be the points in which the circle Σ'' meets the sphero-quartic, then PP′ will be a generator of a paraboloid passing through the quartic. Hence we easily infer the following theorem:—*If* H, H′, *the double foci of a sphero-quartic, be joined to any point* P *of the quartic, and circles described with radii* HP, H′P *cutting the sphero-quartic again in* P′, P″ *respectively, the lines* PP′, PP″ *are parallel to the planes of circular sections of quadrics passing through the quartic, and they are the generators through* P *of one of the paraboloids which can be drawn through the sphero-quartic.*

Cor. Since three paraboloids can be drawn through the sphero-quartic, this theorem affords another proof that a sphero-quartic has six double focal lines.

137. If we take the canonical form of a sphero-quartic $a\alpha^2+b\beta^2+c\gamma^2+d\delta^2=0$, we get precisely, in the same way as in art. 117, the following system of equations, each denoting a sphero-quartic confocal with the given sphero-quartic, that is, each having a quartet of foci common with it:—

$$\frac{(a-b)\beta^2}{(a-b)+k}+\frac{(a-c)\gamma^2}{(a-c)+k}+\frac{(a-d)\delta^2}{(a-d)+k}=0. \qquad (102)$$

$$\frac{(b-c)\gamma^2}{(b-c)+k'}+\frac{(b-d)\delta^2}{(b-d)+k'}+\frac{(b-a)\alpha^2}{(b-a)+k'}=0. \qquad (103)$$

$$\frac{(c-d)\delta^2}{(c-d)+k''}+\frac{(c-a)\alpha^2}{(c-a)+k''}+\frac{(c-b)\beta^2}{(c-b)+k''}=0. \qquad (104)$$

$$\frac{(d-a)\alpha^2}{(d-a)+k'''}+\frac{(d-b)\beta^2}{(d-b)+k'''}+\frac{(d-c)\gamma^2}{(d-c)+k'''}=0. \qquad (105)$$

In these equations the k's may have any value.

138. As in art. 118, we can show *that the reciprocals of two sphero-quartics having one quartet of foci common are two sphero-quartics having quartic contact at the points where they are intersected by their common circle of inversion* α.

139. *The two sphero-quartics*

$$a\alpha^2+b\beta^2+c\gamma^2+d\delta^2,$$

$$\frac{\alpha^2}{a}+\frac{\beta^2}{b}+\frac{\gamma^2}{c}+\frac{\delta^2}{d}$$

have the system of sixteen foci common to both. The proof is exactly the same as that of the corresponding theorem for two cyclides.

140. *Two sphero-quartics having four concyclic common foci can be described through any point, and they intersect orthogonally in their eight points of intersection.*

Demonstration. Let P be the given point, P′ the inverse of P with respect to the circle through the four common foci, then through the four common foci can be described two sphero-conics touching the great circle which bisects PP′ perpendicularly; these will be the focal sphero-conics of the required sphero-quartics, and the proposition is evident.

141. The construction in art. 124 may be proved as follows: from art. 131 we see that from any point can be drawn concyclic planes which will intersect the planes of the nodal conics of Σ in four tetragrams circumscribed to the nodal conics. Now if the points from which the four concyclic tangent planes are drawn be the pole of the plane of one of the nodal conics (that is, in fact, if it be one of the four centres of inversion of the sphero-quartic), the proposition is evident.

CHAPTER VIII.

Anharmonic Properties of the Developable Σ and its Reciprocal.

142. Let us consider the cyclide $W+kU^2=0$, where

$$W\equiv a\alpha^2+b\beta^2+c\gamma^2+d\delta^2,$$

$$U^2\equiv \alpha^2+\beta^2+\gamma^2+\delta^2;$$

then the tangential equation of the focal quadric of $W+kU^2$ is

$$(a+k)\lambda^2+(b+k)\mu^2+(c+k)\nu^2+(d+k)\varrho^2=0,$$

and this in tetrahedral coordinates is

$$\frac{x^2}{a+k}+\frac{y^2}{b+k}+\frac{z^2}{c+k}+\frac{w^2}{d+k}=0; \quad \text{. . . . .} \quad (106)$$

the discriminant of this with respect to k will be the developable Σ circumscribed to U along the sphero-quartic WU.

143. The differential of (106) with respect to k, gives

$$\frac{x^2}{(a+k)^2}+\frac{y^2}{(b+k)^2}+\frac{z^2}{(c+k)^2}+\frac{w^2}{(d+k)^2}=0; \quad \text{. . . . . .} \quad (107)$$

and the intersection of the quadrics (106) and (107) will be the locus of the centres of the generating spheres passing through the sphero-quartic WU of the cyclide $W+kU^2$; and this curve, namely the intersection of (106) and (407), is a cuspidal edge on the surface of centres of $W+kU^2$. *Hence we see that the locus of all the cuspidal edges for all the surfaces* $W+kU^2$ *is the developable* Σ *circumscribed to* U *along* WU.

Cor. 1. The cuspidal edge of the surface of centres of any cyclide of the system $W+kU^2$ is a quartic of the first family. (See SALMON, p. 274.)

Cor. 2. These cuspidal edges have another and a more important geometrical signification; they are the curves in which the quadrics of the system

$$\frac{x^2}{(a+k)}+\frac{y^2}{(b+k)}+\frac{z^2}{(c+k)}+\frac{w^2}{(d+k)}=0$$

touch the envelope; on this account I shall call them *curves of taction.*

144. The envelope of the quadric (107) is CLEBSCH's Surface of Centres (see SALMON, page 399). If we form the tangential equation of this quadric, we get

$$(a+k)^2\lambda^2+(b+k)^2\mu^2+(c+k)^2\nu^2+(d+k)^2\varrho^2;$$

and this is the focal quadric of the cyclide

$$W'+2kW+k^2U^2=0, \quad \ldots \ldots \ldots \ldots \quad (108)$$

where

$$W'\equiv a^2\alpha^2+b^2\beta^2+c^2\gamma^2+d^2\delta^2.$$

The envelope of (108) is the cyclide

$$W'U^2=W^2, \quad \ldots \ldots \ldots \ldots \quad (109)$$

a surface of the eighth degree. This is the envelope of all the spheres whose centres move on CLEBSCH's Surface of Centres, and which cut a given sphere orthogonally.

145. The lines of Σ are cut homographically by its curves of taction.

Demonstration. Σ is the envelope of all the quadrics,

$$(a\lambda^2+b\mu^2+c\nu^2+d\varrho^2)+k(\lambda^2+\mu^2+\nu^2+\varrho^2);$$

and by giving four different values to k, say k', k'', &c., the anharmonic ratio of the four points in which any line of Σ is divided by the corresponding lines of taction is

$$(k'-k'')(k'''-k''''):(k'-k''')(k''-k''''). \quad \ldots \ldots \ldots \quad (110)$$

Hence the proposition is proved.

A particular case is that the anharmonic ratio is constant of the four points in which any line of Σ is divided by its four nodal conics; the value of this anharmonic ratio is

$$(a-b)(c-d):(a-c)(b-d). \quad \ldots \ldots \ldots \ldots \quad (111)$$

146. *The envelope of tangent lines to the curves of taction of Σ at all the points where any line of Σ meets them is a plane conic which touches the cuspidal edge.*

Let L, L′ be two consecutive lines of Σ; then, since L, L′ are divided homographically by the curves of taction, the proposition is evident.

147. If L, L′ be two non-consecutive lines of Σ, the lines joining the points where they meet the curves of taction generate a ruled quadric; this is evident, since the curves of taction divide L, L′ homographically.

Cor. The lines joining the four pairs of points in which L, L′ meet the double lines of Σ are generators of a ruled quadric.

148. The reciprocal of the developable Σ is the developable formed by the tangent lines of the sphero-quartic WU. We shall denote this latter developable by Δ.

If we reciprocate the *Cor.* in the last article we get the following theorem:—

If the four centres of inversion of WU *be joined by planes to two non-consecutive lines of* WU, *the four lines of intersection of the homologous pairs of planes are generators of a ruled quadric.*

149. *The lines of* Σ *divide its nodal conics homographically.*

Demonstration. Let five lines of Σ, namely L, L′, &c., meet its nodal conics in four systems of five points, namely l, l', &c., m, m', &c., n, n', &c., p, p', &c.; then, by art. 147, the four systems of lines

$$\begin{array}{cccc} ll', & mm', & nn', & pp', \\ ll'', & mm'', & nn'', & pp'', \\ ll''', & mm''', & nn''', & pp''', \\ ll'''', & mm'''', & nn'''', & pp'''' \end{array}$$

are generators of four hyperboloids, H, H′, H″, H‴, and the planes $\text{L}ll'$, $\text{L}ll''$, $\text{L}ll'''$, $\text{L}ll''''$ are tangent planes to H, H′, &c.; hence the anharmonic ratios are equal, $l\{l', l'', l''', l''''\}$ $\{\text{H}\,\text{H}'\,\text{H}''\,\text{H}'''\}$. Hence the proposition is proved.

150. By reciprocating art. 149 we get the following theorem:—*If four tangent planes be drawn through any four lines of the system* WU *to one of the four cones through* WU, *the anharmonic ratio of these four tangent planes is equal to the anharmonic ratio of the four tangential planes drawn through the same lines of* WU *to any of the three remaining cones.*

Cor. From this proposition we infer the following theorem:—*The anharmonic ratio of the four edges of one of the four cones of* WU *passing through any four points on* WU *is equal to the anharmonic ratio of the four edges passing through the same points of any of the three remaining cones.*

151. Since the sphero-quartic WU is a curve of taction on Σ, the tangent line to WU at the point where L cuts it (see art. 146) is a tangent line to the conic of art. 146. This theorem may be enunciated as follows:—*A tangent plane to the sphere* U *at any point* P *of the sphero-quartic* WU *intersects the four faces of the tetrahedron whose vertices are the centres of inversion of the sphero-quartic in four lines; and the conic determined by these lines and the tangent line to* WU *at* P *will also touch the line of contact of the plane with* Σ *at the cuspidal edge of* Σ.

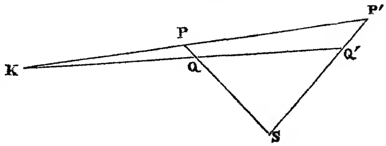
We shall have to refer so frequently to the tetrahedron formed by the centres of inversion of WU, that in order to avoid circumlocution I shall simply call it *the tetrahedron.*

152. By reciprocating with respect to U we get from the theorem of the last article this other theorem:—*The four lines drawn from any point* P *of a sphero-quartic* WU *to its four centres of inversion, the line of* Σ *passing through* P, *and the line of* Δ *through* P *are edges of the same cone of the second degree, which has also the osculating plane of* WU *at* P *for a tangent plane.*

The cone possesses this property also; namely, the anharmonic ratio is constant of the four edges passing through the centres of inversion.

153. Let us now consider the developable Δ reciprocal of Σ. This is formed by the tangent lines of WU. Let K be one of the centres of inversion of WU, P, P′, Q, Q′ two pairs of inverse points of WU; then, if P, Q be consecutive points, PQ, P′Q′ are two lines of Δ, and their point of intersection, S, is a point on two lines, and the locus of S will be a double or nodal line of Δ.

Fig. 4.



Now we have seen that, W being $a\alpha^2+b\beta^2+c\gamma^2+d\delta^2$, $U^2\equiv\alpha^2+\beta^2+\gamma^2+\delta^2$, the sphero-quartic WU will be the intersection of the quadrics

$$ax^2+by^2+cz^2+dw^2=0,$$
$$x^2+y^2+z^2+w^2=0,$$

and the equation of the nodal line of the developable Δ is (see SALMON's 'Geometry of Three Dimensions,' art. 209)

$$\frac{(c-c)^2}{bcx^2}+\frac{(c-a)^2}{cay^2}+\frac{(a-b)^2}{abz^2}=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (112)$$

The same equation may be easily inferred from 'Bicircular Quartics,' art. 43.

Hence each of the four nodal lines of Δ is a quartic curve having three double points, the double points being at the centres of inversion, which are in the plane of the nodal line and passing through the four single foci of the sphero-quartic which lie in the plane of the nodal line.

154. Every line of Δ has a corresponding line in Σ; and by art. 146 any line of Δ and the corresponding one of Σ are tangents to a conic, which also touches the four lines in which their plane intersects the faces of the tetrahedron. Hence any line of Δ, and the corresponding line of Σ, are divided homographically by the faces of the tetrahedron; but the lines of Σ are divided in a given anharmonic ratio by these faces (see art. 145). Hence *the lines of Δ are divided in a given anharmonic ratio by its four nodal lines.*

Cor. If two lines L, L′ of Δ meet its nodal lines in two systems of four points l, m, n, p, l', m', n', p', the corresponding chords of the nodal lines ll', mm', nn', pp' are generators of an hyperboloid; for L, L′ are divided equianharmonically by the nodal lines of Δ.

155. *The nodal or double lines of Δ are homographic figures.*

For let five lines of Δ meet its double lines in the four systems of five points each, then are equal the four pencils

$$l\,\{l'\ l''\ l'''\ l''''\},\quad m\{m'\ m''\ m'''\ m''''\},$$
$$n\{n'\ n''\ n'''\ n''''\},\quad p\,\{p'\ p''\ p'''\ p''''\};$$

that is, the four pencils are equal which are formed by the corresponding chords of the four double lines. This follows exactly in the same way as the corresponding proposition of art. 149. Hence the proposition is proved.

156. By reciprocation we get from the *Cor.* of art. 154 the following theorem:—*If*

L, L', L'', L''', L^{iv} *be five lines of* Σ, *the plane joining* L *to any of the four centres of inversion will intersect the planes joining* L′, L″, *&c. to the same centre in four lines, whose anharmonic ratio will be independent of the centre used in the construction, or, in other words, will be the same for all the centres.*

157. Since the locus of all the points on two lines of Δ is a system of four plane curves, each of the fourth degree, and having three double points, it follows by reciprocation *that the envelope of all the planes through two lines of* Σ *is a system of four cones, each of the fourth class, and each cone having one of the vertices of the tetrahedron for vertex, and the three faces which meet in that vertex as double tangent planes.*

158. If L be a line of Δ, then the anharmonic ratio is constant of the pencil of planes through L to the vertices of the tetrahedron. Hence any face of the tetrahedron will intersect this pencil in four lines whose anharmonic ratio is constant. Now of the rays (four lines), three are lines from the point where L pierces the face of the tetrahedron to the three vertices in that face, and the fourth is a tangent to the cone in which the same face intersects one of the four cones through WU,—the three vertices forming a self-conjugate triangle with respect to that conic. Hence we have the following theorem:—*If from any point in one of the four nodal lines of* Δ *three lines be drawn to the three double points of that nodal line, and a fourth line be drawn tangential to the conic in which the fourth cone through* WU *pierces that face, then is constant the anharmonic ratio of the pencil thus formed.*

159. The following direct proof of the converse of the theorem of the last article, stated as a property of any quartic curve having three double points, was communicated to me by my friend J. C. Malet, Scholar of Trinity College, Dublin. *If from any point of a trinodal plane quartic three rays of a given anharmonic pencil be drawn to the nodes, the envelope of the fourth ray is a conic section.*

Let the quartic be given by the equation

$$x^2y^2+y^2z^2+z^2x^2+2(xyz)(Ax+By+Cz),$$

where (xy), (yz), (zx) are the three nodes, and let the point from which the pencil is drawn be $x'\ y'\ z'$, then three of the rays are evidently the system of determinants

$$\left\| \begin{matrix} x', & y', & z', \\ x, & y, & z, \end{matrix} \right\|$$

and these may be denoted by the concurrent systems $L=0$, $M=0$, $Lz'+My'=0$.

Now, if we denote the fourth ray of the pencil by $L+kM$, the conditions of the question give

$$\frac{y'}{z'}=\frac{k}{c}, \text{ where } c \text{ is constant;}$$

but

$$L+kM=x(kz'-y')+yx'-kzx'$$

$$=\lambda x+\mu y-\nu z \text{ suppose;}$$

and by comparing coefficients we get

$$kz'-y'=\lambda, \quad x'=\mu,$$
$$kx'=\nu, \qquad kz'-cy'=0.$$

Hence we have the following values for x', y', z',

$$x'=\mu\,;\ \ y'=\frac{\lambda}{c-1}\,;\ \ z'=\frac{c\lambda\mu}{(c-1)\nu}\,;$$

and these values, substituted in the equation of the given quartic, give after reduction

$$c^2\lambda^2+c^2(c-1)^2\mu^2+(c-1)^2\nu^2+2c(c-1)(\mathrm{A}\mu\nu+\mathrm{B}\nu\lambda+\mathrm{C}c\lambda\mu)=0,$$

the tangential equation of a conic.

Cor. By reciprocation we get the following not less interesting theorem:—

If any tangent T *to a curve of the fourth class having three double tangents intersect its three double tangents in the three points* A, B, C, *and if a fourth point* D *be taken on* T, *such that the anharmonic ratio* {A B C D} *is given, the locus of* D *is a conic section.*

160. If we take any point P on one of the four nodal lines of Σ, then through P can be drawn two lines of Σ, say L, L′; let these meet the other nodal lines of Σ in the two triads of points a, a', a'', b, b', b''; then, since the lines of Σ are divided equianharmonically by its nodal lines, the two ranges are equal, P$aa'a''$, P$bb'b''$. Hence the lines are concurrent, ab, $a'b'$, $a''b''$, the point of concurrence, being the vertex of the tetrahedron opposite to the plane of the node on which is taken the point P.

161. If we denote the four nodal conics of Σ by N, N′, N″, N‴, and if J be the section of the sphere U made by the face of the tetrahedron on which N lies, P the point where a common tangent PP′ of J and N touches N, then the lines of Σ which can be drawn through P are coincident; in fact the section of Σ made by the plane of N consists of the conic N repeated twice, and of the four common tangents of J and N, the equations of the common tangents being

$$x\sqrt{\frac{b-c}{bc}}\pm y\sqrt{\frac{c-a}{ca}}\pm z\sqrt{\frac{a-b}{ab}}=0 \quad . \ . \ . \ . \ . \ . \ . \quad (113)$$

(see SALMON's 'Geometry of Three Dimensions,' p. 161). Hence it follows from the last article that the common tangent PP′ meets each of the remaining nodal conics N′, N″, N‴, and that the tangents to N′, N″, N‴, at the points where PP′ meets them, are complanar and concurrent, the point of concurrence being the opposite vertex of the tetrahedron. Hence we easily infer the following theorem:—*The three nodal conics* N′, N″, N‴ *pass respectively through the three pairs of opposite intersections of the tetragram found by common tangents of* J *and* N. Compare art. 34, 124, and art. 38, 'Bicircular Quartics.'

162. From the theorems of this Chapter may be easily inferred properties of Bicircular Quartics; I give a couple of instances.

1°. Since the anharmonic ratio is constant of the four planes through any line of Δ to the vertices of the tetrahedron, these planes will cut the sphere U in four circles, which

four circles will belong one to each of the four systems of generating circles of the sphero-quartic WU; but if the sphero-quartic be inverted from any arbitrary point of U, it becomes a bicircular quartic. Hence the *anharmonic ratio is constant of the four generating circles of a bicircular quartic which touch each other at any point of the quartic.* See 'Bicircular Quartics,' art. 99.

2°. *If four points,* l, m, n, p, *be taken on a bicircular quartic and normals be drawn to the quartic at these points, the normals divide the focal conics of the quartic homographically.* This follows from art. 149.

163. Conversely, properties of sphero-quartics may be inferred from those of bicirculars.

If we take any line through two points E, F of the sphero-quartic WU, and through EF draw four planes each tocching WU in another point, these planes intersect U in four circles, which will become, if U be inverted into a plane, four circles intersecting a bicircular quartic in two common points and touching it, each in another point, but the anharmonic ratio of such a pencil of circles is constant (see 'Bicirculars,' art. 99). *Hence is constant the anharmonic ratio of the four planes through* EF.

Cor. 1. *If* A, B, C, D *be the four points where the planes through* EF *touch the sphero-quartic, the tangent lines to the quartic at* A, B, C, D (*that is, the lines of* Δ *through* A, B, C, D) *meet* EF *in four points whose anharmonic ratio is constant.*

Cor. 2. *The four lines of* Δ *of Cor.* 1 *are generators of a ruled quadric.*

Cor. 3. *If through the lines of* Δ *at* A, B, C, D (*that is, the four lines of Cor.* 1) *be drawn four planes intersecting the sphero-quartic in a common chord, if the common chord varies it will generate a ruled quadric.*

CHAPTER IX.

Osculating Circles of Sphero-quartics.

164. If we consider the cyclide

$$W \equiv a\alpha^2+b\beta^2+c\gamma^2+d\delta^2=0,$$

and the sphere U given by the equation

$$U^2 \equiv \alpha^2+\beta^2+\gamma^2+\delta^2=0,$$

then the quadric

$$ax^2+by^2+cz^2+dw^2,$$

which will be the reciprocal of the focal quadric of W, will pass through the sphero-quartic WU, and $x^2+y^2+z^2+w^2=0$ will be the equation of U in the same system of tetrahedral coordinates.

The section of these quadrics by the plane w will be the conic $ax^2+by^2+cz^2=0$ and the circle $x^2+y^2+z^2=0$. Now, following CLEBSCH, let us generalize the method of finding the evolute of $ax^2+by^2+cz^2$ (see SALMON'S 'Geometry of Three Dimensions,' art. 472). We have the following problem to solve, which will be the generalization of drawing a normal to a conic. Let it be required to find a point x, y, z on the conic $ax^2+by^2+cz^2$, such that the pole with respect to the circle $x^2+y^2+z^2$ of the tangent to the conic at

x, y, z shall lie on the line joining x, y, z to a given point x', y', z'; denoting the coordinates of any point on this latter line by $x'-\lambda x$, $y'-\lambda y$, $z'-\lambda z$, we find (as in SALMON, art. 472) that the generalized evolute of $ax^2+by^2+cz^2$ is the discriminant of the conic

$$\frac{ax^2}{(a+\lambda)^2}+\frac{by^2}{(b+\lambda^2)}+\frac{cz^2}{(c+\lambda)^2}=0$$

with respect to λ, and therefore the required evolute is the curve of the sixth degree

$$a^{\frac{1}{3}}(b-c)^{\frac{2}{3}}x^{\frac{2}{3}}+b^{\frac{1}{3}}(c-a)^{\frac{2}{3}}y^{\frac{2}{3}}+c^{\frac{1}{3}}(a-b)^{\frac{2}{3}}z^{\frac{2}{3}}=0; \quad \ldots \ldots \quad (114)$$

and the reciprocal of this with respect to the circle $x^2+y^2+z^2=0$ is the quartic curve

$$\frac{(b-c)^2}{bcx^2}+\frac{(c-a)^2}{cay^2}+\frac{(a-b)^2}{abz^2}=0. \quad \ldots \ldots \ldots \quad (115)$$

165. The equation (115) occurs so frequently in subsequent articles that we shall examine its properties with some detail. If in the equation of the developable Δ formed by the tangent lines of WU we make $w=0$, the result will be the square of (115). Hence we infer the following theorem:—*The nodal lines of the developable Δ are the reciprocals of the generalized evolute of the conics in which the reciprocals of the focal quadric are cut by the faces of the tetrahedron.*

166. If we invert the sphere U from one of the eight centres of inversion (see art. 83) into one of the faces of the tetrahedron, the sphero-quartic WU will invert into a bicircular; and it is easy to see that the nodal line of Δ in that face of the tetrahedron will be the locus of the intersection T of tangents to the bicircular at a pair of inverse points P, P′ (see art. 43, 'Bicirculars'); but the point T is evidently the centre of similitude of two consecutive generating circles of the bicircular. Hence the locus of T is the envelope of the axis of similitude of three consecutive generating circles of the bicircular. Hence we infer the following theorem:—*If a sphero-quartic WU be inverted into a bicircular on the plane of one of the faces of the tetrahedron, the nodal line of the developable Δ formed by the tangent lines of WU is the envelope of the radical axis of a pair of inverse osculating circles of the bicircular.*

167. The equation (115) may, by incorporating constants with the variables, be written in the form

$$\frac{1}{x^2}+\frac{1}{y^2}=\frac{1}{z^2};$$

and in this form it will be satisfied by the coordinate of the point common to the system of determinants

$$\left\|\begin{matrix} x, & y, & z, \\ \sec\phi, & \operatorname{cosec}\phi, & 1. \end{matrix}\right\|$$

If we call this the point ϕ, then the equation of the chord joining the points ϕ, ϕ' will be the determinant

$$\begin{vmatrix} x, & y, & z, \\ \sec\phi, & \operatorname{cosec}\phi, & 1, \\ \sec\phi', & \operatorname{cosec}\phi', & 1, \end{vmatrix}=0. \quad \ldots \ldots \ldots \quad (116)$$

Hence we find without difficulty the tangent to be given by the equation

$$x\cos^3\phi+y\sin^3\phi=z\,; \quad \text{. . . . . (117)}$$

and this is therefore the equation of a tangent to a nodal line of Δ.

168. *If from any point of the curve* $\frac{1}{x^2}+\frac{1}{y^2}=\frac{1}{z^2}$ *four tangents be drawn, the points of contact are in a right line.*

Demonstration. We shall simplify the proof by taking $z=$ unity. Let $x'\ y'$ be the point of contact, then the tangent is

$$\frac{x}{x'^3}+\frac{y}{y'^3}=1;$$

and if $(\alpha\ \beta)$ be a point where this meets the curve again, we have the equations

$$\frac{\alpha}{x'^3}+\frac{\beta}{y'^3}=1, \quad \text{. . . (1)}$$

$$\frac{1}{\alpha^2}+\frac{1}{\beta^2}=1, \quad \text{. . . (2)}$$

$$\frac{1}{x'^2}+\frac{1}{y'^2}=1. \quad \text{. . . (3)}$$

Hence from (1), (3) . . . $\frac{\alpha-x'}{x'^3}+\frac{\beta-y'}{y'^3}=0,$ . . . (4)

" " (2), (3) . . . $\frac{\alpha^2-x'^2}{\alpha^2x'^2}+\frac{\beta^2-y'^2}{\beta^2y'^2}=0,$ . . . (5)

" " (4) and (5) . . $\frac{\alpha^2}{(\alpha+x')x'}+\frac{\beta^2}{(\beta+y')y'}=0,$

or $(\beta x'-\alpha y')(\beta x'+\alpha y'-\alpha\beta)=0$. Therefore the line $\beta x'+\alpha y'-\alpha\beta=0$ passes through the points of contact, and the proposition is proved.

Cor. 1. The envelope of the line through the points of contact is a conic section; for if we seek the envelope of $\frac{x}{\alpha}+\frac{y}{\beta}=1$, subject to the condition $\frac{1}{\alpha^2}+\frac{1}{\beta^2}=1$, we get the conic section

$$x^2+y^2=1.$$

The reader is not to imagine from its form that this equation represents a circle.

Cor. 2. *The anharmonic ratio is constant of the four points in which the chord of contact meets the curve.* This follows at once by considering the pencil of four tangents from a point infinitely near the former one.

Cor. 3. *If four tangents be drawn to the evolute of a conic at the points where any tangent of the evolute meets it, these four tangents are concurrent, and the locus of their points of concurrence is a conic passing through the six cusps of the evolute.*

169. In the sphero-quartic WU, if P, P′ be inverse points with regard to one of its

spheres of inversion, (α) for instance, then the spheres orthogonal to U passing respectively through two triads of consecutive pairs of points at P, P′ will be osculating spheres of W, and their circles of intersection with U will be osculating circles of WU. The radical plane of the inverse pairs of osculating spheres will be a diametral plane of U, and will intersect the face of the tetrahedron in a line which will be a tangent line to the curve (115). Hence we have the following theorem:—*The envelope of the radical plane of a pair of inverse osculating spheres of a sphero-quartic is a cone of the fourth degree possessing the following properties:—*

1°. *It has three double edges passing through three vertices of the tetrahedron.*

2°. *It has six stationary tangent planes.*

3°. *If through ang edge four tangent planes be drawn, their edges of contact are complanar.*

4°. *The anharmonic ratio of the four edges of contact is constant.*

5°. *The envelope of the plane through the four edges of contact is a cone of the second degree touching the six stationary tangent planes.*

170. Let K be one of the vertices of the tetrahedron, and S one of the osculating circles of WU. I say the cone V, whose vertex is K and which stands on S, will have double contact with the cone whose vertex is K and which circumscribes U.

Demonstration. The cone which circumscribes U along S, and the cone whose vertex is at K and which circumscribes U, have plainly two common tangent planes; and these will evidently be tangent planes to V also. Hence the proposition is proved.

171. The cone V osculates the cone through WU having the same vertex as V. This is evident, since S passes through three consecutive points of WU. The planes of circular section of V are parallel to the plane of S, and to the plane of the inverse of S.

172. If we form the reciprocal of the cone V with respect to U, its vertex will be at the centre of U, its intersection with U will be a sphero-conic having double contact with a circle of inversion (see art. 170), (2°) osculating the corresponding focal sphero-conic (art. 171); 3°, the focal lines will pass through two points on the cuspidal edge of the developable Δ circumscribed along WU (art. 171). Hence we may enunciate the following theorem:—*If* J *and* F *denote the two cones whose vertices are at the centre of* U, *and which stand respectively on a circle of inversion and on a focal sphero-conic of the sphero-quartic* WU, *the cone standing on the cuspidal edge of* Δ *is generated by the focal lines of a variable cone which has double contact with* J *and which osculates* F.

173. The theorem of the last article has an analogue in the theory of bicircular quartics. This may be inferred from the one for sphero-quartics; but the following is a direct proof.

First we have to find the locus of the centre of a variable circle which touches one circle and which is orthogonal to another.

Let the variable circle be

$$x^2+y^2+2gx+2fy+c=0,$$

the touched circle

$$x^2+y^2+2g'x+2f'y+c'=0, \qquad (1)$$

the orthogonal circle

$$x^2+y^2+2g''x+2f''y+c''=0. \qquad (2)$$

The given conditions supply the two equations,

$$4(g^2+f^2-c)(g'^2+f'^2-c')=(2gg'+2ff'-c-c')^2,$$
$$2gg''+2ff''-c-c''=0.$$

Hence, eliminating c, and putting x, y in place of $-g$, $-f$, which are the coordinates of the centre of the variable circle, we get for the required locus

$$4(g'^2+f'^2-c')(x^2+y^2+2gx''+2f''y+c'')=\{2(g'-g'')x+2(f'-f'')y+c'-c''\}^2, \quad (118)$$

a conic which has double contact with the circle cut orthogonally, the radical axis of the two fixed circles being the chord of contact.

The focus of the conic is the centre of the fixed circle; this is most easily seen by taking the centre of the fixed circle as origin; then $f'=0$, $g'=0$, and the equation (118) becomes that of a conic having the focus as origin, namely

$$4c'(x^2+y^2)+(2g''x+2f''y+c'+c'')^2=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (119)$$

Now if the circle (1) be an osculating circle of a bicircular quartic, and the circle (2) one of its circles of inversion J, the conic (119) must have three consecutive points common with the focal conic of the quartic which corresponds to J, namely the centres of the three generating circles of the quartic which the circle (1) touches. Hence we see that the proposition is proved, *that the evolute of a bicircular quartic is the locus of the foci of a variable conic which has double contact with a circle of inversion of the quartic, and which osculates the corresponding focal conic.*

174. The theorem proved in the last article enables us to determine the degree of the evolute of a bicircular. For let ν be CHASLES's characteristic; that is, let ν be the number of conics osculating the focal conic F of a bicircular quartic, and having double contact with the corresponding circle of inversion J, which can be described to touch a given line; then the required degree will be 3ν. Hence the degree of the evolute will be known when ν is found. We shall prove in the next article that ν is 12; therefore the degree is 36; but this number suffers a reduction, as we shall prove that it includes the line at infinity taken 24 times. Hence the reduced degree is 12.

175. To find CHASLES's characteristic ν for a system of conics osculating one given conic and having double contact with another given conic. Our solution will depend, 1°, on the question, If a variable conic touch a given line, and have double contact with a given fixed conic, to find the envelope of its chord of contact with the fixed conic.

This is solved as follows. The condition that the conic $S+P''^2$ should touch P' is the tact-invariant

$$(1+S'')S'-R^2=0 \text{ (see art. 46).}$$

Let $S=x^2+y^2+z^2$, $P''=\lambda x+\mu y+\nu z$, $P'=\lambda' x+\mu' y+\nu' z$, and the tact-invariant gives

λ, μ, ν connected by an equation of the second degree. Hence the envelope is a conic section.

2°. On the question, If a variable conic osculate one conic, and have double contact with another given conic, to find the envelope of the chord of contact.

Let the osculated conic be

$$ax^2+by^2+cz^2=0,$$

and the one of double contact

$$x^2+y^2+z^2=0,$$

then the variable conic must be of the form

$$x^2+y^2+z^2-(\lambda x+\mu y+\nu z)^2=0.$$

Now, if we want to describe a conic having double contact with $x^2+y^2+z^2$, where $\lambda x+\mu y+\nu z$ cuts it and touching $ax^2+by^2+cz^2$, the points of contact on $ax^2+by^2+cz^2$ will be given as the points of intersection of $ax^2+by^2+cz^2$ with the Jacobian of $ax^2+by^2+cz^2$, $x^2+y^2+z^2$, and $\lambda x+\mu y+\nu z$; that is, the points of contact will be the points of intersection of $ax^2+by^2+cz^2$ with the conic $\frac{\lambda(b-c)}{x}+\frac{\mu(c-a)}{y}+\frac{\nu(a-b)}{z}$; and if two of these points of intersection coincide, the conic which has double contact with $x^2+y^2+z^2$ will osculate $ax^2+by^2+cz^2$; hence we must form the condition that the conics touch

$$ax^2+by^2+cz^2=0,\quad \frac{\lambda(b-c)}{x}+\frac{\mu(c-a)}{y}+\frac{\nu(a-b)}{z}=0.$$

This is easily found to be

$$a^{\frac{1}{3}}(b-c)^{\frac{2}{3}}\lambda^{\frac{2}{3}}+b^{\frac{1}{3}}(c-a)^{\frac{2}{3}}\mu^{\frac{2}{3}}+c^{\frac{1}{3}}(a-b)^{\frac{2}{3}}\nu^{\frac{2}{3}}=0. \quad \ldots\ldots \quad (120)$$

Now, since this denotes a curve of the sixth class, and the former condition 1° a curve of the second, they will have twelve common tangents; hence $\nu=12$.

Cor. In the same way it may be proved that $\mu=12$.

176. We shall now return from our digression on bicirculars.

At the points where the nodal conic N of the developable Σ (see art. 161) cuts J, the osculating circle of the sphero-quartic WU cuts J orthogonally; and hence it is its own inverse with respect to the sphere α. Therefore the four points in which J cuts N are points of stationary osculation. Hence there are on a sphero-quartic sixteen points of stationary osculation.

Cor. The cone of articles 170, 171 in this case breaks up into two planes; and since the poles of the planes of the osculating circles of WU form the cuspidal edge of the developable Σ, *we see that* Σ *has sixteen stationary points which lie four by four on the four nodal conics* N, N′, N″, N‴, *the four stationary points on* N *being the four points of contact of the common tangents of* N *and* J; *and similarly for* N′, N″, N‴.

177. The sphero-quartic (WU) is the intersection of the two surfaces in tetrahedral coordinates,

$$ax^2+by^2+cz^2+dw^2=0,$$
$$x^2+y^2+z^2+w^2,$$

the first being the reciprocal of the focal quadric of W, and the second the sphere U. Now the osculating plane of WU at any point x', y', z', w' is (see SALMON'S 'Geometry of Three Dimensions,' p. 291)

$$(a-b)(a-c)(a-d)x'^3x+(b-a)(b-c)(b-d)y'^3y$$
$$+(c-a)(c-b)(c-d)z'^3z+(d-a)(d-b)(d-c)z'^3w=0.$$

This may be written in a simpler manner: thus, if $\psi(\lambda)$ denotes a biquadratic whose roots are a, b, c, d, the coefficients of the above equation denote the results of substituting the roots a, b, c, d respectively in $\psi'(\lambda)$, so that the equation becomes

$$\psi'(a)x'^3x+\psi'(b)y'^3y+\psi'(c)z'^3z+\psi'(d)w'^3w=0. \quad . \quad . \quad . \quad . \quad . \quad (121)$$

Hence through any point can be drawn twelve planes to osculate a sphero-quartic.

Cor. 1. *Through any point on the sphere* U *can be described twelve osculating circles of* WU. *Hence* CHASLES'S *characteristic* μ *for the osculating circles of a sphero-quartic is* $\mu=12$.

Cor. 2. *If the point be on the sphero-quartic itself,* $\mu=9$.

Cor. 3. *Every sphero-quartic is osculated by twelve great circles*; for twelve osculating planes can be drawn through the centre of U.

Cor. 4. Let us consider any small circle Z on the surface of U; then, since through the pole of the plane of Z can be drawn twelve planes osculating WU, we have the theorem that *any circle on the surface of* U *is cut orthogonally by twelve osculating circles of* WU.

Cor. 5. By inversion we get the following theorem for bicirculars:—*Any circle in the plane of a bicircular is cut orthogonally by twelve of its osculating circles.*

Cor. 6. The theorems that a bicircular quartic has twelve, and that a circular cubic has nine points of inflection, are the inversions of *Cors.* 1, 2.

178. Since the cuspidal edge of Σ is the locus of the poles of the osculating planes of WU, it is plain that the cone whose vertex is any point of the cuspidal edge, and which circumscribes U, will touch U along an osculating circle of WU, and that it will be an osculating right cone of the cuspidal edge (see SALMON'S 'Geometry of Three Dimensions,' art. 363). Again, since twelve osculating planes of WU pass through any point, we see that the cuspidal edge is of the twelfth degree. This latter part corresponds to the theorem that the evolute of a bicircular quartic is of the twelfth degree.

179. Since the cuspidal edge is of the twelfth degree, any quadric will cut it in 24 points. Hence any cone will in general cut it in 24 points. If the cone circumscribe U, we have, by reciprocation, the theorem that *any circle on the surface of* U *touches in general* 24 *osculating circles of* WU.

Cor. 1. By inversion we get the theorem that *any circle in the plane of a bicircular is in general touched by* 24 *osculating circles of the bicircular.*

Cor. 2. *Any line in the plane of a bicircular is in general touched by* 24 *of its osculating circles.*

Cor. 3. The line at infinity being touched by 24 osculating circles, shows that *the line at infinity is counted* 24 *times* in the evolute of a bicircular (see art. 174).

Cor. 4. CHASLES'S *characteristics for the osculating circles of a bicircular quartic are* $\mu=12$, $\nu=24$.

SECTION II.—*Locus of the Poles of the Osculating Circles of a Sphero-quartic.*

180. The equation (121) is the osculating plane of WU at the point $x'y'z'w'$; and if the coordinates of the pole of this plane with respect to U be X, Y, Z, W, we get

$$x'=\left(\frac{X}{\Psi'(a)}\right)^{\frac{1}{3}},\quad y'=\left(\frac{Y}{\Psi'(b)}\right)^{\frac{1}{3}},\ \&c.;$$

but x', y', z', w' satisfy the two equations

$$ax^2+by^2+cz^2+dw^2=0,\quad x^2+y^2+z^2+w^2=0.$$

Hence, by substitution and replacing X, Y, Z, W by x, y, z, w, we see that the locus of the poles of the osculating circles, or, what is equivalent, that the cuspidal edge of Σ is the intersection of the two surfaces

$$\left(\frac{x}{\Psi'(a)}\right)^{\frac{2}{3}}+\left(\frac{y}{\Psi'(b)}\right)^{\frac{2}{3}}+\left(\frac{z}{\Psi'(c)}\right)^{\frac{2}{3}}+\left(\frac{w}{\Psi'(d)}\right)^{\frac{2}{3}}=0, \quad . \quad . \quad . \quad . \quad . \quad (122)$$

$$a\left(\frac{x}{\Psi'(a)}\right)^{\frac{2}{3}}+b\left(\frac{y}{\Psi'(b)}\right)^{\frac{2}{3}}+c\left(\frac{z}{\Psi'(c)}\right)^{\frac{2}{3}}+d\left(\frac{w}{\Psi'(d)}\right)^{\frac{2}{3}}=0. \quad . \quad . \quad . \quad . \quad (123)$$

181. *Since the equation (121) of the osculating plane is satisfied by the coordinates* of any point in it, we must have

$$\Psi'(a)x'^4+\Psi'(b)y'^4+\Psi'(c)z'^4+\Psi'(d)w'^4=0;$$

and substituting as in the last article, we see that the cuspidal edge of Σ is a curve on the surface

$$\left\{\frac{x^4}{\Psi'(a)}\right\}^{\frac{1}{3}}+\left\{\frac{y^4}{\Psi'(b)}\right\}^{\frac{1}{3}}+\left\{\frac{z^4}{\Psi'(c)}\right\}^{\frac{1}{3}}+\left\{\frac{w^4}{\Psi'(d)}\right\}^{\frac{1}{3}}=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (124)$$

Or we may prove this theorem otherwise. The developable Σ is the envelope of the quadric

$$\frac{x^2}{a+k}+\frac{y^2}{b+k}+\frac{z^2}{c+k}+\frac{w^2}{d+k}=0 \text{ (see art. 142).}$$

Hence the coordinates of any point on the cuspidal edge must satisfy the system of equations:

$$\frac{x^2}{(a+k)^3}+\frac{y^2}{(b+k)^3}+\frac{z^2}{(c+k)^3}+\frac{w^2}{(d+k)^3}=0, \quad \ldots \quad (125)$$

$$\frac{ax^2}{(a+k)^3}+\frac{by^2}{(b+k)^3}+\frac{cz^2}{(c+k)^3}+\frac{dw^2}{(d+k)^3}=0, \quad \ldots \quad (126)$$

$$\frac{a^2x^2}{(a+k)^3}+\frac{b^2y^2}{(b+k)^3}+\frac{c^2z^2}{(c+k)^3}+\frac{d^2w^2}{(d+k)^3}=0. \quad \ldots \quad (127)$$

Hence

$$\frac{x^2}{(a+k)^3}:\frac{y^2}{(b+k)^3}:\frac{z^2}{(c+k)^3}:\frac{w^2}{(d+k)^3}::\frac{1}{\psi'(a)}:\frac{1}{\psi'(b)}:\frac{1}{\psi'(c)}:\frac{1}{\psi'(d)}; \quad \ldots \quad (128)$$

and substituting the values of $(a+k)$, $(b+k)$, &c. from these equations in

$$\frac{x^2}{a+k}+\frac{y^2}{b+k}+\&c.,$$

we get the equation (124). Hence &c.

Cor. 1. By giving k any particular value, we see from the equations (125), (126), (127), *that the points on the cuspidal edge of* Σ *are, eight by eight, the points of intersection of three quadrics.*

Cor. 2. From equations (126), (127) we see that the cuspidal edge is a curve on CLEBSCH's surface of centres; and from equation (109) it follows that *the sphero-quartic* WU *is a double line on the surface which has* CLEBSCH's *surface of centres for a déférente.*

182. By eliminating k from any three of the four equations (128), we get the equations of four cones standing on the cuspidal edge. Thus one of the cones is

$$(b-c)\{\psi'(a)x^2\}^{\frac{1}{3}}+(c-a)\{\psi'(b)y^2\}^{\frac{1}{3}}+(a-b)\{\psi'(c)z^2\}^{\frac{1}{3}}=0. \quad \ldots \quad (129)$$

The vertex of this cone is one of the vertices of the tetrahedron; it possesses several properties. The following are some of the most important:—

1°. *It intersects the opposite face of the tetrahedron in* CLEBSCH's *evolute of a conic* (see art. 164).

2°. *It is the reciprocal of the corresponding double line of* Δ—*that is, of the developable formed by the tangent lines of* WU.

3°. *Every edge of it is a line through two points of the cuspidal edge of* Σ.

4°. *Every tangent plane to it is a plane through two lines of* Σ, *and it is therefore one of the four cones which are the envelopes of all the planes through two lines of that developable.*

5°. The equation, cleared of radicals, is of the form

$$\{A^2+B^2+C^2\}^3=27A^2B^2C^2. \quad \ldots \quad (130)$$

Hence it has six cuspidal edges lying on the cone of the second degree, $A^2+B^2+C^2=0$.

6°. *Any tangent plane to it will intersect it in a pencil of four lines whose anharmonic ratio is constant.*

7°. *The tangent planes, touching it along the lines of intersection of any tangent plane,*

pass through a common line. This common line is an edge of the cone $A^2+B^2+C^2=0$ *passing through the six cuspidal edges.*

183. Let us consider an edge of the cone (129). It pierces U in two points; these are the limiting points of two inverse osculating circles of the sphero-quartic WU. The equation of the locus of these limiting points is easily found; for the tangential equation of the nodal line of Σ is the equation got by substituting λ, μ, ν in place of x, y, z in the equation of the cone. Hence, if α, β, γ be the three circles of inversion of WU, the poles of whose planes are at the three remaining vertices of the tetrahedron, the equation of the required locus will be got by substituting in the equation (129) α, β, γ for x, y, z, and therefore it will be

$$(b-c)\{\psi'(a)\alpha^2\}^{\frac{1}{3}}+(c-a)\{\psi'(b)\beta^2\}^{\frac{1}{3}}+(a-b)\{\psi'(c)\gamma^2\}^{\frac{1}{3}}=0, \quad . \quad . \quad . \quad (131)$$

a curve which has twenty four cusps.

184. If x', y', z', w' be the coordinates of any point in the sphero-quartic WU, then it follows from equation (121), combined with art. 36, that the equation of the osculating circle of WU at the point x' y' z' w' is

$$\psi'(a)x'^3(\alpha)+\psi'(b)(y'^3)(\beta)+\psi'(c)z'^3(\gamma)+\psi'(d)w'^3\delta=0,$$

where α, β, γ, δ are the circles of reference when the sphero-quartic is given by its canonical form.

CHAPTER X.

Classification of Cyclides.

185. Following the analogy of the method given in my memoir on 'Bicircular Quartics,' I shall take as the basis of classification the species of the focal quadric.

The principal varieties of quadrics are:—1°. An ellipsoid or hyperboloid. 2°. A sphere. 3°. A paraboloid. We shall find the cyclides corresponding to these varieties to have fundamental distinctions. We shall therefore devote a section to each.

SECTION I.—*Focal Quadric an Ellipsoid or Hyperboloid.*

186. Figure of cyclide. Let us denote the sphere of inversion by U, and the focal quadric by F. 1°. When the developable circumscribing U and F is imaginary, as for instance when F is an ellipsoid and U entirely within it, the cyclide evidently consists of two distinct sheets, which are inverse to each other with respect to U. One sheet is internal to U, and the other external; each sheet is a closed surface.

2°. When the developable is real, and when U does not intersect F, or else when it does intersect it in a sphero-quartic consisting of two distinct ovals, the cyclide W is made up of two closed surfaces, each of which is an anallagmatic, and divided by U into two parts. The points where W cuts U are the points of contact of the common tangent developable circumscribed to U and F.

3°. When the developable is real and the sphero-quartic of intersection of U and F

consists of one oval, W consists of one closed surface which is divided into two parts by U.

4°. When U touches F, the point of contact will be a nodal point on the cyclide, the cone of contact with the cyclide being real or imaginary according as U touches F on the concave or convex side of F (see art. 76).

Cor. If a cyclide has either a real or imaginary conic node (contracted by Professor CAYLEY into cnic-node), it arises from a real double point or a conjugate point on one of its focal sphero-quartics.

5°. When U has stationary contact with F, the point of osculation will be a biplanar node on the cyclide. In this case the cyclide will be the inverse of a non-central quadric (see art. 76).

6°. When U has double contact with F, the cyclide will be binodal.

7°. When U is inscribed in F (that is, when U touches F along a circle), the cyclide will break up into two spheres.

187. *Double Tangent Cones.*—Let us consider a cyclide whose focal quadric is F; then, taking the limiting points P, P′ of U and any tangent plane to F, the generating sphere through P, P′ will become a plane if its centre be at infinity, and the locus of the points P, P′ will evidently be a sphero-quartic, which is given as the intersection of a sphere concentric with F, and a cone whose edges are perpendicular to the tangent planes of the asymptotic cone of F, the vertex of the cone being the centre of U; this cone will be a double tangent cone. Hence we have the following theorem:—*Every cyclide has as many double tangent cones as it has focal quadrics.*

188. *The lines of intersection of a cyclide with its spheres of inversion are lines of curvature on the cyclide.*

For let us consider any point on the cuspidal edge of Σ, the developable which circumscribes U along WU; then that point is the centre of an osculating sphere of W (see art. 169). Hence WU is a line of curvature on W.

Cor. 1. The cuspidal edge of Σ is a geodesic on the surface of centres of W.

Cor. 2. The sphero-quartic reciprocal to W with respect to U^2 (that is, to $\alpha^2+\beta^2+\gamma^2+\delta^2$) is such that the focal sphero-quartic of W lying on the sphere U is a line of curvature on it.

189. *Binodal Cyclides of a Cyclide.*—We have seen (art. 33) that the cyclide W may be written in five different ways, (I.), (II.), (III.), (IV.), (V.). Now taking the first (I.), its equation is

$$(a-b)\beta^2+(a-c)\gamma^2+(a-d)\delta^2+(a-e)\epsilon^2=0,$$

and the square of the corresponding sphere of inversion is $\beta^2+\gamma^2+\delta^2+\epsilon^2$; and eliminating in succession each of the four letters β^2, γ^2, δ^2, ϵ^2, we get four binodal cyclides, each touching W along the line of curvature WU. *Hence every cyclide has in general four times as many binodals inscribed in it as it has spheres of inversion.*

190. *The imaginary circle at infinity is a flecnodal curve on the surface of centres of a cyclide.* This proposition is an extension of art. 52 in 'Bicircular Quartics.' It is proved as follows:—It is evident that the normal at any point of the imaginary circle at

infinity lies in the plane touching the cyclide along a tangent line to the circle at infinity; hence the tangent plane to the cyclide is also a tangent plane to the surface of centres.

Again, the sphere of curvature at any point P of a cyclide is the quadric through the imaginary circle at infinity and through four consecutive points at P; if P be any point on the circle at infinity, this quadric is indeterminate, and the pole of the circle at infinity is any point on the tangent plane at P. Hence any point on the tangent plane may be regarded as a point of intersection with a consecutive tangent plane; in other words, the tangent plane to the cyclide at any point along the imaginary circle at infinity is a stationary tangent plane to the surface of centres.

Cor. If the imaginary circle at infinity be a cuspidal curve on the cyclide, it will be a cuspidal curve on the surface of centres of the cyclide.

191. The points of contact of tangent lines from any point to a cyclide of the fourth degree W are the points of intersection of W with the polar cubic of the point; but this polar cubic is evidently a cubic cyclide. *Hence the tangent cone which circumscribes a cyclide and has any point for vertex reduces to the eighth degree by rejecting the square of the cone to the imaginary circle at infinity.* Or thus:—Draw any plane through the vertex of the cone; this plane will cut the cyclide in a bicircular quartic; and this quartic being of the eighth class, eight tangents can be drawn to it from the vertex of the cone.

192. *Class of Tangent Cone.*—Let V be the vertex of the tangent cone and V′ any other point, then the class of the tangent cone is plainly equal to the number of points common to W and the polar cubics of the points V, V′. Here we have three cyclides to consider, viz. W and the polar cubics. Let F, G, H be their focal quadrics; then F, G, H have eight common tangent planes; and corresponding to each common tangent plane there will be a pair of inverse points common to the cubics; therefore through the line V V′ sixteen tangent planes can be drawn to the cone; *the class of the tangent cone is therefore sixteen.*

193. The equation of the tangent cone from any point to a cyclide may be found as follows. Taking the point which is to be the vertex of the cone as origin, let the equation of the cyclide in Cartesian coordinates be written in the form

$$A(x^2+y^2+z^2)^2+4B(x^2+y^2+z^2)+6C+4D+E=0, \quad . \quad . \quad . \quad . \quad (132)$$

where A, E are constants,

$$B\equiv lx+my+nz,$$
$$D\equiv px+qy+rz,$$
$$C\equiv(a, b, c, f, g, h\between x, y, z)^2.$$

In polar coordinates this becomes, by putting $\varrho\cos\alpha=x$, $\varrho\cos\beta=y$, $\varrho\cos\gamma=z$, and putting for shortness

$$B'\equiv l\cos\alpha+m\cos\beta+n\cos\gamma,$$
$$D'\equiv p\cos\alpha+q\cos\beta+r\cos\gamma,$$
$$C'\equiv(a, b, c, f, g, h\between\cos\alpha, \cos\beta, \cos\gamma^2),$$
$$A\varrho^4+4B'\varrho^3+6C'\varrho^2+4D'\varrho+E=0;$$

forming the discriminant of this and returning to x, y, z coordinates, we get the equation of the tangent cone to be

$$\mathrm{I}^3-27\mathrm{J}^2=0, \qquad (133)$$

where

$$\mathrm{I} \equiv \mathrm{AE}(x^2+y^2+z^2)^2+4\mathrm{BD}(x^2+y^2+z^2)+3\mathrm{C}^2, \qquad (134)$$

$$\mathrm{J} \equiv \mathrm{ACE}(x^2+y^2+z^2)^2+2\mathrm{BCD}(x^2+y^2+z^2)-\mathrm{AD}^2(x^2+y^2+z^2)^2-\mathrm{EB}^2(x^2+y^2+z^2)^2-\mathrm{C}^3. \qquad (135)$$

194. From the form of the equation of the tangent cone, $\mathrm{I}^3-27\mathrm{J}^2=0$, it has twenty-four cuspidal edges; but from the forms of I and J we see that they have respectively, with the imaginary circle at infinity, contacts of the first and second order at each of the points where the cone C meets that circle. Hence the cuspidal edges coincide six by six with the four lines from the origin to these imaginary points; and it hence follows that, when we omit the factor $(x^2+y^2+z^2)^2$ in the equation (133), the remaining part, which represents a cone of the eighth degree, has no cuspidal edge. This equation is

$$\left.\begin{aligned}&\mathrm{A}^3\mathrm{E}^3\varrho^8-12\mathrm{A}^2\mathrm{BDE}^2\varrho^6\\ &\quad-(6\mathrm{AB}^2\mathrm{D}^2\mathrm{E}+18\mathrm{A}^2\mathrm{C}^2\mathrm{E}^2-54\mathrm{A}^2\mathrm{CD}^2\mathrm{E}+27\mathrm{A}^2\mathrm{D}^4-54\mathrm{AB}^2\mathrm{CE}^2+27\mathrm{B}^4\mathrm{E}^2)\varrho^4\\ &\quad-(180\mathrm{AB}^2\mathrm{C}^2\mathrm{DE}-108\mathrm{ABCD}^3+64\mathrm{B}^3\mathrm{D}^3)\varrho^2\\ &\quad-(54\mathrm{AC}^3\mathrm{D}^2-81\mathrm{AC}^4\mathrm{E}+54\mathrm{B}^2\mathrm{C}^3\mathrm{E}+36\mathrm{B}^2\mathrm{C}^2\mathrm{D}^2)=0\end{aligned}\right\}. \qquad (136)$$

In this equation for shortness we have written ϱ^2 for $x^2+y^2+z^2$.

Cor. If the origin be on the cyclide $\mathrm{E}=0$, and the tangent cone reduces to the square of the tangent plane to the cyclide at the origin and a cone of the sixth degree,

$$27\mathrm{A}^2\mathrm{D}^2\varrho^4+4\mathrm{BD}(16\mathrm{B}^2-27\mathrm{AC})\varrho^2-18\mathrm{C}^2(2\mathrm{B}^2+3\mathrm{AC})=0. \qquad (137)$$

195. The cone J is such that every edge of it is cut harmonically by the cyclide; and therefore, if any edge of it meet the cyclide in two coincident points, there must be a third point coincident; therefore, since the imaginary circle at infinity is a double line on the surface. the points where J meets it are such that every edge which passes through it is an inflectional tangent. Hence from any point can be drawn to a cyclide twelve lines, which are inflectional tangents to it at the imaginary circle at infinity; and these lines are distributed into four sets of three lines each, each triad consisting of three consecutive lines.

196. If the cyclide W has a double point, its class is diminished by two; if it has a biplanar node, its class will be diminished by three; if it has two nodes, its class will be diminished by four. The following Table contains the singularities of the tangent cones for each of these cases:—

No node.	Conic node.	Biplanar node.	Two nodes.
$m=8$,	$m=8$,	$m=8$,	$m=8$,
$n=16$,	$n=14$,	$n=13$,	$n=12$,
$\varkappa=0$,	$\varkappa=6$,	$\varkappa=9$,	$\varkappa=8$,
$\delta=20$,	$\delta=12$,	$\delta=8$,	$\delta=10$,
$\iota=24$,	$\iota=24$,	$\iota=24$,	$\iota=20$,
$\tau=80$,	$\tau=51$,	$\tau=38$,	$\tau=32$.

SECTION II.—*Focal Quadric a Sphere.*

197. When the focal quadric is a sphere, the cyclide has the imaginary circle at infinity as a cuspidal edge; on this account we shall call the surface a Cartesian cyclide.

Figure of the Surface.

1°. When U is external to F, W consists of two distinct sheets, each intersecting U in a circle. Each sheet is a closed surface.

2°. When U is internal to F, W consists of one sheet internal to U, and another sheet the inverse of the former, and therefore external to U. Each sheet is a closed surface.

3°. When U intersects F, W consists of one sheet; this intersects U in one real circle and another imaginary circle. The sheet is a closed surface.

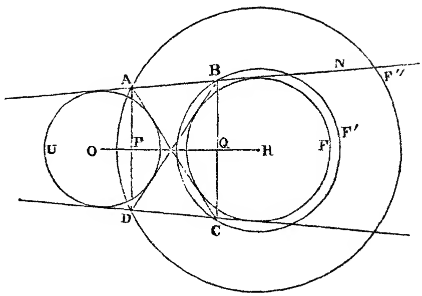
4°. When U touches F internally, W has a conic node, the tangent cone to W at the node being a real cone of revolution.

5°. When U touches externally, W has a conic node at which the tangent cone is imaginary.

6°. When U reduces to a point, W is the pedal of a sphere, and is therefore the inverse of a quadric of revolution from the focus.

198. In the annexed diagram, which is supposed to be a plane section through the centres of U and F, let B, C and A, D be the opposite pairs of the intersections of the tangent cones circumscribed about U and F made by the plane of the section, then the spheres F′, F″ concentric with F, and passing respectively through B, C and A, D, are

Fig. 5.



focal spheres (see art. 34) of the Cartesian cyclide; for the developable circumscribed about U and F reduces in this case to two cones of revolution, and the nodal lines of the geometric system composed of the two cones are two circles which are intersected by the plane of the section in B, C, A, D and the vertices of the cones. Hence the line O H passing through the two vertices must be regarded as a limiting case of an hyperboloid confocal with the spheres F, F′, F″ (see art. 106). Hence the focal quadrics of a Cartesian cyclide are three concentric spheres and a straight line through their centre.

199. Let OH intersect AD and BC in P and Q, then P and Q are the limiting points of U and F; and if we denote the radii of U and F by r and R, we have $\mathrm{OP}.\mathrm{OQ}=r^2$, and $\mathrm{HP}.\mathrm{HQ}=\mathrm{R}^2$; but since the lines AO, AH are evidently the bisectors at A of the supplemental angles made by the tangents, they are at right angles to each other, and in like maner the lines OB, BH are at right angles to each other. Hence the points Q and O are inverse points with respect to F′, and P and O with respect to F″. Again, it is easy to see that Q and O are inverse points with respect to the imaginary sphere U′ whose centre is P, and which cuts U orthogonally, and P and O with respect to U″ whose centre is Q, and which cuts U orthogonally. Hence the limiting points of U′ and F′ are the centres of U″ and U; the limiting points of U″ and F″ are the centres U and U′; so that the limiting points of any U and its corresponding F are the centres of the two remaining U's or spheres of inversion.

200. The centres of inversion of a Cartesian cyclide are foci of the surface.

Demonstration. Let the equations of U and F be

$$x^2+y^2+z^2=r^2, \text{ and } (x+a)^2+y^2+z^2=\mathrm{R}^2,$$

then the perpendicular OT let fall from O, the centre of U on a tangent plane to F, is evidently equal to $\mathrm{R}-a\cos\theta$, where θ is the angle which the perpendicular makes with the axis of x: and if P, P′ be points on OT such that $\mathrm{OT}^2-\mathrm{TP}^2=\mathrm{OT}^2-\mathrm{TP}'^2=r^2$, then P, P′ are points on the Cartesian cyclide; and denoting OP by ϱ, we have

$$2(\mathrm{R}-a\cos\theta)\varrho=r^2+\varrho^2,$$

or

$$2\mathrm{R}\varrho=x^2+y^2+z^2+2ax+r^2;$$

that is,

$$4\mathrm{R}^2(x^2+y^2+z^2)=(x^2+y^2+z^2+2ax+r^2). \quad . \quad . \quad . \quad . \quad . \quad (138)$$

Hence $x^2+y^2+z^2=0$ is an imaginary cone circumscribed to the cyclide. Hence the centre of U is a focus of the surface.

201. The equation (138) may evidently be written in the form $\mathrm{S}^2=b^3\mathrm{L}$, where S is a sphere and L a plane, showing that the imaginary circle at infinity is a cuspidal edge on the surface.

The equation of the sphere S is found to be

$$x^2+y^2+z^2+2ax+r^2-2\mathrm{R}^2=0;$$

and this is concentric with the focal spheres F, F′, F″; but the centre of S is a triple focus, as appears from the equation $\mathrm{S}^2=b^3\mathrm{L}$.

Hence the common centre of the three focal spheres F, F′, F″ *is the triple focus of the Cartesian cyclide. The Cartesian cyclide has a tangent plane which touches it along a circle; the plane is* L*; and the circle of contact is the circle of intersection of* S *and the plane* L.

202. Since being given the sphere of inversion U and the focal sphere F the Cartesian cyclide is determined, we see that a Cartesian cyclide is determined by eight constants. The same thing appears from the equation $\mathrm{S}^2=b^3\mathrm{L}$. From this equation also

we see that if a Cartesian cyclide be intersected by any plane the curve of intersection will be a Cartesian oval; for the equation will be of the form $S^2=b^3l$, where S and l denote the circle and line in which the sphere S and the plane L are intersected by the plane.

203. From the equation (138) we see that the Cartesian cyclide is the envelope of the variable sphere

$$x^2+y^2+z^2+\mu(x^2+y^2+z^2+2ax+r^2)+\mu^2R^2;$$

and if we form the discriminant of this, we get

$$(1+\mu)^2\{(1+\mu)(\mu r^2+\mu^2R^2)-a^2\mu^2\}=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (139)$$

Now the factor $(1+\mu)^2=0$ gives $\mu=-1$; and for this value of μ the variable sphere becomes a plane, namely the tangent plane which touches the cyclide along a circle; the remaining factor,

$$(1+\mu)(\mu r^2+\mu^2R^2)-a^2\mu^2=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (140)$$

gives three values of μ, for each of which the variable sphere becomes an imaginary cone (that is, a point sphere), showing that there are three collinear single foci along the axis of the cyclide. The value $\mu=0$ shows that the origin is a focus, which we knew before; and the values giving the other foci are the roots of the quadratic

$$\mu^2R^2+\mu(R^2+r^2-a^2)+r^2=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (141)$$

204. Since the equation

$$x^2+y^2+z^2+\mu(x^2+y^2+z^2+2ax+r^2)+\mu^2R^2=0$$

is that of a sphere into whose equation an arbitrary constant enters in the second degree, its inverse with respect to any point will be a sphere into whose equation an arbitrary constant enters in the second degree; that is, *the inverse of a Cartesian cyclide will be a cyclide generated as the envelope of a variable sphere whose centre moves along a plane conic. It will therefore be a binodal cyclide.*

This also appears from the fact that the inverse of a focus is a focus; and since the Cartesian cyclide has three collinear single foci, the inverse surface will have four concyclic single foci, namely the inverses of the three collinear foci and the centre of inversion.

205. If we differentiate the equation (138) of the generating sphere with respect to μ, we get

$$x^2+y^2+z^2+2ax+r^2+2\mu R^2=0;$$

and if from $(1+\mu)$ times this result we subtract the equation (138), we get

$$2ax+r^2+(2\mu+\mu^2)R^2=0.$$

Hence the Cartesian cyclide is generated as the locus of the curve of intersection of the sphere

$$x^2+y^2+z^2+2ax+r^2+2\mu R^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (142)$$

with the plane

$$2ax+r^2+(2\mu+\mu^2)R^2=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (143)$$

From this it follows that a Cartesian cyclide is a surface of revolution—a fact which

we knew otherwise, it being the surface generated by the revolution of a Cartesian oval about the axis passing through the three collinear single foci.

206. In order to find the equation of the cone whose vertex is any point x', y', z', and which stands on the circle of intersection of the sphere (142) and the plane (143), let us suppose x'', y'', z'' to be any point on a radius vector from x', y', z' to any point of the circle; then, if the circle divides the distance between these points in the ratio $l:m$, we must substitute, by JOACHIMSTAL's method, $\frac{lx'+mx''}{l+m}\ \frac{ly'+my''}{l+m}\ \frac{lz'+mz''}{l+m}$ for x, y, z in (142) and (143); the results will be of the form

$$l^2S'+2lmP+m^2S''=0, \text{ and } lL'+mL''=0;$$

hence by eliminating $l:m$ and suppressing the double accents, we get the required equation after restoring the values of S', P, &c.:—

$$\left.\begin{aligned}&(2ax+r^2+(2\mu+\mu^2)R^2)^2(x'^2+y'^2+z'^2+2ax'+r^2+2\mu R^2)\\-&2(2ax+r^2+(2\mu+\mu^2)R^2)(2ax'+r^2+(2\mu+\mu^2)R^2)(xx'+yy'+zz'+ax+ax'+r^2+2\mu R^2)\\+&(2ax'+r^2+(2\mu+\mu^2)R^2)(x^2+y^2+z^2+2ax+r^2+2\mu R^2)=0.\end{aligned}\right\}(144)$$

207. Since the equation (144) involves the undetermined μ in the fourth degree, its discriminant with respect to μ will involve x, y, z in the twelfth degree, and this discriminant will be the equation of the tangent cone; but this will contain as a factor the cube of the imaginary cone from (x', y', z') to the imaginary circle at infinity (see SALMON's 'Geometry of Three Dimensions,' art. 521). Hence the reduced degree is six; and it can be shown, as in art. 94, that the reduced cone has no cuspidal edges.

We can show otherwise that the reduced degree is six; for any section of the cyclide made by a plane through the vertex of the cone is a Cartesian oval, and the class of a Cartesian oval is six; the degree of the cone therefore is six.

208. *Class of Tangent Cone.*—Let us take the equation $S^2=b^3L$ and find the polar cubic of the point x', y', z'. This will be of the form

$$2S\left(x'\frac{d}{dx}+y'\frac{d}{dy}+z'\frac{d}{dz}\right)S=b^3\left(x'\frac{d}{dx}+y'\frac{d}{dy}+z'\frac{d}{dz}\right)L;$$

and eliminating b^3 between this and the equation $S^2=b^3L$, we get

$$S\left(x'\frac{d}{dx}+y'\frac{d}{dy}+z'\frac{d}{dz}\right)L=2L\left(x'\frac{d}{dx}+y'\frac{d}{dy}+z'\frac{d}{dz}\right)S;$$

and since the operation $x'\frac{d}{dx}+y'\frac{d}{dy}+z'\frac{d}{dz}$ performed upon L reduces it to a constant, and performed upon S reduces it to a plane, this equation represents a quadric. Hence it is easy to see that the points of contact of tangent planes drawn through a line to the tangent cone are the intersections of three surfaces of the degrees 3, 2, 2; hence the class of the tangent cone is 12; and we have shown that its degree is six, and that it has no cuspidal edges. Hence all the singularities are determined.

SECTION III.—*Focal Quadric a Paraboloid.*

209. When the focal quadric is a paraboloid, the cyclide becomes a cubic surface passing through the imaginary circle at infinity. The varieties of this surface correspond to those of quartic cyclides, and may be briefly enumerated as follows:—

1°. When the developable circumscribed to U and F is imaginary, the surface consists of two sheets, one of which is a closed surface passing through the centre of U and altogether within U. The other sheet, which is its inverse of the first, is an open sheet, extending to infinity, which it intersects in a right line. The case considered here would occur if F were an elliptic paraboloid, and U in the concavity of it without meeting it.

2°. When the developable is real the surface consists, as in 1°, of two sheets, one of which is a closed surface and passes through the centre of U, the other sheet is infinite. Each sheet intersects U; and the part of each sheet internal to U is the inverse of the external part.

3°. When U intersects F in a single oval, the cyclide consists of one infinite sheet passing through the centre of U.

4°. When U touches F, we have to consider separately the cases where F is an elliptic paraboloid and where a hyperbolic paraboloid.

If F be an elliptic paraboloid, and U touch it on the convex side, the cyclide has a conic node, whose tangent cone is imaginary.

If U touch F on the concave side, the cyclide has a node whose tangent cone is real; and if U touch F at an umbilic, the tangent cone to the node is one of revolution: lastly, if U osculate F at an umbilic, the tangent cone becomes a plane; that is, each sheet of the cone opens out into a plane, and the node is a biplanar node whose planes coincide. In the case where F is a hyperbolic paraboloid, when U touches it we have a conic node whose tangent cone is always real, but which becomes a pair of planes if U osculate F.

5°. When U has double contact with F, the cyclide will be binodal.

210. In the examination we have given in this and the previous sections of this chapter, we have seen that a cyclide of any class can have but three species of node (namely, the conic node, the biplanar node, and the uniplanar node), and that these correspond respectively to ordinary contact of the sphere of inversion and the focal quadric, oscular contact, and oscular contact at an umbilic. From this it follows that a cyclide can at most have but two real nodes; and if it has two, they must be conic nodes; for if it had a conic node and a biplanar node, U should touch and osculate F at the same time; that is, U should intersect F in a quartic curve having a double point and a cusp, and the plane through the cuspidal tangent and the double point would intersect a quartic curve in five points, which it cannot do; and of course a cyclide cannot, for a like reason, have two biplanar nodes.

By a different mode of reasoning it may be shown that a cubic cyclide cannot have three real nodes; for if it had, the plane through the nodes must intersect the cubic cyclide

in three right lines, and then, by inversion, from any point in the plane we should have the absurdity of a quartic cyclide being intersected by a plane in three circles.

211. *Parallel Tangent Planes.*—Let us consider a cubic cyclide whose sphere of inversion is U and focal paraboloid F. If P, P′ be the limiting points of U and the tangent plane to F at infinity, then of the two points P, P′ one must be at the centre of U and the other at infinity; and it is plain that the generating sphere which touches the cyclide at P, P′ must break up into two planes, namely the tangent planes to the cyclide at P, P′. The tangent plane at the centre of U must evidently be parallel to the principal plane of the paraboloid which does not intersect it in a parabola; and since the cyclide has five centres of inversion, *we see that every cubic cyclide has five parallel tangent planes, and these are the five tangent planes which can be drawn to a cubic cyclide from the line at infinity on the cyclide.* Hence we infer the following theorem:—*The five tangent planes to a cubic cyclide from the line at infinity on the cyclide have the five centres of inversion as points of contact.*

212. The property of the last article may be shown otherwise. Thus, consider any quartic cyclide; then at any point Q five generating spheres touch, namely one belonging to each of the five systems of generating spheres. Now, since a generating sphere intersects a cyclide in two circles, if we invert the quartic cyclide from the point Q we get a system of five parallel planes, each intersecting the cubic cyclide into which the quartic inverts in two lines, and therefore having the points of intersection of these lines as points of contact with the cyclide.

213. The section of a cubic cyclide made by a plane passing through any line on the cyclide except the line at infinity must consist of the line and a circle; for it must consist of a line and a conic, and by inverting from any point the line and conic must invert into a bicircular quartic; hence the conic must be a circle.

This reasoning will not apply in the case of a section made by a plane through the line at infinity; for when we invert the line at infinity it becomes a point, which accounts for the double point which results when a conic is inverted; so that when we say the inverse of a plane conic is a bicircular quartic, this includes the inverse of the line at infinity together with that of the conic.

214. Since the five centres of inversion of a cubic cyclide form a pentahedron, such that, taking any four of them forming a tetrahedron, the perpendiculars of that tetrahedron are concurrent and intersect in the fifth point, we see without difficulty that the feet of the perpendiculars of the pentahedron are points on the cubic cyclide, and we may easily infer the following theorem:—

Being given eight homospheric points (say, eight points on the sphere U), *three cubic cyclides can be described having these eight points as foci. These cyclides intersect two by two orthogonally, they have the same five centres of inversion, and each passes through the feet of the perpendiculars of the pentahedron.*

215. In order to find the equation of the tangent cone to a cubic cyclide from any given point, let the given point be taken as the origin of Cartesian coordinates, and

we shall have

$$W\equiv A(x^2+y^2+z^2)+3B+3C+D, \quad \ldots \quad (145)$$

where

$$A\equiv lx+my+nz, \qquad B\equiv(a, b, c, f, g, h\between x, y, z)^2,$$

$$C\equiv px+qy+rz, \qquad D\equiv \text{constant};$$

then, by the method of art. 193, we find the tangent cone

$$G^2-4H^3\equiv \text{discriminant} \times A^6, \quad \ldots \quad (146)$$

where

$$G\equiv A^2D(x^2+y^2+z^2)^2+2B^3+3ABC(x^2+y^2+z^2)=0,$$

$$H\equiv B^2-AC(x^2+y^2+z^2).$$

Hence the tangent cone is

$$A^2D^2(x^2+y^2+z^2)^2+(4AC^3-6ABCD)(x^2+y^2+z^2)+4DB^3-3B^2C^2=0. \quad . \quad (147)$$

Cor. The plane A is parallel to the tangent planes to the cyclide at its centres of inversion.

216. The cone G possesses the property that any edge of it meets the cyclide in three points, whose distances from the vertex are in arithmetical progression. Now, if we invert a cubic cyclide from the vertex of the cone, we get a quartic cyclide; and since the cone G meets the cubic cyclide in points whose distances from the vertex are in arithmetical progression, it will meet the quartic cyclide in points whose distances are in harmonical progression. Hence the cone G is identical with the cone J of article 193, when the vertex of J is on the surface.

217. Since a cubic cyclide is determined when U and F are given, and U is determined by four and F by eight conditions, we see that a cubic cyclide is determined by 4+8=12 conditions. Hence it follows that every cubic cyclide can be written in the form

$$A\alpha=B\beta, \quad \ldots \quad (148)$$

where A and B are planes, and α and β spheres; and in this form it is evident that the *intersection of the radical plane of the spheres α, β with the two planes A, B is a centre of inversion of the cyclide.* From the equation (148), being the result of eliminating k between the equations $A-kB=0$ and $k\alpha-\beta=0$, *we infer that if we have a system of planes passing through the same line, and a homographic system of spheres passing through the same circle, the locus of the circle of intersection of a sphere and its corresponding plane is a cubic cyclide.*

CHAPTER XI.

Classification of Sphero-quartics.

218. We have seen that if $W\equiv a\alpha^2+b\beta^2+c\gamma^2+d\delta^2$, and $U^2\equiv\alpha^2+\beta^2+\gamma^2+\delta^2$, the sphero-quartic WU is also the curve of intersection of the quadric $V\equiv ax^2+by^2+cz^2+dw^2$ and the sphere $U\equiv x^2+y^2+z^2+w^2$, the tetrahedron of reference being the one formed by the four planes of intersection of U with the four orthogonal spheres α, β, γ, δ.

Hence in this section we shall discuss the curve WU by regarding it as the intersection of V and U.

For the purpose of classification I shall, following CAYLEY and SALMON, consider the curve (UV) as made up of points; then the points of UV will be the points of the system, the line joining two consecutive points will be a line of the system, and the plane of two consecutive lines will be a plane of the system. If a plane of the system contains four consecutive points it will be a stationary plane; and reciprocally, if four consecutive planes of the system intersect in a point of the system, it will be a stationary point. Again, if a line join two non-consecutive points it will be a line through two points; reciprocally, if a line be the intersection of two non-consecutive planes, it will be a line in two planes; finally, if two non-consecutive lines intersect, their point of intersection will be a point on two lines, and their plane a plane through two lines. For the purpose of denoting these singularities the following notation will be used.

Thus we shall denote by

r,	the number of	lines of the system which meet an arbitrary line.
m,	,,	points of the system which lie in any plane.
α,	,,	stationary planes of the system.
x,	,,	points on two lines which lie in a given plane.
g,	,,	lines in two planes which lie in a given plane.

The reciprocals of m, α, x, g will be denoted by the letters respectively consecutive to them, namely, n. β, y, h.

m	is called the	degree	of the system.
n	,,	class	,,
r	,,	rank	,,

219. The complete surface formed by the lines of the system (UV) is the developable Δ, whose properties we have discussed in Chapter VIII., the curve (UV) being the cuspidal edge. Again, the developable Σ of Chapter VIII. formed by the tangent planes to U along the curve UV is the reciprocal of Δ, and the *point, lines, and planes of Δ are* respectively the reciprocals of the planes, lines, and points of Σ, with respect to U; so that when we have the characteristics of Δ, by reciprocation we shall have the characteristics of Σ.

220. Let us consider the cone whose vertex is at any point and which stands on the curve (UV). If, for the sake of distinction, we denote by Greek letters the characteristics of a plane curve (that is, if μ, ν, δ, τ, $\varkappa$, ι denote the degree, class, double points, double tangents, cusps, points of inflection of the curve), then, for a cone standing on that plane curve, it is evident that the same letters will denote the degree, class, double edges, double tangent planes, cuspidal edges, and stationary tangent planes; then for the cone on (UV) $\mu=m$, $\nu=r$, $\delta=h$, $\tau=y$, $\varkappa=\beta$, $\iota=n$. (See SALMON'S 'Geometry of Three Dimensions,' art. 321.)

Hence we get, by PLÜCKER's equations,

$$r=m(m-1)-2h-3\beta, \qquad m=r(r-1)-2y-3n,$$
$$n=3m(m-2)-6h-8\beta, \qquad \beta=3r(r-2)-6y-8n,$$
$$(n-\beta)=3(r-m), \qquad 2(y-h)=(r-m)(r+m-9).$$

221. Again, let us consider a plane section of Δ. Professor CAYLEY has shown that for such a section

$$\mu=r,\ \nu=n,\ \delta=x,\ \tau=g,\ \varkappa=m,\ \iota=\alpha.$$

Hence, from PLÜCKER's equations, we get

$$n=r(r-1)-2x-3m, \qquad r=n(n-1)-2g-3\alpha,$$
$$\alpha=3r(r-2)-6x-8m, \qquad m=3n(n-2)-6g-8\alpha.$$

Whence also Dr. SALMON gets

$$(m-\alpha)=3(r-n), \qquad 2(x-g)=(r-n)(r+n-9).$$

It is plain this system might be got from the former by considering the cone whose vertex is any point and which stands on the cuspidal edge of Σ, and then reciprocating.

If we combine the equations of this article with those of the last, we get

$$(\alpha-\beta)=2(n-m), \quad x-y=(n-m), \quad 2(g-h)=(n-m)(m+n-7),$$

since PLÜCKER's equations enable us, being given any three singularities of a plane curve, to determine all the rest. The equations of this and the preceding article enable us, being given any three singularities of a twisted curve, to determine all the rest. In a succeeding article I shall point out how they may be employed to determine the singularities of the evolute of a plane curve when three of the singularities of the original curve are given.

222. *Eight lines of Δ meet an arbitrary line.*

Demonstration. Let (LM) be the arbitrary line and P a point in (LM) where one of the lines of Δ meets it; then it is plain, if P be the common vertex of two cones tangential to U and V respectively, one of the four common edges of the two cones will be a line of Δ, and also that the intersection of the polar planes of P with respect to U and V will pass through the curve UV. Now the intersection of the polar planes of U and V with respect to all the points of LM is a quadric. For let $x', y', z', w', x'', y'', z'', w''$ be any two fixed points on (LM), and U', U'', V', V'' be their polar planes with respect to U and V; then the coordinates of any other point on (LM) will be $lx'+mx'', ly'+my'', lz'+mz'', lw'+mw''$, and therefore the polar planes will be $lU'+mU''$, $lV'+mV''$, and, eliminating linearly, the locus required is $U'V''-U''V'=0$, a quadric which intersects the curve UV in eight points. Hence the proposition is proved.

223. *The eight planes determined by the line* LM *with the eight lines of Δ which meet it are tangent planes to the reciprocal of the quadric* $U'V''-U''V'$ *with respect to* U. For since eight lines of Δ meet LM, eight lines of Σ meet the polar line of LM with

respect to U, and the eight planes are the reciprocals of the eight points of meeting; but these eight points of meeting lie on a generator of $U'V''-U''V$. Hence the proposition is proved.

Cor. 1. The line LM is a generator of the reciprocal of $U'V''-U''V'$.

Cor. 2. The quadric $U'V'-U''V'$ is the hyperboloid generated by the polar lines of LM with respect to all the quadrics of the pencil $U+kV$. In fact the polar lines form one system of generators, and the intersection of the polar planes of art. 222 the other system of generators.

Cor. 3. The eight planes of art. 222 are homographic with the eight points in which the corresponding lines of Δ meet LM.

224. From art. 161 it is evident that Σ has sixteen stationary points. These are the points of contact with N, N', N'', N''' of the common tangents of J and N, J', N'; J'', N''; J''', N'''. Hence it follows that Δ has sixteen stationary planes. Hence we have three of the characteristics of Δ; for m evidently is equal to 4, and $r=8$ from art. 222. Therefore we have

$$m=4, \quad \alpha=16, \quad r=8.$$

Hence by CAYLEY's equation we get

$$n=12, \quad \beta=0, \quad x=16, \quad y=8, \quad g=38, \quad h=2.$$

225. We can now show the connexion which exists between the singularities determined in the last article and those of bicircular quartics. Let us suppose a cone whose vertex is any point on U, and which stands on UV; then for such a cone we have the singularities (see art. 220)

$$\mu=4, \quad \nu=8, \quad \delta=2, \quad \tau=8, \quad \varkappa=0, \quad \iota=12;$$

but if we invert the sphere U into a plane, taking the vertex of the cone as centre of inversion, UV will be inverted into a bicircular quartic, which will be the curve of intersection of the cone with the plane into which U inverts, and therefore having the same singularities as the cone. The numbers here determined are therefore the singularities of a bicircular quartic (see 'Bicircular Quartics,' art. 46).

226. Again, to show the connexion with the evolute of a bicircular, let us consider a plane section of Δ; the characteristics are (see art. 221)

$$\mu=8, \quad \nu=12, \quad \delta=16, \quad \tau=38, \quad \varkappa=4, \quad \iota=16.$$

Now the cone whose vertex is the pole with respect to U of the plane of sections, and which stands on the cuspidal edge of Σ, will be the reciprocal of the section; and if the plane of section be a tangent plane to U, its pole will be a point on U; therefore the singularities of this cone will be

$$\mu=12, \quad \nu=8, \quad \delta=38, \quad \tau=16, \quad \varkappa=16, \quad \iota=4;$$

but when the sphere is inverted into a plane as in the last article, it has been shown in art. 92 that the cone here considered, viz. the one standing on the cuspidal edge of Σ, intersects the plane into which the sphere inverts in the evolute of the bicircular into

which UV inverts; consequently the evolute has the same singularities as the cone (see art. 51, 'Bicircular Quartics').

227. If U and V touch, the singularities of Δ are (see SALMON's 'Geometry of Three Dimensions,' art. 342)

$$\begin{aligned} m&=4, & g&=6, & \beta&=0, \\ n&=6, & h&=3, & x&=6, \\ r&=6, & \alpha&=4, & y&=4. \end{aligned}$$

Hence, by the method of the last two articles, we get for the bicircular and its evolute the following singularities:—

Bicircular, $\mu=4,\ \nu=6,\ \delta=3,\ \tau=4,\ \varkappa=0,\ \iota=6.$

Evolute, $\mu=6,\ \nu=6,\ \delta=6,\ \tau=6,\ \varkappa=4,\ \iota=4.$

This bicircular is the inverse of an ellipse or hyperbola; and the characteristics of the bicircular are the reciprocals of the characteristics of the evolute of an ellipse or hyperbola.

228. If U and V osculate, we have (see SALMON's 'Geometry of Three Dimensions,' art. 342) the singularities

$$\begin{aligned} m&=4, & g&=2, & \beta&=1, \\ n&=4, & h&=2, & x&=2, \\ r&=5, & \alpha&=1, & y&=2. \end{aligned}$$

Hence for the bicircular and its evolute we get

Bicircular, $\mu=4,\ \nu=5,\ \delta=2,\ \tau=2,\ \varkappa=1,\ \iota=4.$

Evolute, $\mu=4,\ \nu=5,\ \delta=2,\ \tau=2,\ \varkappa=1,\ \iota=4.$

This bicircular is the inverse of a parabola; and we see that it has the same characteristics as its evolute (see foot-note, art. 70, 'Bicircular Quartics').

229. By considering special sections of Δ and Σ we get, as in the preceding articles, the singularities of bicircular quartics, Cartesian ovals, circular cubics, and the evolutes of these respective species of curves. The reader who has followed out the method of reasoning in recent articles can easily account for the result in each case. In order to save space, I shall give only the particular cone whose intersection with the inverse of the sphere U gives the bicircular and the evolute. The numbers for the several cones are taken from SALMON's 'Geometry of Three Dimensions,' art. 324 (see also Cambridge and Dublin Mathematical Journal, vol. v. p. (23)–(46)).

I. When U and V do not touch.

1°. Cone whose vertex is a point of the system Δ:

$$\left.\begin{aligned} \mu&=3, & \varkappa&=0, \\ \nu&=6, & \tau&=0, \\ \iota&=9, & \delta&=0. \end{aligned}\right\}\ \text{A circular cubic of the sixth class.}$$

2°. Cone whose vertex is a point on two lines of Δ:

$$\left.\begin{array}{ll}\mu=4, & \kappa=2,\\ \nu=6, & \tau=1,\\ \iota=9, & \delta=0.\end{array}\right\}$$ A Cartesian oval, sixth class.

3°. Cone whose vertex is a point on a line of Σ and which stands on the cuspidal edge of Σ:

$$\left.\begin{array}{ll}\mu=12, & \varkappa=37,\\ \nu=\ 7, & \tau=\ 2,\\ \iota=\ 2, & \delta=37.\end{array}\right\}$$ Evolute of circular cubic of sixth class.

4°. Cone whose vertex is a point on two lines of Σ and which stands on the cuspidal edge of Σ:

$$\left.\begin{array}{ll}\mu=12, & \varkappa=18,\\ \nu=\ 6, & \tau=\ 9,\\ \iota=\ 0, & \delta=36.\end{array}\right\}$$ Evolute of Cartesian oval of sixth class.

II. When U and V touch.

1°. Cone whose vertex is a point of Δ:

$$\left.\begin{array}{ll}\mu=3, & \varkappa=0,\\ \nu=4, & \tau=0,\\ \iota=3, & \delta=1.\end{array}\right\}$$ A circular cubic of fourth class.

2°. Cone whose vertex is a point on two lines of Δ:

$$\left.\begin{array}{ll}\mu=4, & \varkappa=2,\\ \nu=4, & \tau=1,\\ \iota=2, & \delta=1.\end{array}\right\}$$ A Cartesian oval of fourth class.

3°. Cone whose vertex is a point on a line of Σ and which stands on the cuspidal edge of Σ:

$$\left.\begin{array}{ll}\mu=6, & \varkappa=5,\\ \nu=5, & \tau=4,\\ \iota=2, & \delta=5.\end{array}\right\}$$ Evolute of circular cubic of fourth class.

4°. Cone whose vertex is a point on two lines of Σ and which stands on the cuspidal edge of Σ:

$$\left.\begin{array}{ll}\mu=6, & \varkappa=6,\\ \nu=4, & \tau=3,\\ \iota=0, & \delta=4.\end{array}\right\}$$ Evolute of Cartesian oval of fourth class.

III. When U and V osculate.

1°. Cone whose vertex is a point of Δ:

$$\left.\begin{array}{ll}\mu=3, & \varkappa=1,\\ \nu=3, & \tau=0,\\ \iota=1, & \delta=0.\end{array}\right\}\text{ Circular cubic of third class.}$$

2°. Cone whose vertex is a point on two lines of Δ:

$$\left.\begin{array}{ll}\mu=4, & \varkappa=3,\\ \nu=3, & \tau=1,\\ \iota=0, & \delta=0.\end{array}\right\}\text{ A Cartesian oval of third class—that is, a cardioide.}$$

3°. Cone whose vertex is a point on a line of Σ and which stands on the cuspidal edge of Σ:

$$\left.\begin{array}{ll}\mu=4, & \varkappa=2,\\ \nu=4, & \tau=1,\\ \iota=2, & \delta=1.\end{array}\right\}\text{ Evolute of circular cubic of third class.}$$

4°. Cone whose vertex is a point on two lines of Σ and which stands on the cuspidal edge of Σ:

$$\left.\begin{array}{ll}\mu=4, & \varkappa=3,\\ \nu=3, & \tau=1,\\ \iota=0, & \delta=0.\end{array}\right\}\text{ Evolute of a cardioide.}$$

230. If a plane curve whose degree is N be inverted from any point out of the plane of the curve, it will invert into a twisted curve, whose characteristics are easily found.

1°. Let us suppose that the plane curve does not pass through the circular points at infinity. Then, since the curve passes through N points at infinity, the inverse curve will have this order of multiplicity at the origin of inversion; and since the plane of the curve will invert into a sphere, the inverse curve will be the intersection of two surfaces of the degrees N and 2 respectively, having a multiple contact equivalent to $\frac{N(N-1)}{2}$ points of ordinary contact. Hence we have $m=2N$, $\beta=0$, $2h=3N^2-3N$ (see SALMON, p. 273). Hence, by CAYLEY's equations,

$$\begin{array}{lll} r=N^2+N, & n=3N^2-3N, & 2y=N^4+2N^3-9N^2+6N,\\ \alpha=6N^2-10N, & 2x=N^4+2N^3-3N^2-4N, & 2g=9N^4-18N^3-13N^2+32N.\end{array}$$

231. Next, let us suppose that the plane curve passes k times through each of the circular points at infinity; then it is easy to see that the inverse curve will be the intersection of a sphere and a surface of the degree $N-k$. Hence we have in this case for determining the singularities, $m=2(N-k)$, $\beta=0$, and $2k=3N^2-8Nk+6k^2-3N+4k$.

Hence by CAYLEY's equations we get the following results for the other singularities:

$$r= N^2- 2k^2+N-2k,$$
$$n=3N^2- 6k^2-3N,$$
$$\alpha=6N^2-12k^2-10N+4k,$$
$$2y= N^4+ 2N^3- 9N^2+ 6N-4(N^2+N-1)k-4(N^2+N-6)k^2+8k^3+4k^4,$$
$$2x= N^4+ 2N^3- 3N^2- 4N-4(N^2+N-2)k-4(N^2+N-3)k^2+8k^3+4k^4,$$
$$2g=9N^4-18N^3-13N^2+32N-10k-4(9N^2-9N-11)k^2+36k^4.$$

We can easily verify these results in the case of $N=4$ and $k=2$, which is that of a bicircular quartic; they give the results previously obtained (see art. 224).

232. Let us now find the singularities of the cone whose vertex is the point we invert from, and which stands on the inverse curve. The vertex of the cone is a multiple point on the curve, the degree of multiplicity being of the order $N-2k$; but since the twisted curve is of the degree $2N-2k$, and the multiplicity of the vertex is $N-2k$, it follows that the degree of the cone is N, $\therefore \mu=N$. Again, the class of the cone is the same as the number of tangent planes which pass through an arbitrary line through the vertex;

$$\therefore \nu=r-2(N-2k);$$

and using the value of r in the last article we get

$$\nu=N^2-N-2k(k-1), \text{ and } k=\beta=0.$$

Hence by PLÜCKER's equations the other singularities of the cone are determined. It is evident we could get all these results at once, since evidently the singularities of the cone are the same as those of the original plane curve; but getting them as done here verifies the equations of the last article.

233. Let us find the singularities of the section of the developable circumscribed to the sphere into which the plane inverts along the inverse curve made by the tangent plane to the sphere at the origin of inversion. The characteristics of the developable considered here are got from those of art. 231, by leaving r unaltered and by changing m, α, g, x into n, β, h, y, and *vice versâ*; and the plane in question will be a plane of multiple contact of the degree $N-2k$; that is, it will touch the developable along $N-2k$ lines, and will intersect it besides in a curve of the degree

$$r-2(N-2k)=N^2-N-2k(k-1).$$

We have therefore

$$\mu=N(N-1)-2k(k-1).$$

The class of the curve is determined by the number of planes of the system which can be drawn through any point of the section; and since in this case $N-2k$ planes coincide with the plane of the section itself, the number of remaining planes $\nu=N$, and we have $\iota=\alpha=0$, and by PLÜCKER's equation the remaining characteristics can be found. These results are the reciprocals of those found in the last article, as they evidently ought to be.

234. The most important problem in this inquiry is to find the singularities of the

cone whose vertex is the origin of inversion, and which stands on the cuspidal edge of the developable formed by the tangent planes of the sphere along the inverse curve—that is, the sphere into which the plane inverts. These singularities will be those of the evolute of the original plane curve.

The singularities of the developable will be got by the changing of letters as in the last article. Since the origin is a multiple point of the degree $N-2k$, therefore the class of the cone will be $r-(N-2k)=N^2-2k^2$, since it is evident that, in finding the number of lines of the system which meets an arbitrary line through the vertex of the cone, we must subtract from the rank of the system the number denoting the multiplicity of the vertex.

Since any arbitrary plane meets the cuspidal edge in a number of points equal to the degree of the system (that is, $3N^2-6k^2-3N$), it is evident the degree of the cone is equal to this number diminished by $N-2k$; therefore the degree of the cone is

$$3N^2-6k^2-4N+2k.$$

Again, the cuspidal edges of the cone will be equal to the number of stationary points of the system—that is, equal to

$$6N^2-12k^2-10N+4k.$$

In order to find the corresponding singularities for the evolute, we must plainly add $N-2k$ to the last two singularities; for, $N-2k$ branches of the inverse curve passing through the origin of inversion, each branch will add one to the number of cusps, and one to the degree, and we shall have for the evolute of a curve of degree N which passes k times through each of the circular points at infinity, but which has no finite double point or cusp, the following singularities,

$$\begin{aligned}\text{Class} &= N^2-2k^2,\\ \text{Degree} &= 3N^2-6k^2-3N,\\ \text{Cusps} &= 6N^2-12k^2-9N+2k;\end{aligned}$$

and by PLÜCKER'S equations the other singularities can be determined.

By putting $N=4$ and $k=2$, we find class, degree, and cusps of a bicircular quartic to be 8, 12, 16, and our formula is verified for the bicircular.

Again, putting $N=3$ and $k=1$, we find the numbers for the circular cubic to be 7, 12, 17, which we know otherwise to be the characteristics for the evolute of a circular cubic. If we put $k=0$ in the above formulæ, the numbers coincide with those in SALMON'S higher curves.

The foregoing numbers are to be modified when the curve of the Nth degree has cusps at the circular points at infinity. In that case for each cusp at a circular point at infinity the class of the evolute will be diminished by unity, and the number of its cusps increased by unity, the degree remaining the same. If the original curve had finite double points or cusps, the surfaces, viz. the sphere and the surface of the Nth degree, will have ordinary contact for each double point on the curve, and stationary contact for each

cusp; and we see that there is no difficulty in completing the investigation—that is, being given the degree, the finite double points or cusps, and the double points or cusps at the circular points at infinity of a plane curve, to find the characteristics of the evolute.

CHAPTER XII.

Sphero-Cartesians.

235. If a Cartesian cyclide be intersected by any sphere, I shall call the curve of intersection a *sphero-Cartesian.* It is evident, if the intersecting sphere become a plane, that the sphero-Cartesian will become a Cartesian oval. We have seen that, being given a sphere U and a quadric F, the cyclide which has U for a sphere of inversion and F for a focal quadric will intersect U in the same sphero-quartic as the reciprocal of F with respect to U intersects U. Now, when F is a sphere its reciprocal with respect to U is a quadric of revolution. Hence we have the following fundamental theorem:—

A sphero-Cartesian is the curve of intersection of a sphere and a quadric of revolution.

236. *The focal sphero-conics of a sphero-Cartesian are circles.*

Demonstration. Let the sphero-Cartesian be the intersection of the sphere U and quadric V. Then, since V is a quadric of revolution, the cones which can be described through (UV) have but one system of circular sections, and therefore the cones reciprocal to them have but one system of focal lines; but the reciprocal cones with respect to U intersect U in the focal sphero-conics of UV; therefore the focal sphero-conics of UV are circles.

237. *One of the four cones through* (UV) *is a right cylinder on a parabolic base, the plane of the base being perpendicular to the planes of circular sections of* V.

Demonstration. Let

$$U\equiv(x+\alpha)^2+(y+\beta)^2+(z+\gamma)^2-r^2=0,$$

$$V\equiv\left(\frac{x-\alpha}{a}\right)^2+\left(\frac{y-\beta}{a}\right)^2+\left(\frac{z-\gamma}{c}\right)^2-1=0;$$

then

$$U-a^2V\equiv4\alpha x+4\beta y+(z+\gamma)^2-\frac{a^2}{c^2}(z-\gamma)^2+a^2-r^2=0,$$

and this will be of the form

$$a'z^2+f'y+d=0$$

by a change of axes. Hence the proposition is proved; or we may show it thus: the biquadratic

$$\frac{\alpha^2}{a^2+\lambda}+\frac{\beta^2}{b^2+\lambda}+\frac{\gamma^2}{c^2+\lambda}=1+\frac{\varrho^2}{\lambda}$$

(see SALMON'S 'Geometry of Three Dimensions,' page 146), whose roots are the values of λ, for which

$$(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2-\varrho^2+\lambda\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1\right)=0$$

represents a cone, reduces to a cubic when $a=b$, showing that in this case there are only three cones.

Again, the equation

$$\frac{\lambda a\alpha^2}{1+a\lambda}+\frac{\lambda b\beta^2}{1+b\lambda}+2\lambda r\gamma=\lambda^2r^2+\varrho^2,$$

which is the biquadratic for the paraboloid $ax^2+by^2+2rz=0$ and the sphere

$$(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2-\varrho^2,$$

becomes a cubic when $a=b$.

238. *If a sphero-Cartesian be projected on the plane of circular section of* V *by lines parallel to the axis of revolution, the projection will be a Cartesian oval.*

Demonstration. Let $U\equiv x^2+y^2+z^2-r^2=0$,

$$V\equiv\frac{(x-\alpha)^2}{a^2}+\frac{(y-\beta)^2}{a^2}+\frac{(z-\gamma)^2}{c^2}-1=0.$$

Now, putting $x^2+y^2-r^2\equiv-S$ and $c^2\left(1-\frac{(x-\alpha)^2+(y-\beta)^2}{a^2}\right)\equiv S'$, these equations are equivalent to $z-S^{\frac{1}{2}}=0$, $z-\gamma-S'^{\frac{1}{2}}=0$. Hence, eliminating z, we get the equation of a Cartesian oval.

239. *If a plane parallel to the planes of circular section of* V *intersect* U *and* V *in two circles* u *and* v, *the locus of the radical axis of* u *and* v *will be the cylinder on the parabolic base.*

Demonstration. Put $z=k$ in the equations of U and V, and we have their sections by the plane $z=k$; thus

$$u\equiv x^2+y^2+k^2-r^2=0,\quad v\equiv\frac{(x-\alpha)^2}{a^2}+\frac{(y-\beta)^2}{a^2}+\frac{(k-\gamma)^2}{c^2}-1=0.$$

Hence the radical axis of u and v is $u-a^2v=0$; therefore the same value of λ for which $U+\lambda V$ becomes a parabolic cylinder reduces $u+\lambda v$ to the radical axis of u and v, and the proposition is proved.

Cor. 1. The curve of intersection of a sphere with a cylinder on a parabolic base is a sphero-Cartesian.

Cor. 2. From recent articles we infer the following method of generating sphero-Cartesians.

Let J be a circle and F a parabola in the same plane (say, in the plane of the paper); then from any point P in F erect two perpendiculars in opposite directions to the plane of the paper and equal respectively to $\pm T\sqrt{-1}$, where T is the length of the tangent drawn from P to J; then the locus of the extremities of the perpendicular will be a sphero-Cartesian.

240. Since one of the four cones passing through the sphero-Cartesian (UV) is a parabolic cylinder, it follows that one of the nodal conics of the developable Σ formed by tangent planes to U along (UV) will pass through the centre of U. Hence we have the following theorem:—

The binodal cyclide $(a, b, c, f, g, h \between \alpha, \beta, \gamma)^2=0$ *will be intersected by the sphere* U *orthogonal to* α, β, γ, *and whose centre is complanar with their centres in a sphero-Cartesian if the conic* $(a, b, c, f, g, h \between \lambda, \mu, \nu)^2$ *pass through the centre of* U.

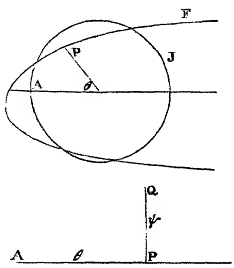
241. From the method of generating sphero-Cartesians given in art. 239, *Cor.* 2, we can get one form of its equation considered as a curve described on a sphere.

Thus, let the equation of the sphere of which the circle J is a great circle be $x^2+y^2+z^2=1$, and the equation of F be $(y+k)^2=4a(h+x)$, or, in polar coordinates,

$$(\rho \sin\theta+k)^2=4a(h+\rho\cos\theta),$$

and it is clear that the perpendicular to the plane of the paper at P will cut the sphere in a point Q whose spherical coordinates are thus determined.

Fig. 6.



Taking the great circle J as a circle of reference, making $AP=\theta$, PQ perpendicular to it $=\psi$, then we have $\cos\psi=\rho$, and the equation required is

$$(\sin\theta\cos\psi+k)^2=4a(h+\cos\theta\cos\psi). \quad . \quad . \quad . \quad (149)$$

242. Let the great circle J of the sphere intersect the parabolic base of the cylinder in four points, and let K, K′, K″ be the points of intersection of the three pairs of lines through these four points, the sides of the triangle K K′ K″ will cut off from the sphere three arcs, and the three small circles which have these three arcs as spherical diameters will be the three circles of inversion of the sphero-Cartesian. Again, the three pairs of perpendiculars from the centre of the sphere on the three pairs of opposite connectors will cut the sphere in three pairs of points which will be the extremities of the diameters of the three focal circles of the sphero-Cartesian. Hence, being given on the surface of a sphere U a focal circle F and a circle of inversion J of a sphero-Cartesian, we infer the following construction for the two remaining focal circles and circles of inversion:—

Let H *and* O *be the centres of* J *and* F, *and let* HO *intersect the circles* J *and* F *in the points* A, B, C, D: *if the points* P *and* Q *be taken so as to be common points of harmonic section of* AB *and* CD, *then* P *and* Q *are plainly the points in which radii from the centre of* U *to the points* K′, K″ *pierce* U; *they are therefore the centres of inversion of the sphero-Cartesian.*

243. Again (see fig. 5, art. 198), if the circles F′, F″ be described on the sphere as they are in the diagram referred to on the plane, we shall have the three focal circles and their radii given by the following equations:—

$$\left.\begin{aligned} \tan^2 r &= \tan PH . \tan QH, \\ \tan^2 r' &= \tan OH . \tan QH, \\ \tan^2 r'' &= \tan OH . \tan PH. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (150)$$

The first equation is evident, since P, Q are conjugate points with respect to J; and

the second follows from the fact that the bisectors of the angles ABD and DBN pass respectively through O and H.

244. The points O, P, Q are the centres of J, J′, J″. Let their distances from H be denoted by δ, δ', δ'' respectively, and the preceding equations may be written

$$\left.\begin{aligned} \tan^2 r &= \tan\delta' \ \tan\delta'', \\ \tan^2 r' &= \tan\delta'' \tan\delta \ , \\ \tan^2 r'' &= \tan\delta \ \tan\delta' . \end{aligned}\right\} \quad \ldots \ldots \ldots \ldots \quad (151)$$

Hence we get the three following equations:

$$\left.\begin{aligned} \tan\delta &= \tan r' \ \tan r'' : \tan r \ , \\ \tan\delta' &= \tan r'' \tan r \ : \tan r' , \\ \tan\delta'' &= \tan r \ \tan r' \ : \tan r'' . \end{aligned}\right\} \quad \ldots \ldots \ldots \ldots \quad (152)$$

Hence also we get

$$\tan\delta \ \tan\delta' \ \tan\delta'' = \tan r \ \tan r' \ \tan r'' . \quad \ldots \ldots \ldots \quad (153)$$

245. If we denote the radii of the circles of inversion J, J′, J″ by ϱ, ϱ', ϱ'', we easily get the system of three equations,

$$\left.\begin{aligned} \tan^2\varrho &= \tan(\delta - \delta') \tan(\delta - \delta''), \\ \tan^2\varrho' &= \tan(\delta' - \delta'') \tan(\delta' - \delta), \\ \tan^2\varrho'' &= \tan(\delta'' - \delta) \tan(\delta'' - \delta'), \end{aligned}\right\} \quad \ldots \ldots \ldots \ldots \quad (154)$$

with this other system of equations,

$$\left.\begin{aligned} \tan(\delta - \delta') &= \tan\varrho \ \tan\varrho' \ : \tan\varrho''\sqrt{-1}, \\ \tan(\delta' - \delta'') &= \tan\varrho' \tan\varrho'' : \tan\varrho'\sqrt{-1}, \\ \tan(\delta'' - \delta) &= \tan\varrho'' \tan\varrho \ : \tan\varrho''\sqrt{-1}. \end{aligned}\right\} \quad \ldots \ldots \ldots \quad (155)$$

Hence also

$$\tan(\delta - \delta')\tan(\delta' - \delta'')\tan(\delta'' - \delta') = \tan\varrho \ \tan\varrho' \ \tan\varrho''\sqrt{-1}. \quad . \quad (156)$$

From either system it follows, as we know otherwise, that one of the circles of inversion is imaginary.

246. From combining the equations of the two preceding articles, we easily get the relations between the radii of the F's and the J's; thus

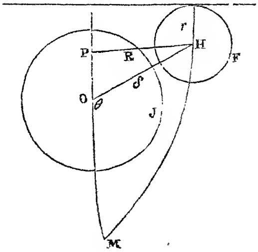
$$\begin{aligned} \tan^2\varrho &= \tan(\delta - \delta')\tan(\delta - \delta'') \\ &= \frac{\tan^2\delta - \tan\delta(\tan\delta' + \tan\delta'') + \tan\delta' \tan\delta''}{(1 + \tan\delta \tan\delta')(1 + \tan\delta \tan\delta'')} \\ &= \frac{(\tan^2 r - \tan^2 r')(\tan^2 r - \tan^2 r'')}{\tan^2 r \sec^2 r' \sec^2 r''} \\ &= \frac{4(\sin^2 r - \sin^2 r')(\sin^2 r - \sin^2 r'')}{\sin^2 2r}. \end{aligned}$$

Hence we have the system of equations:

$$\left.\begin{aligned}\tan^2\rho &= 4(\sin^2 r - \sin^2 r')(\sin^2 r - \sin^2 r'') : \sin^2 2r,\\ \tan^2\rho' &= 4(\sin^2 r' - \sin^2 r'')(\sin^2 r' - \sin^2 r) : \sin^2 2r',\\ \tan^2\rho'' &= 4(\sin^2 r'' - \sin^2 r)(\sin^2 r'' - \sin^2 r') : \sin^2 2r''.\end{aligned}\right\} \quad . \quad . \quad . \quad (157)$$

247. Let us denote the radii of the three circles of inversion by J, J′, J″, the radii of the focal circles by r, r', r'', and the distances by δ, δ', δ'', as in recent articles. Now denoting the perpendicular from the centre of J to any tangent to F by p, then taking $\mathrm{OP}=\rho$ such that

Fig. 7.



$$\cos p : \cos(p-\rho) = \cos \mathrm{J} = \frac{1}{k} \text{ suppose};$$

$$\therefore\ (\cos\rho - k)\cos p + \sin\rho \sin p = 0;$$

but from the spherical triangles O H M and O P H we get

$$\cos p \sin\delta \cos\theta + \cos\delta \sin p = \sin r,$$

$$\cos\rho \cos\delta + \sin\rho \sin\delta \cos\theta = \cos \mathrm{R}.$$

Hence, eliminating p and θ, we get

$$(1+k^2-2k\cos\rho)^{\frac{1}{2}} \sin r = k\cos\delta - \cos \mathrm{R},$$

with two similar expressions involving k', k''; δ', δ''. Hence we have the determinant

$$\begin{vmatrix} (1+k^2-2k\cos\rho)^{\frac{1}{2}}\sin r, & k\cos\delta, & 1, \\ (1+k'^2-2k'\cos\rho')^{\frac{1}{2}}\sin r', & k'\cos\delta', & 1, \\ (1+k''^2-2k''\cos\rho'')^{\frac{1}{2}}\sin r'', & k''\cos\delta'', & 1, \end{vmatrix} = 0;$$

and restoring the value of k, k', k'', we get

$$\begin{vmatrix} (1+\cos^2 \mathrm{J} - 2\cos\rho\cos \mathrm{J})^{\frac{1}{2}}\sin r, & \cos\delta, & \cos \mathrm{J}, \\ (1+\cos^2 \mathrm{J}' - 2\cos\rho'\cos \mathrm{J}')^{\frac{1}{2}}\sin r', & \cos\delta', & \cos \mathrm{J}', \\ (1+\cos^2 \mathrm{J}'' - 2\cos\rho''\cos \mathrm{J}'')^{\frac{1}{2}}\sin r'', & \cos\delta'', & \cos \mathrm{J}'', \end{vmatrix} = 0. \quad . \quad . \quad (158)$$

248. It is evident that $1+k^2-2k\cos\rho$ is equal to the square of the distance of the point P of the sphero-Cartesian from the pole of the plane of J with respect to the sphere U. Hence, if this distance be denoted by D, the first determinant of the last article may be written

$$\begin{vmatrix} \mathrm{D}\sin r, & \cos\delta\sec \mathrm{J}, & 1, \\ \mathrm{D}'\sin r', & \cos\delta'\sec \mathrm{J}', & 1, \\ \mathrm{D}''\sin r'', & \cos\delta''\sec \mathrm{J}'', & 1, \end{vmatrix} = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (159)$$

249. From the equation, art. 247,

$$(1+k^2-2k\cos\varrho)^{\frac{1}{2}}\sin r=k\cos\delta-\cos R,$$

which we may put in the form

$$(1+k^2-2k\cos\varrho)^{\frac{1}{2}}\sin r=C,$$

where C is the small circle cutting J orthogonally and concentric with F, we have this other equation,

$$(k-e^{\varrho\sqrt{-1}})(k-e^{-\varrho\sqrt{-1}})\sin^2 r=C^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (160)$$

Hence the imaginary lines $k-e^{\varrho\sqrt{-1}}$, $k-e^{-\varrho\sqrt{-1}}$ are tangents to the sphero-Cartesian. Hence the centre of J is a focus; and similarly the centres of J′, J″ are foci.

250. The equation

$$(1+k^2-2k\cos\varrho)\sin^2 r=C^2$$

is the envelope of the circle

$$1+k^2-2k\cos\varrho+\mu C+\mu^2\sin^2 r=0\,;$$

but

$$C=k\cos\delta-(\cos\varrho\cos\delta-\sin\varrho\sin\delta\cos\theta)=0,$$

$$\therefore 1+k^2+\mu k\cos\delta+\mu^2\sin^2 r=(2k+\mu\cos\delta)\cos\varrho-\mu\sin\delta\sin\varrho\cos\theta.$$

Now the equation of a circle (see art. 36) is

$$\cos R=\cos n\cos\varrho+\sin n\sin\varrho\cos\theta.$$

Multiplying by an indeterminate constant, we get

$$\lambda\cos R=1+k^2+\mu k\cos\delta+\mu^2\sin^2 r,$$

$$\lambda\cos n=2k+\mu\cos\delta,$$

$$\lambda\sin n=-\mu\sin\delta.$$

Hence

$$\cos R=\frac{1+k^2+\mu k\cos\delta+\mu^2\sin^2 r}{\sqrt{\mu^2+4\mu k\cos\delta+4k^2}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (161)$$

Now if R be equal to zero, the circle whose radii is R will be a focus, since it will have imaginary double contact with the sphero-Cartesian; but R=0 gives the biquadratic in μ,

$$(1+k^2+\mu k\cos\delta+\mu^2\sin^2 r)^2=(\mu^2+4\mu k\cos\delta+4k^2), \quad . \quad . \quad (162)$$

showing that there are four foci, as we know otherwise, since there are three single foci and one triple focus.

251. If $R=\frac{\pi}{2}$, we have the equation

$$1+k^2+\mu k\cos\delta+\mu^2\sin^2 r=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (163)$$

a quadratic showing that a sphero-Cartesian has two great circles which have double contact with it. These are not, however, the only great circles which have double contact with the sphero-Cartesian. These correspond to the great circle passing

through the three single foci of the curve; and we will now show that there are two great circles of double contact corresponding to each of the three small circles of inversion, J, J′, J″. For let F be the focal circle corresponding to J, then the great circle whose pole is the centre of J will intersect F in two points; these will be the poles of two great circles, each having double contact with the sphero-Cartesian. Hence a *sphero-Cartesian has eight great circles of double contact.*

Cor. *It is evident that a similar property holds for a sphero-quartic.*

252. If in the equation of art. 250

$$1+k^2+\mu k\cos\delta+\mu^2\sin^2 r=(2k+\mu\cos\delta)\cos\rho-\mu\sin\delta\sin\rho\cos\theta$$

we substitute the spherical coordinates ρ', θ' of any point (see art. 36), we get a quadratic for μ, showing that through any point ρ', θ' two generating circles pass. Hence, reasoning as in the last article, *through any point may be described eight circles each having double contact with the sphero-Cartesian.* Hence, if we invert the sphere into a plane, *the inverse of the sphero-Cartesian will not be a Cartesian oval but a bicircular quartic.*

253. The following properties of sphero-Cartesians are the analogues of properties of plane Cartesians which have appeared in the 'Educational Times':—

1°. *Being given two small circles such that a spherical triangle can be inscribed in one and circumscribed to the other, the envelope of the small circle which has the spherical triangle as a self-conjugate, or, as it may more appropriately be called, an harmonic triangle, is a sphero-Cartesian.*

2°. *Through any point on a sphero-conic can be described three circles which osculate the sphero-conic; the envelope of the circle through the three points of osculation is a sphero-quartic.*

3°. *If a sphero-quartic with a double point* O *be cut by a circle in four points* A, B, C, D, *and if* OA, OB, OC, OD *cut the circle again in* E, F, G, H, *any pair of great circles through these points will be equally inclined to the bisectors of the angles between the tangents at* O.

4°. *If a sphero-conic be turned through* 90° *round the principal axis of the cone which cuts the sphere in the sphero-conic, the locus of the intersection of any tangent with the same tangent in its new position is a sphero-quartic.*

5°. The locus of one set of foci of all the conics which have double contact with a given circle at given points is another circle passing through those points and through the centre of the given circle. Hence, by inversion, *the locus of one set of foci of sphero-quartics with a double point which have a given generating circle, and which have given points of contact with it, is a circle through the points of contact.*

6°. The three points in which a circular cubic is cut by any transversal are the foci of a Cartesian oval passing through four concyclic foci of the cubic. Hence, by inversion, *four concyclic points on a sphero-quartic* A *are the foci of another sphero-quartic* B *passing through four concyclic foci of* A. It is evident that this property is analogous to that of pole and polar, and that a similar use may be made of it.

254. The following properties are the inverses of properties of conics &c. :—

1°. A circular cubic is the locus of one set of foci of all the conics that can be drawn through four concyclic points. Hence, by inversion, a *sphero-quartic is the locus of the locus of one set of foci of all the sphero-quartics with a double point which can be drawn through four concyclic points.*

A more general proposition than this can be easily inferred from art. 253, 6°.

2°. If two tangents to a conic intersect at a given angle, the locus of their intersection is a bicircular quartic. Hence, by inversion,

If two generating circles of a sphero-quartic with a double point (including cusps and conjugate points) intersect at a given angle, the locus of their intersection, if they belong to the system of generating circles which passes through the double point, is a sphero-quartic.

Cor. If the angle of intersection be a right angle the locus will be a circle.

3°. A cardioide can be inverted into a cissoid. *Hence a cusped sphero-quartic will be got by inverting a cusped sphero-Cartesian.*

255. Particular spherical sections of a general cyclide will be sphero-Cartesians.

The following is an example:—*Let* W *be a cyclide,* U *and* F *a sphere of inversion and corresponding focal quadric; then if any sphere has its centre on the focal hyperbola of* F *and cuts* U *orthogonally, it will intersect* W *in a sphero-Cartesian.*

CHAPTER XIII.

SECTION I.—*Substitutions.*

256. If $W=(a, b, c, d, l, m, n, p, q, r\between\alpha, \beta, \gamma, \delta)^2=0$ be the equation of a cyclide, and if the equation W be satisfied by the values x', y', z', w' of α, β, γ, δ, we can state it thus: the system of six spheres denoted by the matrix

$$\left\|\begin{matrix} \alpha, & \beta, & \gamma, & \delta, \\ x', & y', & z', & w', \end{matrix}\right\| \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (164)$$

have the two points which are common to them on W; and if the ratios $x' : y' : z' : w'$ be supposed to vary, but subject to the condition

$$(a, b, c, d, l, m, n, p, q, r\between x', y', z', w')^2=0,$$

then the pair of points denoted by the matrix (164) will vary, and the locus will be the cyclide W. Hence we may call $(a, b, c, \ldots . r\between\alpha, \beta, \gamma, \delta)^2=0$ the local equation of a cyclide.

I remark that whenever I shall speak of a pair of inverse points on a cyclide it will be a pair determined by a matrix such as (164).

257. We have seen that the tangential equation of the focal quadric of a cyclide is the same in form as the local equation (see last article) of the cyclide, and that to a tangent plane of the quadric will correspond a pair of inverse points of the cyclide, and generally

to any plane L related to the quadric will correspond a pair of inverse points having a correlative reference to the cyclide, and these inverse points will be the limiting points of the sphere U (the Jacobian of α, β, γ, δ) and the plane L.

258. We have determined in art. 5 the condition that the sphere $x\alpha+y\beta+z\gamma+w\delta$ should be a generating sphere of W to be given by the determinant (7), and that this determinant in tetrahedral coordinates is the equation of the focal quadric F of W. Now since for any system of values of x, y, z, w which satisfies the determinant (7) we get a point on F, we see that to any point on F will correspond a generating sphere of W, and generally to any point P having any special relation to F will correspond a sphere Q having a similar relation to W; in fact the sphere Q will have the point P for centre, and will be orthogonal to U.

259. Since the tetrahedral coordinates of the centre of $x_1\alpha+y_1\beta+z_1\gamma+w_1\delta$ are x_1, y_1, z_1, w_1, and if four spheres orthogonal to U pass through the same pair of inverse points, with respect to U we know that their centres are complanar. Hence we have the following theorem:—

The condition that the four spheres

$$x_1\alpha+y_1\beta+z_1\gamma+w_1\delta,\ x_2\alpha+y_2\beta+z_2\gamma+w_2\delta,\ \&c.$$

should pass through the same pair of inverse points is the vanishing of the value of the determinant

$$\begin{vmatrix} x_1, & y_1, & z_1, & w_1, \\ x_2, & y_2, & z_2, & w_2, \\ x_3, & y_3, & z_3, & w_3, \\ x_4, & y_4, & z_4, & w_4. \end{vmatrix} \quad \ldots \ldots \ldots \ldots \quad (165)$$

260. In art. 257 it is proved that the pair of inverse points given by the matrix (164) correspond to a plane, and in art. 27 it is shown that the perpendiculars from the centres of α, β, γ, δ on the plane are proportional to x_1, y_1, z_1, w_1 of the matrix, that is, in other words, the coordinates of the plane are x_1, y_1, z_1, w_1. Hence we infer the following theorem:—

The four pairs of inverse points given by the matrices

$$\begin{Vmatrix} \alpha, & \beta, & \gamma, & \delta, \\ x_1, & y_1, & z_1, & w_1, \end{Vmatrix}; \quad \begin{Vmatrix} \alpha, & \beta, & \gamma, & \delta, \\ x_2, & y_2, & z_2, & w_2, \end{Vmatrix}; \quad \begin{Vmatrix} \alpha, & \beta, & \gamma, & \delta, \\ x_3, & y_3, & z_3, & w_3, \end{Vmatrix}; \quad \begin{Vmatrix} \alpha, & \beta, & \gamma, & \delta, \\ x_4, & y_4, & z_4, & w_4, \end{Vmatrix}$$

are homospheric, if vanishes the determinant

$$\begin{vmatrix} x_1, & y_1, & z_1, & w_1, \\ x_2, & y_2, & z_2, & w_2, \\ x_3, & y_3, & z_3, & w_3, \\ x_4, & y_4, & z_4, & w_4. \end{vmatrix} \quad \ldots \ldots \ldots \ldots \quad (166)$$

261. Since to any system of complanar points corresponds a system of spheres passing through a pair of inverse points, *to a plane conic on the focal quadric of a cyclide W will*

correspond a binodal cyclide circumscribed about W; *the nodes of the binodal cyclide will correspond to the plane of the conic.*

262. Since to any system of planes passing through a point corresponds a system of homospheric pairs of inverse points, *to a cone circumscribed about the focal quadric of a cyclide* W *will correspond a sphero-quartic on* W; *and if the cone be one of revolution, the sphero-quartic will become a sphero-Cartesian.*

Cor. 1. *If the vertex of the cone be at infinity, that is, if the cone become a cylinder, to it will correspond a bicircular quartic; and if the cylinder be one of revolution, to it will correspond a section of the cyclide, which will be a Cartesian oval.*

Cor. 2. *Since two cylinders of revolution can be described about a quadric, through each centre of inversion of a cyclide can be drawn two planes which will intersect it in Cartesian ovals.*

263. Since to a point on F corresponds a generating sphere of W, to the line joining two points on F will correspond the circle of intersection of two generating spheres; and if every point of the line be on F, every point of the circle will be on W. Hence to a rectilinear generator of F will correspond a circular generator of W; and since through any point on F can be drawn two rectilinear generators, *hence in general can be drawn two circular generators corresponding to each focal quadric of* W *through any point of* W.

264. The last article may be established differently as follows. Thus if perpendiculars from the centres of the spheres of reference α, β, γ, δ of the cyclide $W = a\alpha^2 + b\beta^2 + c\gamma^2 + d\delta^2$ on any plane be denoted by λ, μ, ν, ϱ, then the points whose equations are

$$\begin{vmatrix} A, & B, & C, & D, \\ A', & B', & C', & D', \end{vmatrix} (\lambda, \mu, \nu, \varrho)$$

correspond to the spheres whose equations are

$$\begin{vmatrix} A, & B, & C, & D, \\ A', & B', & C', & D', \end{vmatrix} (\alpha, \beta, \gamma, \delta),$$

and consequently the line joining the points corresponds to the circle of intersection of the spheres. Now the six determinants of the matrix

$$\begin{Vmatrix} A, & B, & C, & D, \\ A', & B', & C', & D', \end{Vmatrix}$$

or their mutual ratios, are called by Professor CAYLEY the six coordinates of a line in space, and are denoted by the notation (a, b, c, f, g, h). Hence we see that we can in our extension call the same ratios the six coordinates of a circle in space (see CAYLEY "On the Six Coordinates of a Line," Cambridge Philosophical Transactions, vol. xi. pt. 2). Hence the same investigation which in Professor CAYLEY's system proves any property of a system of lines in space, will, with our interpretation, give a corresponding property of a system of circles in space. It is to be remembered, however, that all our circles are cut orthogonally by the same sphere, namely the sphere U, the Jacobian of α, β, γ, δ

(compare art. 19). As an example, three lines in space determine a ruled quadric; to which we have the corresponding theorem:—Three circles in space orthogonal to the same sphere determine a cyclide. Again, every ruled quadric has two systems of rectilinear generators; to this corresponds the theorem:—Every cyclide has two systems of circular generators corresponding to each sphere of inversion; or any four rectilinear generators of one system on a ruled quadric are cut equianharmonically by all the rectilinear generators of the opposite. Hence any four circular generators of one system belonging to a cyclide are cut equianharmonically by all the circular generators of the opposite system.

265. From recent articles we see that, being given any graphic property of the focal quadric F of a cyclide W, we can get a corresponding property of W by the following substitutions:—

	F	W
I.	*a.* For a point on F, ...	*a.* A generating sphere of W.
	b. A point having any special relation to F, ...	*b.* A sphere having a corresponding relation to W.
	c. A system of complanar points,	*c.* A system of spheres through the same two points.
	d. A system of collinear points, .	*d.* A system of spheres through the same circle.
II.	*a.* A tangent line to	*a.* A circle having double contact with
	b. A line having any special relation to	*b.* A circle having a corresponding relation to
	c. A system of concurrent lines,	*c.* A system of homospheric circles.
	d. A system of complanar lines,	*d.* A system of circles through the same two points.
III.	*a.* A tangent plane to	*a.* A pair of inverse points on
	b. A plane having any special relation to	*b.* A pair of inverse points having a corresponding relation to
	c. A system of planes through the same point,	*c.* A system of homospheric pairs of inverse points.
	d. A system of planes through the same line,	*d.* A system of inverse pairs of points on the same circle.

266. In order to give illustrations of this system of substitutions, I give the theorems derived by them from a splendid paper of Mr. TOWNSEND's in the 'Quarterly Journal,' vol. viii. p. 10. For this purpose the following proposition is necessary:—If three tangent planes to a quadric be mutually perpendicular the locus of their point of intersection is a sphere, called the director sphere of the quadric. Hence, by substitution, we get the following theorem:—*If three lines mutually perpendicular be drawn through a centre of inversion of a cyclide, and if* P, P′, Q, Q′, R, R *be three pairs of inverse points*

in which these lines intersect the cyclide, the envelope of the sphere through P, P′, Q, Q′, R, R′ *is a Cartesian cyclide, which by analogy I shall call the director cyclide of the given cyclide.*

1°. "If a system of quadrics touch a common system of eight planes, their director spheres have a common radical plane." Hence, *if a system of cyclides pass through a common system of eight inverse pairs of points, their director Cartesian cyclides are inscribed in a common binodal Cartesian cyclide.*

2°. "If a system of quadrics touch a common system of seven planes, their director spheres have a common radical axis." Hence, *if a system of cyclides pass through a common system of seven inverse pairs of points, their director Cartesians have two generating spheres common to all.*

3°. "If a system of quadrics touch a common system of six planes, their director spheres have a common radical centre." Hence, *if a system of cyclides pass through a common system of six inverse pairs of points, their director Cartesian cyclides are such that any of them can be expressed as a linear function of four others,* because their director spheres in the property of the quadrics having a common radical centre are coorthogonal, and any of them can be expressed as a linear function of four others.

Cor. The property in 1° may be expressed thus:—any director Cartesian can be expressed linearly in terms of two others; and the property in 2°, any director Cartesian can be expressed linearly in terms of three others.

267. When a quadric becomes a paraboloid, the director sphere becomes a director plane. Hence if three lines mutually perpendicular be drawn through a centre of inversion of a cubic cyclide intersecting it in three pairs of inverse points P, P′, Q, Q′, R, R′, the sphere determined by these three pairs of inverse points passes through a fixed pair of points. *I shall call these points the director points of the cubic cyclide.*

1°. "If a system of quadrics touch the same eight planes, the common radical plane of their director spheres is the director plane of the paraboloid which touches the planes." Hence, *if a system of cyclides pass through a common system of eight inverse pairs of points, the nodes of the binodal Cartesian cyclide which is circumscribed to their director Cartesian cyclides are the director points of the cubic cyclide which passes through the system of eight pairs of inverse points.*

2°. "If a system of paraboloids touch the same seven planes, their director planes have a common line of intersection." Hence, *if a system of cubic cyclides pass through a common system of seven pairs of inverse points, their director points are concyclic.*

3°. "If a system of quadrics having a common rectilinear generator touch five planes, their director spheres have a common radical plane." Hence, *if a system of cyclides having a common circular generator pass through five inverse pairs of points, their director cyclides are inscribed in the same binodal Cartesian cyclide.*

4°. "If a system of ruled quadrics have two common rectilinear generators and touch two common planes, their director spheres have a common circle of intersection with that of the ruled quadric passing through the two lines and through the intersection of

the two planes." Hence, *if a system of cyclides having two common circular generators pass through two inverse pairs of points, their director cyclides are inscribed in a binodal Cartesian cyclide, in which is also inscribed the cyclide determined by the two circular generators and the circle through the two inverse pairs of points.*

5°. "The director sphere of every ruled quadric passing through the four sides L, M, N, P of any skew quadrilateral passes through the circle of intersection of the two spheres of which the two diagonals are diameters." Hence, *if* L, M, N, P *be four circles in space, such that each of the four pairs* (LM), (MN), (NP), (PL) *is homospheric, and if* L, M, N, P *be circles on a cyclide* W, *the director cyclide of* W *can be expressed as a linear function of two Cartesian cyclides, viz. the cyclide which has the spheres* (LM) and (NP) *as generating spheres, and the middle point between their centres as a triple focus, and the cyclide similarly determined by the spheres* (MN) *and* (PL).

6°. "When eight planes pass in pairs through any generators of the same ruled quadric, the director spheres of all quadrics touching them all have a common circle of intersection with that of the original quadric." Hence, *when eight pairs of inverse points lie two by two on four circular generators of a given cyclide, the director cyclides of all cyclides passing through the eight pairs are such that the director cyclide of the given cyclide can be expressed as a linear function of any two of them; in other words, the director cyclides of the variable cyclides and the director cyclides of the given cyclides are all inscribed in the same binodal Cartesian cyclide.*

7°. *If a system of paraboloids touch the same six planes, their director planes have a common point of intersection.* Hence, *if a system of cubic cyclides pass through the same six pairs of inverse points, all their director pairs of points are homospheric.*

8°. "If a system of ruled quadrics passing through a common line touch four common planes, their director spheres have a common radical axis." Hence, *if a system of cyclides having a common circular generator pass through four inverse pairs of common points, their director cyclides have two common generating spheres.*

268. I shall for the next illustration take the properties of quadrics given in a paper of Dr. SALMON's in the same volume of the Quarterly Journal, "On the Number of Surfaces of the Second Degree which satisfy nine conditions."

1°. Dr. SALMON proves "that two quadrics can be described through eight given points to touch a given line." Hence *two cyclides having a given sphere of inversion can be described having eight given generating spheres to have double contact with a given circle.*

2°. "Three quadrics can be described through eight given points to touch a given plane." Hence *three cyclides can be described having eight given generating spheres to pass through a given pair of inverse points.*

3°. "Twenty-one quadrics can be described through five given points to touch four planes." Hence *twenty-one cyclides can be described having five given generating spheres to pass through four given inverse pairs of points.*

4°. *In general if* N(r, s, t) *denote the number of quadrics which can be described to pass*

through r *points, to touch* s *lines, and to touch* t *planes, where* r+s+t=9, *then precisely the same number* N *of cyclides can be described, being given* r *generating spheres to have double contact with* s *circles and to pass through* t *inverse pairs of points.*

269. If V be the reciprocal of the focal quadric F with respect to U, or, in other words, if V be a quadric of the system passing through the sphero-quartic WU, then the planes, lines, and points of V will correspond to the points, lines, and planes of F; and hence by substitutions reciprocal to those of art. 265, being given any graphic property of V, we can get a corresponding graphic property of W.

Cor. Hence, being given any graphic of a quadric, we can get two correlative graphic properties of a cyclide.

SECTION II.—*Substitutions. Sphero-quartics.*

270. We have seen, if in the equation of a cyclide $W\equiv\phi(\alpha, \beta, \gamma, \delta)$, where ϕ represents a homogeneous function of the second degree, we regard $\alpha, \beta, \gamma, \delta$ as the small circles in which the spheres $\alpha, \beta, \gamma, \delta$ intersect U, that we get the equation of the sphero-quartic (WU); also that the sphero-quartic is generated as the envelope of a variable circle, whose centre moves along a sphero-conic, and which cuts a given circle orthogonally; and we might investigate, as in the last section, a system of substitutions by which, from known properties of sphero-conics, we could infer properties of sphero-quartics; but there is a simpler system of substitutions by which we may arrive at the latter, namely, by means of substitutions from known properties of plane conics. This method I shall explain briefly in the following articles.

271. Let $W\equiv a\alpha^2+b\beta^2+c\gamma^2+d\delta^2=0$ be a cyclide, then the Jacobian of $\alpha, \beta, \gamma, \delta$ is given by the equation

$$U^2\equiv\alpha^2+\beta^2+\gamma^2+\delta^2=0.$$

Hence the sphero-quartic (WU) will be the curve of intersection of U, and either of the binodal cyclides

$$W-aU^2, \quad W-bU^2, \quad W-cU^2, \quad W-dU^2.$$

Now let us consider $W-aU^2$, or $(b-a)\beta^2+(c-a)\gamma^2+(d-a)\delta^2$; this cyclide has four focal quadrics, of which one reduces to a plane conic, and this conic is a focal conic of each of three remaining focal quadrics (see art. 113). The conic is one of the nodal lines of the developable Σ (see Chapter VIII.), and is the reciprocal of one of the four cones through (WU). Now any tangent plane to the cone will intersect U in a circle, which will be a generating circle of WU, and this tangent plane will intersect the plane of the nodal conic of Σ, that is, the plane of the conic whose equation in tangential coordinates is

$$(b-a)\mu^2+(c-a)^2\nu+(d-a)\varrho^2,$$

in a line, which will be a tangent line to the reciprocal conic, that is, to the conic whose trilinear equation is $(b-a)y^2+(c-a)z^2+(d-a)w^2=0$. Hence a tangent line to this conic corresponds to a generating circle of WU. Again, any edge of the cone intersects the conic $(b-a)y^2+(c-a)z^2+(d-a)w^2$ in a point, and passes through a pair of points

of WU; this pair of points will be inverse to each other with respect to the vertex of the cone. Hence a point on the conic corresponds to a pair of inverse points on the sphero-quartic. Again, a point and its polar with respect to the conic correspond to a pair of inverse points and a circle, which are related to each other with respect to the sphero-quartic, as poles and polars are in ordinary geometry. For example, the point and polar with respect to the conic are such that any line through the point meets the conic in two points such that tangents to the conic drawn through them meet on the polar; and the pair of inverse points and their polar circle are such that any circle through the inverse points meets the sphero-quartic in two pairs of points, such that the generating circles which touch the sphero-quartic at these points intersect on the polar circle of the pair of inverse points.

If we have any system of collinear points on the plane of $(b-a)y^2+(c-a)z^2+(d-a)w^2$, it is evident we shall have to correspond with them a system of inverse pairs of points which are concyclic. Lastly, to a system of concurrent lines we shall have a corresponding system of coaxal circles on the sphere U.

272. From the last article we see that, being given any graphic property of the conic

$$(b-a)y^2+(c-a)z^2+(d-a)w^2=0,$$

we shall get a corresponding graphic property of the cyclide WU by the following system of substitutions:—

	$(b-a)y^2+(c-a)z^2+(d-a)w^2$.	(WU).
I.	a. For a point on,	A. A pair of inverse points.
	b. A point having any permanent relation to,	B. A pair of points having a corresponding relation.
	c. A system of collinear points,	C. A system of concyclic inverse pairs of points.
II.	a'. A tangent to,	A′. A generating circle of
	b'. A line having any permanent relation to,	B′. A circle having a corresponding relation to
	c'. A system of concurrent lines.	C′. A system of coaxal circles.

273. If we take the reciprocal of the conic $(b-a)y^2+(c-a)z^2+(d-a)w^2=0$, that is, the conic in tangential coordinates $(b-a)\mu^2+(c-a)\nu^2+(d-a)\varsigma^2$, we get properties of WU, by substitutions, reciprocal to the foregoing; hence we are to substitute from the last article, for

$$a,\ b,\ c;\ \mathrm{A}',\ \mathrm{B}',\ \mathrm{C}'.$$
$$a',\ b',\ c';\ \mathrm{A},\ \mathrm{B},\ \mathrm{C}.$$

Cor. 1. Hence, being given any graphic property of a plane conic, we can get two correlative properties of a sphero-quartic.

Cor. 2. The properties of bicircular quartics which are derived by substitutions from those of conics have their analogues in sphero-quartics.

Cor. 3. If two sphero-quartics have one centre of inversion common to both, they

have four common generating circles; for the two conics which lie on the polar plane of the common centre of inversion have four common tangents.

Cor. 4. If $W=a\alpha^2+b\beta^2+c\gamma^2+d\delta^2=0$, $W'=a'\alpha^2+b'\beta^2+c'\gamma^2+d'\delta^2=0$ be two cyclides, the sphero-quartics (WU) und (W′U) have sixteen common generating circles; for they have four centres of inversion common, namely, the centres of the spheres α, β, γ, δ.

CHAPTER XIV.

SECTION I.—*Poles and Polars.*

Observation. All the spheres which we shall have occasion to use in this and the following chapter will be of the form $x\alpha+y\beta+z\gamma+w\delta$, where x, y, z, w are numerical coefficients.

274. If $(a, b, c, d, l, m, n, p, q, r)(\alpha, \beta, \gamma, \delta)^2=0$ be a cyclide, and

$$x_1\alpha+y_1\beta+z_1\gamma+w_1\delta=S_1, \quad x_2\alpha+y_2\beta+z_2\gamma+w_2\delta=S_2$$

be two spheres, then the condition that $\lambda S_1+\mu S_2=0$ may be a generating sphere of the cyclide is given by the determinant

$$\begin{vmatrix} a, & n, & m, & p, & \lambda x_1+\mu x_2, \\ n, & b, & l, & q, & \lambda y_1+\mu y_2, \\ m, & l, & c, & r, & \lambda z_1+\mu z_2, \\ p, & q, & r, & d, & \lambda w_1+\mu w_2, \\ \lambda x_1+\mu x_2, & \lambda y_1+\mu y_2, & \lambda z_1+\mu z_2, & \lambda w_1+\mu w_2, & 0, \end{vmatrix}=0. \qquad (167)$$

This determinant may be written $\lambda^2\Sigma'+2\lambda\mu\phi+\mu^2\Sigma''=0$, and we have a quadratic for determining the ratio $\lambda:\mu$. Now if $\phi=0$ we shall have the two values $\lambda:\mu$ equal, but with contrary signs. Hence $\phi=0$ is the condition that the spheres S_1, S_2 and the two generating spheres of the cyclide whose centres are collinear with their centres, or, in other words, the two generating spheres which are coaxal with them, should form an harmonic system of spheres.

Def. An harmonic system of spheres is a system passing through a common circle and whose centres form an harmonic row of points; this system possesses the property that four tangent planes through any common tangent line form an harmonic system; or again, that the segments which these spheres intercept on the line of collinearity of their centres may be an harmonic system of segments on that axis. See CHASLES, 'Sections Coniques,' art. 136.

275. The equation $\phi=0$ is the determinant

$$\begin{vmatrix} a, & n, & m, & p, & x_1, \\ n, & b, & l, & q, & y_1, \\ m, & l, & c, & r, & z_1, \\ p, & q, & r, & d, & w_1, \\ x_2, & y_2, & z_2, & w_2, & 0, \end{vmatrix}=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (168)$$

And if we suppose x_1, y_1, z_1, w_1 constant while x_2, y_2, z_2, w_2 vary, removing the suffixes from the lower row in the determinant, we see that if the centre of a variable sphere $S=0$ moves along the plane

$$\begin{vmatrix} a, & n, & m, & p, & x_1, \\ n, & b, & l, & q, & y_1, \\ m, & l, & c, & r, & z_1, \\ p, & q, & r, & d, & w_1, \\ x, & y, & z, & w, & 0, \end{vmatrix}=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (169)$$

then the sphere S, the sphere $S_1=x_1\alpha+y_1\beta+z_1\gamma+w_1\delta=0$, and the coaxal generating spheres form an harmonic pencil of spheres. Now the sphere $S\equiv x\alpha+y\beta+z\gamma+w\delta$, whose centre moves in the plane (169), evidently passes through two fixed points, namely, the two limiting points of the Jacobian sphere U of α, β, γ, δ, and of the plane (169); I shall call these points the pole points of the sphere $x_1\alpha+y_1\beta+z_1\gamma+w_1\delta$.

Cor. The plane (169) is the polar plane of the centre of S_1 with respect to the focal quadric of the cyclide.

276. If two spheres be such that one of them, A, passes through the pole points of the other, B, then, conversely, B passes through the pole points of A. This is evident from the determinants of the last article, from which it appears that the relation between the spheres is reciprocal. I shall extend the known terms of conics and quadrics, and call two such spheres conjugate spheres, and their two pairs of pole points conjugate pairs of pole points.

277. If two circles in space be such that the pole points of any sphere passing through one lie on the other, then, conversely, the pole points of any sphere passing through the latter lie on the former. This is analogous to the theorem in quadrics, that if two lines A and B be such that the polar plane of any point of A passes through B, then, conversely, the polar plane of any point of B passes through A, and may be derived from it by the substitution explained in the last chapter.

278. If $W=(\ast \between \alpha, \beta, \gamma, \delta)^2=0$ be a cyclide, and α', β', γ', δ' the sphero-coordinates of a pair of inverse points of W, that is, the pair of points given as common to the matrix

$$\begin{Vmatrix} \alpha, & \beta, & \gamma, & \delta, \\ \alpha', & \beta', & \gamma', & \delta', \end{Vmatrix} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (170)$$

and α'', β'', γ'', δ'' the sphero-coordinates of another pair of inverse points, then $\lambda\alpha'+\mu\alpha''$ &c. will be the sphero-coordinates of a pair of points concyclic with α', β', γ', δ' and α'', β'', γ'', δ''; and if these satisfy the equation of W, we shall have

$$\lambda^2W'+2\lambda\mu P+\mu^2W''=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (171)$$

Now if $P=0$, the circle through $\alpha', \beta', \gamma', \delta'$; $\alpha'', \beta'', \gamma'', \delta''$ meets the cyclide in two pairs of inverse points, which are harmonic conjugates to the two pairs $\alpha', \beta', \gamma', \delta'$; $\alpha'', \beta'', \gamma'', \delta''$ (see

CHASLES, 'Sections Coniques,' art. 136, also 'Bicircular Quartics,' arts. 153–155); but P is

$$=\left(\alpha'\frac{d}{d\alpha''}+\beta'\frac{d}{d\beta''}+\gamma'\frac{d}{d\gamma''}+\delta'\frac{d}{d\delta''}\right)W''.$$

Hence, omitting the double accents, we see that the equation of the sphere of which α', β', γ', δ' are the pole points is

$$\left(\alpha'\frac{d}{d\alpha}+\beta'\frac{d}{d\beta}+\gamma'\frac{d}{d\gamma}+\delta'\frac{d}{d\delta}\right)W=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (172)$$

And we have evidently the following theorem:—*If through a pair of inverse points* $(\alpha', \beta', \gamma', \delta')$ *we describe any circle* X *cutting the polar sphere of* $(\alpha', \beta', \gamma', \delta')$ *in a pair of inverse points* $(\alpha'', \beta'', \gamma'', \delta'')$, X *will cut the cyclide in two other pairs of inverse points, such that the four segments made on* X *by the pairs of points* $(\alpha', \beta', \gamma', \delta')$, $(\alpha'', \beta'', \gamma'', \delta'')$ *and by the cyclide are harmonic* (see 'Bicircular Quartics,' art. 153).

279. From the last article we see that, if $(\alpha', \beta', \gamma', \delta')$ be a pair of inverse points on the cyclide, the equation of the generating sphere which touches the cyclide at $(\alpha', \beta', \gamma', \delta')$ is

$$\left(\alpha'\frac{d}{d\alpha}+\beta'\frac{d}{d\beta}+\gamma'\frac{d}{d\gamma}+\delta'\frac{d}{d\delta}\right)W=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (173)$$

This equation establishes a relation between the coordinates $(\alpha', \beta', \gamma', \delta')$ of a pair of inverse points of the cyclide, and $(\alpha, \beta, \gamma, \delta)$ the sphero-coordinates of any other pair of inverse points on the generating sphere which touches the cyclide at $(\alpha', \beta', \gamma', \delta')$; and since the relation is symmetrical with respect to $(\alpha', \beta', \gamma', \delta')$ and $(\alpha, \beta, \gamma, \delta)$, we infer the following theorem:—*If through any pair of inverse points we describe a generating sphere of the cyclide, the locus of all their points of contact is the sphero-quartic which is given as the curve of intersection of* W *with the sphere* (173), *or the polar sphere of* $(\alpha', \beta', \gamma', \delta')$.

280. The discriminant of the equation (171) is

$$W'W''=P^2;$$

and by omitting the double accents we see that the equation of the binodal cyclide which circumscribes W, and which has the pair of points $(\alpha', \beta', \gamma', \delta')$ as nodes, is

$$WW'=\left\{\left(\alpha'\frac{d}{d\alpha}+\beta'\frac{d}{d\beta}+\gamma'\frac{d}{d\gamma}+\delta'\frac{d}{d\delta}\right)W\right\}^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (174)$$

281. Since a cyclide has five spheres of inversion, taking any point A, we get five points, namely, the inverses of A with respect to the five spheres of inversion of the cyclide. Let the inverse points be A_1, A_2, A_3, A_4, A_5; and, with the five pairs of inverse points (AA_1), (AA_2), (AA_3), (AA_4), (AA_5), we get by the last article five binodal cyclides circumscribed to W, and these binodals will have one common node, namely, the point A; the other nodes of these circumscribed cyclides will be the five points A_1, A_2, A_3, A_4, A_5.

282. If we invert the cyclide W from the point A, the five binodal cyclides of the last article invert into five cones of the second degree, each having double contact with the inverse cyclide (see art. 187). Now all the points of contact of the five double tangent cones lie on five concentric spheres. Hence we have the following theorem:—

If five binodal cyclides circumscribed to a cyclide W *have one common node, their curves of contact with* W *are five sphero-quartics lying on five spheres having a common radical plane.*

283. If $\alpha', \beta', \gamma', \delta'$; $\alpha'', \beta'', \gamma'', \delta''$ be two inverse pairs of points with respect to the sphere of inversion U of the cyclide W, the binodal cyclides which have these pairs of points respectively as nodes, and which circumscribe W, touch W along the sphero-quartics in which it is intersected by the polar spheres of $(\alpha', \beta', \gamma', \delta')$, $(\alpha'', \beta'', \gamma'', \delta'')$; hence the points of contact of generating spheres through both pairs lie on the circle of intersection of the polar spheres: but the plane of this circle intersects W in a bicircular quartic, and the bicircular quartic and the circle intersect in four points; hence there will be four points of contact, and consequently only two generating spheres can be described through two pairs of inverse points.

This theorem may be otherwise stated. Thus, it is plain that the circle through two pairs of inverse points is reciprocal to the circle in which their polar spheres intersect, and then we have the theorem, *that through any circle can be described two generating spheres; their points of contact are concyclic, and lie on the reciprocal circle.*

284. Since when a sphere of inversion U and a focal quadric F are given the cyclide is determined, and if nine points are given the quadric is determined, it hence follows *that, being given a sphere* U *and nine spheres which are orthogonal to it, a cyclide can be described having* U *for a sphere of inversion, and the nine spheres as generating spheres.*

285. Since, being given any eight points, three quadrics can be described to touch a given plane, we have the theorem, *that, being given any eight generating spheres of a cyclide, three cyclides can be described through the same pair of inverse points with respect to* U (see art. 268, 2°), *and the cyclides are mutually orthogonal* (see art. 119).

286. If two quadrics intersect in the same eight points, all quadrics passing through these eight points have a common curve of intersection. Hence, *if two cyclides* W, W′ *have eight generating spheres common, every cyclide having these eight spheres as generators will have also as generators all the generators common to* W, W′. We shall in the next chapter find the equation of the surface formed by all the generators common to two cyclides, and also give some of its properties.

287. Given seven points or tangent planes common to a series of quadrics, then an eighth point or tangent plane common to the system is determined. Hence, *being given seven generating spheres or pairs of inverse points common to a system of cyclides, then an eighth generating sphere or pair of inverse points common to the whole system is determined.*

288. *If a system of cyclides pass through the same eight pairs of inverse points, their polar spheres with respect to a given pair of inverse points have a common radical plane.*

For if P and Q be the polar spheres of a given pair of inverse points with respect to W and W′, then $P + \lambda Q$ is the polar sphere of the same pair of inverse points with respect to $W + \lambda W'$.

289. By reciprocating the theorem of the last article we get the theorem:—*If a system*

of cyclides have eight common generating spheres, the locus of the pole points of a fixed sphere is a circle.

290. *If a system of cyclides pass through the same eight pairs of inverse points, that is, if they have a common curve of intersection, the polar circles of a fixed circle generate a cyclide.*

Let the polar spheres of two fixed pairs of inverse points be $P+\lambda Q$ and $P'+\lambda Q'$; eliminating λ, we get the cyclide $PQ'-P'Q=0$.

291. *Reciprocally, if a system have eight common generating spheres, the polar circles of a fixed circle generate a cyclide.*

292. *If a system of cyclides pass through a common curve, the locus of the pole points of a fixed sphere is a torse curve of the sixth degree.*

Demonstration. Let the polar spheres of three pairs of inverse points lying in the fixed sphere be $P+\lambda Q$, $P'+\lambda Q'$, $P''+\lambda Q''$; then, eliminating λ, we get the system of determinants

$$\left\| \begin{matrix} P, & P', & P'', \\ Q, & Q', & Q'', \end{matrix} \right\| \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (175)$$

which represents a twisted curve of the sixth degree. For the intersection of the cyclides $PQ'-P'Q$, $PQ''-P''Q$, each of which has the imaginary circle at infinity as a double line, is a twisted curve of the eighth degree; but this includes the circle (PQ), which is not part of the intersection of the cyclides $PQ''-P''Q$, $P'Q''-P''Q'$; there is, therefore, only a curve of the sixth degree common to the three determinants of the matrix (175).

Cor. The cone whose vertex is the centre of U, the common sphere of inversion of the cyclides, and which stands on the curve of the sixth degree, is only of the third degree. For any plane through the vertex of the cone meets the curve in six points; but these are inverse two by two, since the curve is evidently an anallagmatic, and therefore only three edges of the cone lie in the plane.

293. *Given seven pairs of inverse points of a cyclide, the polar spheres of a given inverse pair of points pass through a given pair of inverse points.*

For evidently the polar sphere of a given fixed pair of inverse points with respect to $W+\lambda W'+\mu W''$ will be of the form $P+\lambda P'+\mu P''$, and will therefore pass through a given fixed pair of inverse points, namely, the two points common to the spheres P, P', P''.

Reciprocally, given seven generating spheres of a cyclide, the locus of the pole points of a fixed sphere is a fixed sphere.

294. If $W\equiv(*\between\alpha,\beta,\gamma,\delta)^2=0$ be a given cyclide, we have seen that $(*\between\lambda,\mu,\nu,\varrho)^2=0$ *is the tangential equation* of the focal quadric; but if the discriminant vanish of the equation of a quadric in tangential coordinates, it represents a conic in space, and the corresponding cyclide will be binodal. Hence we have the theorem, *if the discriminant vanish of the equation of a cyclide, the cyclide will be a binodal cyclide.*

295. Since the discriminant contains the coefficients in the fourth degree, it follows

that we have a biquadratic to solve to determine λ, in order that $W+\lambda W'$ may represent a binodal cyclide. Hence, through the curve of intersection of two cyclides, four binodal cyclides may be described. The binodes of these binodals are thus determined. If we denote by W_1, W_2, W_3, W_4 the differentials of W with respect to α, β, γ, δ respectively, and by λ_1, λ_2, λ_3, λ_4 the four roots of the biquadratic in λ, then any three of the four spheres

$$W_1+\lambda_1 W'_1, \quad W_2+\lambda_1 W'_2, \quad W_3+\lambda_1 W'_3, \quad W_4+\lambda_1 W'_4$$

will determine by their mutual intersections the binodes of one of the binodals; and the binodes of the remaining binodals are got from these by using the remaining roots of the biquadratic in place of λ_1, namely, λ_2, λ_3, λ_4 respectively.

296. *There are four spheres whose pole points are the same with respect to all the cyclides passing through a common curve of intersection of two cyclides, namely, the polar spheres of the four pairs of nodes of the four binodals of the last article.* For to express the condition that

$$\alpha W'_1+\beta W'_2+\gamma W'_3+\delta W'_4,$$
$$\alpha X'_1+\beta X'_2+\gamma X'_3+\delta X'_4$$

should represent the same spheres, we find the following set of determinants:

$$\left\| \begin{matrix} W'_1, & W'_2, & W'_3, & W'_4, \\ X'_1, & X'_2, & X'_3, & X'_4, \end{matrix} \right\|$$

and these satisfy by the last article the binodes of the binodal cyclides.

297. *The four spheres are such that the two points common to any three are the pole points of the fourth; and, conversely, the four pairs of binodal points are such that the sphere determined by any three is the polar sphere of the fourth pair.* Thus, if the two cyclides be W and W', their equations in terms of the four spheres will be of the form

$$aX^2+bY^2+cZ^2+dV^2,$$
$$a'X^2+b'Y^2+c'Z^2+d'V^2,$$

and the nodes of the binodal cyclides are the four pairs of points

$$(XYZ), \quad (XYV), \quad (XZV), \quad (YZV).$$

It is to be remembered that the spheres X, Y, Z, V are not mutually orthogonal.

298. If the cyclide W' break up into two spheres, the form $W+\lambda W'$ becomes $W+\lambda LM$. In general the intersection of two cyclides is a twisted curve of the eighth degree; but if one of the two cyclides reduce to two spheres, the intersection becomes two sphero-quartics. Any pair of inverse points on the circle LM has the same polar sphere with respect to all the cyclides of the system $W+\lambda LM$; and in particular all the cyclides of the system have the same generating spheres at the four points where W is met by the circle LM.

Lastly, all the cyclides of the system are enveloped by four binodal cyclides. For if

the four common generating spheres be α, β, γ, δ, then the nodes of the four enveloping binodals are the four pairs of inverse points (α, β, γ), (α, γ, δ), (α, β, δ), (β, γ, δ).

299. If a sphero-quartic be common to three cyclides, each pair must have another sphero-quartic, and the spheres through these sphero-quartics are coaxal.

300. W, W′ are two cyclides having a common sphere of inversion U; it is required to find the locus of the pole points of the generating spheres of W′ with respect to W.

Let W, W′ be reduced to their canonical forms,

$$W \equiv a\alpha^2+b\beta^2+c\gamma^2+d\delta^2, \quad W' \equiv a'\alpha^2+b'\beta^2+c'\gamma^2+d'\delta^2.$$

Now let $(\alpha', \beta', \gamma', \delta')$ be a pair of inverse points with respect to U, then the polar sphere of $(\alpha', \beta', \gamma', \delta')$ with respect to W is

$$a\alpha'\alpha+b\beta'\beta+c\gamma'\gamma+d\delta'\delta=0;$$

and the condition that this should be a generating sphere of W′ is

$$\frac{a^2\alpha'^2}{a'}+\frac{b^2\beta'^2}{b'}+\frac{c^2\gamma'^2}{c'}+\frac{d^2\delta'^2}{d'}=0;$$

and omitting the accents, we have the locus required,

$$\left(\frac{a^2}{a'}\right)\alpha^2+\left(\frac{b^2}{b'}\right)\beta^2+\left(\frac{c^2}{c'}\right)\gamma^2+\left(\frac{d^2}{d'}\right)\delta^2=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (176)$$

301. If we denote the cyclide (176) by $W''=0$, we see that the equations of the focal quadrics of W, W′, W″, in tangential coordinates, are

$$a\lambda^2+b\mu^2+c\nu^2+d\varrho^2=0,$$
$$a'\lambda^2+b'\mu^2+c'\nu^2+d'\varrho^2=0,$$
$$\frac{a^2}{a'}\lambda^2+\frac{b^2}{b'}\mu^2+\frac{c^2}{c'}\nu^2+\frac{d^2}{d'}\varrho^2=0;$$

and the third is the reciprocal of the second with respect to the first. Hence we have the following theorem:—*If* W, W′, W″ *be three cyclides having a common sphere of inversion* U, *and* F, F′, F″ *be the focal quadrics of* W, W′, W″ *corresponding to* U, *then if* F″ *be the reciprocal of* F′ *with respect to* F, W″ *will be the reciprocal of* W′ *with respect to* W.

302. Since the reciprocal of a circle in space with respect to a cyclide is another circle in space, hence, if a variable circle move along three circles, its reciprocal will move along three circles reciprocal to the former; *so that the reciprocal of a cyclide described by the motion of a variable circle is another cyclide described by the motion of another variable circle.* This corresponds to the theorem that the reciprocal of a ruled surface is a ruled surface.

303. *If a pair of inverse points move along a fixed sphere, the locus of the pair of inverse points common to their polar spheres, with respect to three cyclides having a common sphere of inversion* U, *is a cyclide of the sixth degree, having* U *for a sphere of inversion.*

Demonstration. If the polar spheres of the three cyclides with respect to the three pairs of inverse points $(\alpha', \beta', \gamma', \delta')$, $(\alpha'', \beta'', \gamma'', \delta'')$, $(\alpha''', \beta''', \gamma''', \delta''')$ be $X', Y', Z', X'', Y'', Z'', X''', Y''', Z'''$, then the polar spheres of the cyclides with respect to the pair of inverse points whose sphero-coordinates are $k\alpha'+l\alpha''+m\alpha'''$, $k\beta'+l\beta''+m\beta''$, $k\gamma'+l\gamma''+m\gamma'''$, $k\delta'+l\delta''+m\delta''$ will plainly be $kX'+lX''+mX'''$, $kY'+lY''+mY'''$, $kZ'+lZ''+mZ'''$. Hence, eliminating k, l, m, we get the required locus,

$$\begin{vmatrix} X', & X'', & X''', \\ Y', & Y'', & Y''', \\ Z', & Z'', & Z''', \end{vmatrix} = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (177)$$

a cyclide of the sixth degree having the circle at infinity as a triple line.

Cor. *If the pair of inverse points move along a circle, the locus of the intersection of their polar spheres with respect to two cyclides will be a cyclide of the fourth degree.*

304. From article 301, we can easily infer theorems in cyclide reciprocation from known theorems in quadric reciprocation. Thus, if two spheres be concentric, the reciprocal of one with respect to the other is a concentric sphere. Hence, *if two Cartesian cyclides having a common sphere of inversion have a common triple focus, the reciprocal of one with respect to the other is a Cartesian cyclide having the same triple focus;* or, since the theorem concerning the spheres may be enunciated thus, if tangent planes to a sphere intersect at given angles, the locus of their point of intersection is a concentric sphere, and the envelope of the plane through their points of contact is another concentric sphere. Hence we infer that *if three generating spheres of a Cartesian cyclide intersect at given angles, the locus of their points of intersection is a Cartesian cyclide having the same triple focus, and the envelope of the sphere through their six points of contact is another Cartesian cyclide having also the same triple focus.* These theorems may evidently be inferred by the methods of substitution given in the last Chapter.

305. If we reciprocate one sphere with respect to another not concentric, we get a quadric of revolution. Hence *the reciprocal of a Cartesian cyclide with respect to another Cartesian cyclide having a different triple focus is a symmetrical cyclide, that is, a cyclide having one of its spheres of inversion opened out into a plane, the corresponding focal quadric being one of revolution.*

306. If we reciprocate a surface of revolution with respect to a sphere, we get a general quadric. Hence, *if we reciprocate a cyclide having a plane of symmetry with respect to a Cartesian cyclide, we get a general cyclide.*

307. The principles explained in recent articles will obviously give some of the systems of substitutions explained in the last Chapter; and, conversely, the results of this Chapter may be derived from the substitutions of the last. It is unnecessary to pursue the subject further, and I shall conclude the section with the two following theorems:—

1°. The locus of the intersection of three rectangular tangent planes to a quadric is a sphere. Hence *the locus of the pairs of points common to three generating spheres of a cyclide which are mutually orthogonal is a Cartesian cyclide.*

2°. Every central quadric has two systems of circular sections. Hence *every quartic cyclide has two systems circumscribed to it of binodal Cartesian cyclides, and the locus of their nodes is two right lines respectively perpendicular to the directions of the planes of circular sections of the focal quadric.*

SECTION II.—*Poles and Polars. Sphero-quartics.*

308. The investigation of the polar properties of sphero-quartics is analogous to that employed in the last section for cyclides.

Thus if $(a, b, c, f, g, h \between \alpha, \beta, \gamma)^2=0$, where α, β, γ are circles on a sphere U, be the equation of a sphero-quartic, and if $\lambda_1\alpha+\mu_1\beta+\nu_1\gamma=C_1$ and $\lambda_2\alpha+\mu_2\beta+\nu_2\gamma=C_2$ be two circles, then the condition that $lC_1+mC_2=0$ should be a generating circle is given by the determinant

$$\begin{vmatrix} a, & h, & g, & l\lambda_1+m\lambda_2, \\ h, & b, & f, & l\mu_1+m\mu_2, \\ g, & f, & c, & l\nu_1+m\nu_2, \\ l_1\lambda+m\lambda_2, & l\mu_1+m\mu_2, & l\nu_1+m\nu_2, & 0. \end{vmatrix} \quad . \quad . \quad . \quad . \quad (178)$$

This determinant may be written in the form

$$l^2\Sigma'+2lm\phi+m^2\Sigma''=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (179)$$

Hence $\phi=0$ is the condition that the circles C_1, C_2, and the two generating circles whose centres lie on the same great circle with their centres, should form an harmonic pencil of circles, or it is the condition that their centres should form an harmonic row of points; or, again, it is the condition that their diameters should form an harmonic system of segments on the same great circle of U.

309. The equation $Q=0$ is the determinant

$$\begin{vmatrix} a, & h, & g, & \lambda_1, \\ h, & b, & f, & \mu_1, \\ g, & f, & c, & \nu_1, \\ \lambda_2 & \mu_2 & \nu_2, & 0, \end{vmatrix}=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (180)$$

This is the condition that the circles C_1 and C_2 may be conjugate circles with respect to the quartic; if the suffixes be removed from the lower row, we see that, if the centre of a variable circle $C=\lambda\alpha+\mu\beta+\nu\gamma=0$ move along the great circle

$$\begin{vmatrix} a, & h, & g, & \lambda_1, \\ h, & b, & f, & \mu_1, \\ g, & f, & c, & \nu_1, \\ \lambda, & \mu, & \nu, & 0, \end{vmatrix}=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (181)$$

then $C=0$, $C_1=\lambda_1\alpha+\mu_1\beta+\nu_1\gamma$, and the two generating circles whose centres are on the same great circle with their centres form an harmonic pencil; but if a variable circle

whose centre moves along a great circle cuts a given circle J orthogonally, it will pass through two fixed points; these fixed points are the limiting points of J and the great circle; or if we denote, as Dr. SALMON does, the equation of a great circle by an equation of the first degree in $x\ y\ z$, say $L=0$, and the circle J by the equation $S^{\frac{1}{2}}-M=0$, then the two limiting points will be given as those for which the discriminant of $(S^{\frac{1}{2}}-M+kL)$ vanishes. These points will be the pole points of the circle $\lambda_1\alpha+\mu_1\beta+\nu_1\gamma=0$ with respect to the quartic $(a, b, c, f, g, h \between \alpha, \beta, \gamma)^2=0$.

Cor. 1. The great circle (181) is the polar of the centre of $C_1=\lambda_1\alpha+\mu_1\beta+\nu_1\gamma$ with respect to the sphero-conic whose tangential equation is

$$A\lambda^2+B\mu^2+C\nu^2+2H\lambda\mu+2F\mu\nu+2G\nu\lambda=0,$$

where, as usual, $A=bc-f^2$, $B=ca-g^2$, &c.

Cor. 2. If two circles be such that one of them, A, passes through the pole points of B, then, conversely, B passes through the pole points of A.

310. If $(\alpha', \beta', \gamma')$ be the cyclic coordinates of a pair of inverse points, that is, the pair of points given by the system of circles

$$\left\| \begin{matrix} \alpha, & \beta, & \gamma, \\ \alpha', & \beta', & \gamma', \end{matrix} \right\|$$

and α'', β'', γ'' the cyclic coordinates of another pair of points, then $l\alpha'+m\alpha''$, $l\beta'+m\beta''$, $l\gamma'+m\gamma''$ will be the coordinates of a pair of points concyclic with them; and if these satisfy the equation of the sphero-quartic, which we may denote by Q, we shall have

$$l^2Q'+2lmP+m^2Q''=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (182)$$

Now, if $P=0$ the circle through $(\alpha', \beta', \gamma')$, $(\alpha'', \beta'', \gamma'')$ meets the quartic in two pairs of points which are harmonic conjugates with respect to the two pairs $(\alpha', \beta', \gamma')$, $(\alpha'', \beta'', \gamma'')$; but P is

$$\left(\alpha'\frac{d}{d\alpha''}+\beta'\frac{d}{d\beta''}+\gamma'\frac{d}{d\gamma''}\right)Q''.$$

Hence the equation of the polar circle of the points $(\alpha', \beta', \gamma')$ is

$$\left(\alpha'\frac{d}{d\alpha}+\beta'\frac{d}{d\beta}+\gamma'\frac{d}{d\gamma}\right)Q=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (182\,a)$$

Cor. 1. From this article we have evidently the following theorem:—If through a pair of inverse points α', β', γ' we describe any circle Z cutting the polar circle of $(\alpha', \beta', \gamma')$ in a pair of inverse points $(\alpha'', \beta'', \gamma'')$, Z will cut the quartic in two other pairs of points, such that the four segments made on Z by $(\alpha', \beta', \gamma')$, $(\alpha'', \beta'', \gamma'')$ and by the quartic are harmonic.

Cor. 2. If $(\alpha', \beta', \gamma')$ be a pair of points, the generating circle which touches Q at $(\alpha', \beta', \gamma')$ is

$$\left(\alpha'\frac{d}{d\alpha}+\beta'\frac{d}{d\beta}+\gamma'\frac{d}{d\gamma}\right)Q=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (183)$$

Cor. 3. If through a pair of inverse points we describe two generating circles of the

quartic, their points of contact with the quartic lie on the polar circle of the given pair of inverse points.

311. The discriminant of the equation (182) is

$$Q'Q''=P^2.$$

Hence the equation of the pair of generating circles of the sphero-quartic which pass through $(\alpha', \beta', \gamma')$ is

$$QQ'=\left\{\left(\alpha'\frac{d}{d\alpha}+\beta'\frac{d}{d\beta}+\gamma'\frac{d}{d\gamma}\right)Q\right\}^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (184)$$

This equation can also, as is evident, be written in the form

$$\begin{vmatrix} a, & h, & g, & \beta'\gamma-\beta\gamma', \\ h, & b, & f, & \gamma'\alpha-\gamma\alpha', \\ g, & f, & c, & \alpha'\beta-\alpha'\beta, \\ \beta'\gamma-\beta\gamma', & \gamma'\alpha-\gamma\alpha', & \alpha'\beta-\alpha\beta' & 0, \end{vmatrix}=0. \quad . \quad . \quad . \quad (185)$$

In like manner the equation of the four points in which the circle $\lambda_1\alpha+\mu_1\beta+\nu_1\gamma$ cuts the quartic is

$$\Sigma'\,\Sigma''-\phi^2=0 \text{ (see art. 308)}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (186)$$

which may also be written in the form

$$\begin{vmatrix} A, & H, & G, & \mu_1\nu-\mu\nu_1, \\ H, & B, & F, & \nu_1\lambda-\nu\lambda_1, \\ G, & F, & C, & \lambda_1\mu-\lambda\mu_1, \\ \mu_1\nu-\mu\nu_1, & \nu_1\lambda-\nu\lambda_1, & \lambda_1\mu-\lambda\mu_1 & 0. \end{vmatrix} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (187)$$

CHAPTER XV.

Invariants and Covariants of Cyclides.

312. It is always possible in an infinity of ways to choose four spheres $\alpha, \beta, \gamma, \delta$ so that the equations of two cyclides having a common sphere of inversion can be thrown into the forms

$$W=(a, b, c, d, l, m, n, p, q, r)(\alpha, \beta, \gamma, \delta)^2=0,$$

$$W'=(a', b', c', d', l', m', n', p', q', r')(\alpha, \beta, \gamma, \delta)^2=0.$$

For each of these equations contains explicitly nine constants, and each of the spheres $\alpha, \beta, \gamma, \delta$ contains implicitly four constants, so that we have thirty-four constants at our disposal, and we require but twenty-two. For the two cyclides are determined when the common sphere of inversion and the two focal quadrics are given; hence the number of constants required is $4+9\times 2=22$.

313. Denoting the discriminants of W, W′ by Δ, Δ', we have Δ, Δ' given by the determinants

$$\Delta=\begin{vmatrix} a, & n, & m, & p, \\ n, & b, & l, & q, \\ m, & l, & c, & r, \\ p, & q, & r, & d, \end{vmatrix}, \qquad \Delta=\begin{vmatrix} a', & n', & m', & p', \\ n', & b', & l', & q', \\ m', & l', & c', & r', \\ p', & q', & r', & d'. \end{vmatrix} \quad . \quad . \quad . \quad (188)$$

Then the discriminant of $k\mathrm{W}+\mathrm{W}'$ will be got from Δ by writing in place of a, b, &c. $ka+a'$, $kb+b'$, &c.; the result will be a quartic in k, which I shall call the invariant equation of the two cyclides, and write in the form

$$k^4\Delta+k^3\Theta+k^2\Phi+k\Theta'+\Delta'=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (189)$$

Since there are four values of k which satisfy this equation, we see that through the curve (WW′) can be drawn four binodal cyclides, that is, four cyclides each having two conic nodes. If we eliminate k between $k\mathrm{W}+\mathrm{W}'$ and (189), we shall get the equation of these four binodal cyclides, namely,

$$\Delta'\mathrm{W}^4-\Theta'\mathrm{W}^3\mathrm{W}'+\Phi\mathrm{W}^2\mathrm{W}'^2-\Theta\mathrm{W}\mathrm{W}'^3+\Delta\mathrm{W}'^4=0. \quad . \quad . \quad . \quad (190)$$

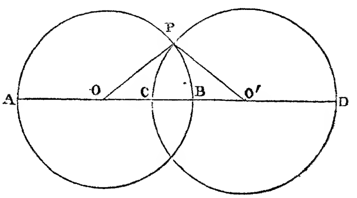
314. Since the equations of W, W′ are the same in form as the tangential equation of their focal quadrics F, F′, and if F, F′ touch, W, W′ will have double contact, hence it follows that the condition of W, W′ having double contact is the vanishing of the discriminant of the invariant equation (189); $\therefore$ the tact-invariant of W, W′ is

$$4(12\Delta\Delta'-3\Theta\Theta'+\Phi^2)^3-(72\Delta\Delta'\Phi+9\Theta\Theta'\Phi-27\Delta\Theta'^2-27\Delta'\Theta^2-2\Phi^3)^2. \quad . \quad (191)$$

315. The tact-invariants of two conics and two quadrics are the analytical expression of remarkable geometrical properties which have not been hitherto noticed by any writer so far as I am aware; on this account, and because extensions of them hold for the tact-invariants of two bicircular quartics and two cyclides, I shall give their investigations here, and we shall incidentally find results that are important independently of the properties that we have alluded to, and which we now proceed to demonstrate.

316. If A, B, C, D be four points ranged in alternate order on a right line, the six anharmonic ratios of A, B, C, D can be expressed in a way that bears a remarkable analogy to the six trigonometrical functions of an angle.

Fig. 8.



On A B and C D describe circles; let O, O′ be their centres, P one of their points of intersection, then the angle O P O′ equal angle of intersection of the circles; and taking the six anharmonic ratios of A, B, C, D, as given in TOWNSEND's 'Modern Geometry,' or CHASLES's 'Géométrie Supérieure,' it is easy to see, if we denote the angle O P O′ by θ, that we shall have the equations:

$$
\begin{array}{ll}
(1) & CA \,.\, BD : BA \,.\, CD = \sin^2 \tfrac{1}{2}\theta, \\
(2) & CB \,.\, AD : AB \,.\, CD = \cos^2 \tfrac{1}{2}\theta, \\
(3) & AC \,.\, BD : BC \,.\, AD = -\tan^2 \tfrac{1}{2}\theta, \\
(4) & BA \,.\, CD : CA \,.\, BD = \operatorname{cosec}^2 \tfrac{1}{2}\theta, \\
(5) & AB \,.\, CD : CB \,.\, AD = \sec^2 \tfrac{1}{2}\theta, \\
(6) & BC \,.\, AD : AC \,.\, BD = -\cot^2 \tfrac{1}{2}\theta.
\end{array}
$$

317. Let there be given two conics referred to their common self-conjugate triangle $S \equiv x^2+y^2+z^2$, $S' \equiv ax^2+by^2+cz^2$, and let us denote by θ', θ'', θ''' the angles (see last article) of the anharmonic ratios of the three quartets of points in which the sides of the self-conjugate triangle is intersected by the two conics. Then for determining θ' we must find the anharmonic ratio in which the side $x=0$ is intersected by the two conics; for that purpose we have the pencil formed by the two pairs of lines $y^2+z^2=0$ and $by^2+cz^2=0$, and we easily get

$$\sin^2 \tfrac{1}{2}\theta' = -\frac{(b^{\frac{1}{2}}-c^{\frac{1}{2}})^2}{4b^{\frac{1}{2}}c^{\frac{1}{2}}},$$

$$\cos^2 \tfrac{1}{2}\theta' = \frac{(b^{\frac{1}{2}}+c^{\frac{1}{2}})^2}{4b^{\frac{1}{2}}c^{\frac{1}{2}}}.$$

Hence

$$\sin^2 \theta' = -\frac{(b-c)^2}{4bc}.$$

Now if we form the invariant equation in k for the two conics S, S′, that is, if we form the discriminant of $kS+S'$, and denote its roots by k', k'', k''', these roots are known to be $-a$, $-b$, $-c$. Hence we have the following system of equations:—

$$\left.\begin{aligned}
\sin^2\theta' &= -(k''-k''')^2 : 4k''k''', \\
\sin^2\theta'' &= -(k'''-k')^2 : 4k'''k', \\
\sin^2\theta''' &= -(k'-k'')^2 : 4k'k''.
\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (192)$$

Hence the discriminant for the invariant equation of the two conics S, S′ is $\frac{-64\Delta'^2}{\Delta^2}(\sin^2\theta' . \sin^2\theta'' . \sin^2\theta''')$, or omitting the multiplier $\frac{-64\Delta'^2}{\Delta^2}$, which is numerical, the discriminant is $\sin^2\theta' . \sin^2\theta'' . \sin^2\theta'''$; and as each sine squared is the product of two anharmonic ratios (see art. 316), we have the following theorem, which is the one referred to in art. 315:—

The tact-invariant of two conics is the product of six anharmonic ratios, and the vanishing of any one of these six ratios is a condition of contact of the two conics.

Cor. From the values given for the invariant angles θ', θ'', θ''' in this article, we get

$$e^{2\theta'\sqrt{-1}} = k'' : k''', \quad e^{2\theta''\sqrt{-1}} = k''' : k', \quad e^{2\theta'''\sqrt{-1}} = k' : k''.$$

Hence $\theta'+\theta''+\theta'''=0$, that is, *the sum of the three invariant angles of two conics is equal to zero.*

318. If we take the original conics S, S′, and form the reciprocal of S with respect to S′, we get $a^2x^2+b^2y^2+c^2z^2$; if we denote this by S″, and form the invariant angles of S, S″, we find them to be $2\theta'$, $2\theta''$, $2\theta'''$; similarly, if S‴ be the reciprocal of S′ with respect to S″, the invariant angles of S, S‴ are $3\theta'$, $3\theta''$, $3\theta'''$, and so on. Again, if we denote by $S_{\frac{1}{2}}$ the conic which reciprocates S into S′, the invariant angles of S, $S_{\frac{1}{2}}$ are $\frac{1}{2}\theta'$, $\frac{1}{2}\theta''$, $\frac{1}{2}\theta'''$, &c.

Cor. *If two conics* S, S′ *be so related that a triangle circumscribed to* S *will be inscribed in* S′, *and if we reciprocate* S *with respect to* S′, *the reciprocal conic* S″ *will be related to* S *by the condition that any triangle inscribed in* S″ *will be self-conjugate with respect to* S.

319. From the values given (art. 317), we get $\cos\theta'=\frac{b+c}{2\sqrt{bc}}$; and since $-a=k'$, we have $\cos\theta'\div\sqrt{k'}=(b+c):2\sqrt{-abc}$, with similar values for $\cos\theta''\div\sqrt{k''}$, $\cos\theta'''\div\sqrt{k'''}$. Hence we may write the equation of the conic ϕ, which is the envelope of a line cut harmonically by S, S′ (see SALMON's 'Conics,' p. 334), in the following manner:—

$$\left(\frac{\cos\theta'}{\sqrt{k'}}\right)\lambda^2+\left(\frac{\cos\theta''}{\quad}\right)\mu^2+\left(\frac{\cos\theta'''}{\sqrt{k'''}}\right)\nu^2=0. \quad . \quad . \quad . \quad . \quad . \quad (193)$$

This equation is altogether metrical, having no reference to any particular system of axes, being in fact true for any system whatever of trilinear axes.

Cor. 1. In like manner the equation of SALMON's conic F, which is the locus of points whence tangents to S, S′ form an harmonic pencil, may be written in the form

$$\sqrt{k'}\cos\theta' x^2+\sqrt{k''}\cos\theta'' y^2+\sqrt{k'''}\cos\theta''' z^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (194)$$

Cor. 2. The discriminants of the covariant conics ϕ, F are the quotient and product of the expressions $\cos\theta'.\cos\theta''.\cos\theta'''$ and $\sqrt{k'.k''.k'''}$.

Cor. 3. The reciprocal of S′ with respect to F, that is, with respect to the conic (194), is

$$\cos^2\theta' x^2+\cos^2\theta'' y^2+\cos^2\theta''' z^2=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (195)$$

320. It is easy now to extend the results we have arrived at to the case of two quadrics. Let them be

$$U\equiv ax^2+by^2+cz^2+dw^2=0,$$
$$V\equiv x^2\ +y^2\ +z^2\ +w^2\ =0;$$

and if the angles be determined thus,

$$e^{2\theta'\sqrt{-1}}=k'':k''',\qquad e^{2\theta^{(\text{IV})}\sqrt{-1}}=k':k^{(\text{IV})},$$
$$e^{2\theta''\sqrt{-1}}=k''':k',\qquad e^{2\theta^{(\text{V})}\sqrt{-1}}=k'':k^{(\text{IV})},$$
$$e^{2\theta'''\sqrt{-1}}=k':k'',\qquad e^{2\theta^{(\text{VI})}\sqrt{-1}}=k''':k^{(\text{IV})},$$

the following Table gives the angles for the pairs of conics in which the faces of the self-conjugate tetrahedron intersect the quadrics:—

Faces.	Angles.		
x,	θ',	θ'',	$\theta^{(\mathrm{vi})}$,
y,	θ''',	$\theta^{(\mathrm{iv})}$,	$\theta^{(\mathrm{vi})}$,
z,	θ''',	$\theta^{(\mathrm{iv})}$,	$\theta^{(\mathrm{v})}$,
w,	θ';	θ'',	θ''';

and then the discriminant of the invariant equation of the two quadrics is

$$\sin^2\theta'.\sin^2\theta''.\sin^2\theta'''.\sin^2\theta^{(\mathrm{iv})}.\sin^2\theta^{(\mathrm{v})}.\sin^2\theta^{(\mathrm{vi})},$$

which, as in art. 317, is the product of twelve anharmonic ratios. Hence *the tact-invariant of two quadrics is the product of twelve anharmonic ratios, and the vanishing of any one of these ratios is the condition of contact of the two quadrics.*

Cor. *It follows from art.* 317 *that the condition of double contact of two bicircular or two sphero-quartics is expressible as the product of six anharmonic ratios, and, from the present article, of twelve anharmonic ratios, for the double contact of two cyclides.*

321. We now return from this digression (articles 315–320). If the cyclide W$'$ (see art. 313) be a binodal cyclide, we have $\Delta'=0$; and we proceed to examine the meaning in this case of Θ, Φ, Θ'. Let us take the nodes of W$'$ as the points common to three of the spheres of reference α, β, γ, then in the equation of W$'$ (see art. 313) p', q', r', d' all vanish, and we get $\Theta'=d(a'b'c'+2l'm'n'-a'l'^2-b'm'^2-c'n'^2)$, or Θ' vanishes if W$'$ break up into two spheres, or if the nodes of W$'$ be on the surface of W. Let the binodal cyclide which circumscribes W, and whose nodes coincide with those of W$'$, viz.

$$d(a\alpha^2+b\beta^2+c\gamma^2+2l\beta\gamma+2m\gamma\alpha+2n\alpha\beta)-(p\alpha+q\beta+r\gamma)^2=0,$$

be written

$$a''\alpha^2+b''\beta^2+c''\gamma^2+2l''\beta\gamma+2m''\gamma\alpha+2n''\alpha\beta=0,$$

then Φ may be written

$$a''(b'c'-l'^2)+b''(c'a'-m'^2)+c''(a'b'-n'^2)+2l''(m'n'-a'l')+2m''(n'l'-b'm')+2n''(l'm'-c'n'). \quad (196)$$

Hence, by the theory of bicircular quartics (art. 174), Φ *vanishes when the intersections of three harmonic spheres of* W$'$ *are three circles having double contact with* W. In like manner

$$\left.\begin{aligned}d\Theta=a'(b''c''-l''^2)+b'(c''a''-m''^2)+c'(a''b''-n''^2)+2l'(m''n''-a''l'')\\+2m'(n''l''-b''m'')+2n'(l''m''-c''n''),\end{aligned}\right\} \quad . \quad (197)$$

or Θ vanishes when the generators of W are harmonic spheres of W$'$ (see 'Bicircular Quartics,' art. 218).

When W$'$ breaks up into two spheres, both Δ' and Θ' vanish. Let the two spheres be α, β, then W$'$ reduces to $n\alpha\beta$, and Φ reduces to $n'^2(r^2-cd)$, or Φ will vanish when the intersection of the two spheres is a circle having double contact with W. In like manner Θ vanishes when the two spheres are conjugate spheres with respect to W. The condition will be satisfied, $\Theta^2=4\Delta\Phi$, if either of the two spheres be a generating sphere of W.

322. Given nine cyclides, W$_1$, W$_2$ .... W$_9$, it is possible in an infinity of ways to deter-

mine nine constants $l_1, l_2, \ldots l_9$ so that $l_1W_1+l_2W_2\ldots+l_9W_9$ may be a perfect square L^2, or the product of two spheres, M and N; it is required to prove that the envelope of the sphere L is a cyclide, and that M and N are conjugate spheres with respect to it.

Demonstration. We can determine a cyclide $W=(a, b\ldots)(\alpha, \beta, \gamma, \delta)^2$ so that the invariant Θ shall vanish for W and each of the nine cyclides, since we have nine equations of the form

$$Aa_1+Bb_1+Cc_1+Dd_1+2Ll_1+2Mm_1+2Nn_1+2Pp_1+2Qq_1+2Rr_1=0, \quad . \quad . \quad (198)$$

A, B, C, &c. being the minors of the determinant Δ (see art. 313), and a_1, b_1, c_1, &c. the coefficients of W_1; hence the mutual ratios of A, B, C are determined. Now if we have separately nine equations of the form (198), we have plainly also

$$A(l_1a_1+l_2a_2\ldots l_9a_9)+\text{\&c.}=0,$$

that is, Θ vanishes for W and every cyclide of the system $l_1W_1+l_2W_2\ldots l_9W_9$. Hence the theorem is proved.

Cor. If the sphere M be given, N passes through a given pair of inverse points, namely, the pole points of M with respect to W.

323. If we are given only eight cyclides, $W_1, W_2\ldots W_8$, and seek to determine the cyclide W as in art. 322, so that the invariant Θ shall vanish for W and each of the eight cyclides, then, since we have only eight conditions, one of the tangential coefficients A, &c. remain undetermined; but we can determine all the rest in terms of that one, so that the tangential equation of W is $\Omega+K\Omega'=0$. Hence the focal quadric of W contains an indeterminate constant in the first degree, and therefore it passes through a given curve.

324. If ten spheres, $\alpha_1, \alpha_2\ldots\alpha_{10}$, be all generators of the same cyclide W, their equations are connected by a linear relation,

$$l_1\alpha_1^2+l_2\alpha_2^2\ldots l_{10}\alpha_{10}^2=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (199)$$

Demonstration. Let $\alpha_1=\lambda_1\alpha+\mu_1\beta+\nu_1\gamma+\varrho_1\delta=0$, &c.; and writing down the conditions that α_1, α_2, &c. are generating spheres of W, and eliminating linearly the ten quantities

$$\alpha^2,\ \beta^2,\ \gamma^2,\ \delta^2,\ \alpha\beta,\ \alpha\gamma,\ \alpha\delta,\ \beta\gamma,\ \beta\delta,\ \gamma\delta,$$

we get the following determinant:—

$$\left|\begin{array}{cccccccccc} \lambda_1^2 & \mu_1^2 & \nu_1^2 & \varrho_1^2 & \lambda_1\mu_1 & \lambda_1\nu_1 & \lambda_1\varrho_1 & \mu_1\nu_1 & \nu_1\varrho_1 & \varrho_1\mu_1 \\ \lambda_2^2 & \mu_2^2 & \nu_2^2 & \varrho_2^2 & \lambda_2\mu_2 & \lambda_2\nu_2 & \lambda_2\varrho_2 & \mu_2\nu_2 & \nu_2\varrho_2 & \varrho_2\mu_2 \\ 3 & , & , & , & , & , & , & , & , & , \\ 4 & , & , & , & , & , & , & , & , & , \\ 5 & , & , & , & , & , & , & , & , & , \\ 6 & , & , & , & , & , & , & , & , & , \\ 7 & , & , & , & , & , & , & , & , & , \\ 8 & , & , & , & , & , & , & , & , & , \\ 9 & , & , & , & , & , & , & , & , & , \\ 10 & , & , & , & , & , & , & , & , & , \end{array}\right|=0. \quad . \quad . \quad . \quad . \quad (200)$$

but this is also the condition that the squares should be connected by the linear relation.

325. Propositions similar to those of the three last articles hold for sphero-quartics and also for bicirculars. Thus the analogue of art. 322 is, being given five sphero-quartics S_1, S_2, S_3, S_4, S_5, and if multiples l_1, l_2 . . l_5 be determined so that $l_1S_1+l_2S_2+l_3S_3+l_4S_4+l_5S_5$ may be the square of a circle ν, then the envelope of ν is a sphero-quartic. The analogue of art. 324 is, if six circles be generators of the same sphero-quartic, their equations are connected by a linear relation.

326. To find the equation of the binodal cyclide formed by the generating spheres which touch W along the sphero-quartic in which W is intersected by the sphere $\lambda\alpha+\mu\beta+\nu\gamma+\varrho\delta$, where λ, μ, ν, ϱ are multiples.

The equation of any cyclide touching W along this curve will be of the form

$$kW+(\lambda\alpha+\mu\beta+\nu\gamma+\varrho\delta)^2,$$

and it is required to determine k so that this cyclide will be binodal. We find in this case Φ, Θ', Δ' all $=0$; the invariant equation has therefore three roots $=0$; and if we denote by σ the tangential expression $(A, B, C, D, L, M, N, P, Q, R\between\lambda, \mu, \nu, \varrho)^2$, the equation of the required binodal will be

$$\sigma W=\Delta(\lambda\alpha+\mu\beta+\nu\gamma+\varrho\delta)^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (201)$$

Cor. 1. When $\lambda\alpha+\mu\beta+\nu\gamma+\varrho\delta$ is a generating sphere of W, the binodal (201) reduces to $\sigma=0$.

Cor. 2. If α', β', γ', δ' be the sphero-coordinates of the points polar to $\lambda\alpha+\mu\beta+\nu\gamma+\varrho\delta$ with respect to W, and if W′ be the result of substituting α', β', γ', δ' in W, then we have

$$\sigma=\Delta W'. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (202)$$

For the binodal circumscribed to W, whose nodes are α', β', γ', δ', is

$$WW'=(\lambda\alpha+\mu\beta+\nu\gamma+\varrho\delta)^2$$

(see art. 280), and eliminating $\lambda\alpha+\mu\beta+\nu\gamma+\varrho\delta$ between this and (201), we get (202).

327. To find the condition that the circle of intersection of two spheres shall have double contact with W. Let W be given by the general equation, and let the spheres be $\lambda\alpha+\mu\beta+\nu\gamma+\varrho\delta$, $\lambda'\alpha+\mu'\beta+\nu'\gamma+\varrho'\delta$, then the required condition is the determinant

$$\begin{vmatrix} a, & n, & m, & p, & \lambda, & \lambda', \\ n, & b, & l, & q, & \mu, & \mu', \\ m, & l, & c, & r, & \nu, & \nu', \\ p, & q, & r, & d, & \varrho, & \varrho', \\ \lambda, & \mu, & \nu, & \varrho, & & \\ \lambda', & \mu', & \nu', & \varrho', & & \end{vmatrix}=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (203)$$

328. The condition $\sigma=0$ (see art. 326), that the sphere $\lambda\alpha+\mu\beta+\nu\gamma+\varrho\delta$ should be a generator of W, is a contravariant of the third order in the coefficients of W. Hence, if

we substitute for each coefficient a, $a+ka'$, we get the condition that $\lambda\alpha+\mu\beta+\nu\gamma+\varrho\delta$ shall be a generator of the cyclide $W+kW'$. The condition will be of the form

$$\sigma+k\tau+k^2\tau'+k^3\sigma'=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (204)$$

In terms of the functions σ, τ, τ', σ' can be expressed the condition that the sphere $\lambda\alpha+\mu\beta+\nu\gamma+\varrho\delta$ shall have any permanent relation with the cyclides W, W'; as, for instance, that it should intersect them in sphero-quartics w, w' connected by such permanent relations as can be expressed by relations between the coefficients of the discriminant of $w+kw'$. Thus if we form the discriminant with respect to k of equation (204), we get the condition that the sphero-quartics in which $\lambda\alpha+\mu\beta+\nu\gamma+\varrho\delta$ intersects W and W' shall have double contact; in other words, the discriminant is the condition that the sphero-quartics shall have a common generating circle which touches both quartics at the same points. Again, $\tau=0$ is the condition that the sphero-quartics w and w' are so related that the harmonic circles (see 'Bicircular Quartics,' art. 184) of one are generators of the other.

329. The coefficients σ, τ, τ', σ' of equation (204), and the discriminant of the same equation, have another geometrical interpretation. Thus σ and σ' are the equations in tetrahedral coordinates of the focal quadric of W and W', λ', μ', ν, ϱ being the current coordinates (see art. 27). and τ, τ' are quadrics covariant with σ, σ'. Thus τ "is the locus of a point whence cones circumscribing σ and σ' are so related that three edges of one can be found forming a self-conjugate system with regard to the second, and three tangent planes of the second which form a self-conjugate system with regard to the first" (see SALMON'S 'Geometry of Three Dimensions,' page 159). The discriminant of (204) is the developable circumscribed to σ and σ'; in other words, the locus of the centre of $\lambda\alpha+\mu\beta+\nu\gamma+\varrho\delta$ is the developable. Hence we infer:—*The locus of the centre of a variable sphere which cuts two cyclides in sphero-quartics having double contact is the developable circumscribed about the focal quadrics of the cyclides which correspond to their common inversion sphere.*

330. If we suppose the cyclide W' of the last article to become U^2, we have the following theorem:—*The locus of the centre of a sphere* S *intersecting the cyclide* W *in a sphero-quartic* WS *which has double contact with the sphero-quartic* WU *is the developable* Σ *formed by tangent planes to* U *along* WU (see Chapter VIII., art. 142).

331. If $W = a\alpha^2+b\beta^2+c\gamma^2+d\delta^2=0$, then $\sigma=A\lambda^2+B\mu^2+C\nu^2+D\varrho^2$, where $A=bcd$, $B=cda$, &c.; and changing a into $a+ka'$, &c., we get

$$\tau=\left(\frac{b'}{b}+\frac{c'}{c}+\frac{d'}{d}\right)A\lambda^2+\left(\frac{c'}{c}+\frac{d'}{d}+\frac{a'}{a}\right)B\mu^2+\left(\frac{d'}{d}+\frac{a'}{a}+\frac{b'}{b}\right)C\nu^2+\left(\frac{a'}{a}+\frac{b'}{b}+\frac{c'}{c}\right)D\varrho^2; \quad . \qquad (205)$$

and the cyclide which has (205) for a focal quadric will be got by reciprocating (205) and substituting α, β, γ, δ for the variables. Hence the required cyclide will be

$$\frac{\alpha^2}{b'cd+bc'd+bcd'}+\frac{\beta^2}{c'da+cd'a+cda'}+\frac{\gamma^2}{d'ab+da'b+dab'}+\frac{\delta^2}{a'bc+ab'c+abc'}. \quad . \quad . \qquad (206)$$

This will be the locus of the nodes of binodals circumscribed to W W', *the same points*

being nodes for both, if three harmonic spheres of one binodal be three generating spheres of the other.

332. We can reciprocate the process of recent articles. Thus, let V, V′ be the focal quadrics of two cyclides W, W′ having a common inversion sphere, then $V+kV'$ will be the focal quadric of a cyclide whose equation we can find as follows, viz. form the tangential equation of $V+kV'$, and substitute α, β, γ, δ for the variables, the required equation will be

$$\frac{\alpha^2}{a^{-1}+ka'^{-1}}+\frac{\beta^2}{b^{-1}+kb'^{-1}}+\frac{\gamma^2}{c^{-1}+kc'^{-1}}+\frac{\delta^2}{d^{-1}+kd'^{-1}}=0; \quad . \quad . \quad . \quad (207)$$

a^{-1}, b^{-1}, &c. are evidently the coefficients of V, since a, b, &c. are the coefficients of W. The discriminant of (207) with respect to k will be the envelope of all the cyclides which it can represent by varying k, that is, it will be the *tore* with which they all have double contact. The curve of *taction* of any of them with it will be of the eighth degree, being the intersection of two cyclides of the fourth degree.

The geometrical interpretation of the discriminant is, that it is the envelope of a variable sphere cutting U *orthogonally, and whose centre moves along the twisted quartic* (V V′).

333. We can get the equation of the cuspidal edge as follows: differentiate (207) twice with respect to k, and we get a system reducible to the following equations:—

$$\frac{a^{-2}\alpha^2}{(a^{-1}+ka'^{-1})^3}+\frac{b^{-2}\beta^2}{(b^{-1}+kb'^{-1})^3}+\frac{c^{-2}\gamma^2}{(c^{-1}+kc'^{-1})^3}+\frac{d^{-2}\delta^2}{(d^{-1}+kd'^{-1})^3}=0. \quad . \quad . \quad . \quad (208)$$

$$\frac{(aa')^{-1}\alpha^2}{(a^{-1}+ka'^{-1})^3}+\frac{(bb')^{-1}\beta^2}{(b^{-1}+kb'^{-1})^3}+\frac{(cc')^{-1}\gamma^2}{(c^{-1}+kc'^{-1})^3}+\frac{(dd')^{-1}\delta^2}{(d^{-1}+kd'^{-1})^3}=0. \quad . \quad . \quad . \quad (209)$$

$$\frac{(a')^{-2}\alpha^2}{(a^{-1}+ka'^{-1})^3}+\frac{(b')^{-2}\beta^2}{(b^{-1}+kb'^{-1})^3}+\frac{(c')^{-2}\gamma^2}{(c^{-1}+kc'^{-1})^3}+\frac{(d')^{-2}\delta^2}{(d^{-1}+kd'^{-1})^3}=0. \quad . \quad . \quad . \quad (210)$$

The result of eliminating k between these equations will be a pair of equations representing two surfaces whose curve of intersection will be the cuspidal edge.

Now solving the equations (208), (209), (210), we get

$$\frac{\alpha^2}{(a^{-1}+ka'^{-1})^3}=\begin{vmatrix} \frac{1}{b^2}, & \frac{1}{c^2}, & \frac{1}{d^2}, \\ \frac{1}{bb'}, & \frac{1}{cc'}, & \frac{1}{dd'}, \\ \frac{1}{b'^2}, & \frac{1}{c'^2}, & \frac{1}{d'^2}, \end{vmatrix}=\text{A}^2 \text{ suppose.}$$

Hence $(a^{-1}+ka'^{-1})=\left(\frac{\alpha^2}{\text{A}^2}\right)^{\frac{1}{3}}$, with similar values for $(b^{-1}+kb'^{-1})$, &c.; and substituting in the equation (207), we get

$$(\text{A}^2\alpha^4)^{\frac{1}{3}}+(\text{B}^2\beta^4)^{\frac{1}{3}}+(\text{C}^2\gamma^4)^{\frac{1}{3}}+(\text{D}^2\delta^4)^{\frac{1}{3}}=0\,. \quad . \quad . \quad . \quad . \quad . \quad (211)$$

as another surface on which the cuspidal edge lies. But if we eliminate k between any three of the equations for $a^{-1}+ka'^{-1}$, $b^{-1}+kb'^{-1}$, &c., we get four equations of binodal

cyclides, each of the twelfth degree, on which the cuspidal edge also lies. These equations will be similar in form to the sextic cones containing the cuspidal edge of the developable circumscribed about two quadrics (see art. 182).

Cor. *If any of these binodals be inverted from one of its nodes, it becomes one of the sextic cones of* art. 182.

334. The equation (207), cleared of fractions, becomes

$$\Delta' W + kT + k^2 T' + k^3 \Delta W'. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (212)$$

If in this equation we put $k=\frac{\Delta'}{\Delta}\lambda$, we get

$$\Delta^2 W + \lambda\Delta T + \lambda^2 \Delta' T' + \lambda^3 \Delta'^2 W' = 0. \quad . \quad . \quad . \quad . \quad . \quad (213)$$

Compare SALMON's 'Geometry of Three Dimensions,' art. 206. The value of T is

$$\Delta'\left\{a\left(\frac{b}{b'}+\frac{c}{c'}+\frac{d}{d'}\right)\alpha^2+b\left(\frac{c}{c'}+\frac{d}{d'}+\frac{a}{a'}\right)\beta^2+c\left(\frac{d}{d'}+\frac{a}{a'}+\frac{b}{b'}\right)\gamma^2+d\left(\frac{a}{a'}+\frac{b}{b'}+\frac{c}{c'}\right)\delta^2\right\}; \quad . \quad (214)$$

and T′ is got from T by interchanging accented and unaccented letters.

In terms of the cyclides T, T′ can be expressed all the cyclides having permanent relations to W, W′. Thus if

S be the reciprocal of W with respect to W′,

S′ be the reciprocal of W′ with respect to W,

then

$$T = \Theta' W - S', \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (215)$$

$$T' = \Theta W' - S. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (216)$$

Hence W, S′, T have a common curve of intersection.

335. The discriminant of (212) is

$$27\Delta^2\Delta'^2 W^2 W'^2 + 4(\Delta' W T'^3 + \Delta W' T^3) - TT'(TT' + 18\Delta\Delta' W W'), \quad . \quad . \quad (217)$$

an equation of the sixteenth degree, since it contains α, β, γ, δ in the eighth degree. *The imaginary circle at infinity is on this surface a multiple curve of the eighth degree, so that it is an octavic cyclide.*

By making W=0, we see that the surface touches W along the curve WT, and that it meets W again in the curve of intersection of W with $T'^2 - 4\Delta W' T$; this represents a system of eight circles which are generators of W. The sections of (217) by the spheres of reference are easily obtained; for, by a known process, the section of the discriminant of (207) by the sphere δ will be the sphero-quartic squared,

$$\left(\frac{aa'\alpha^2}{ad'-a'd}+\frac{bb'\beta^2}{bd'-b'd}+\frac{cc'\gamma^2}{cd'-c'd}\right)^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad (218)$$

multiplied by the discriminant of

$$\frac{\alpha^2}{a^{-1}+ka'^{-1}}+\frac{\beta^2}{b^{-1}+kb'^{-1}}+\frac{\gamma^2}{c^{-1}+kc'^{-1}},$$

or the system of four circles,

$$\alpha\sqrt{aa'(bc'-b'c)}\pm\beta\sqrt{bb'(ca'-c'a)}\pm\gamma\sqrt{cc'(ab'-a'b)}=0. \quad . \quad . \quad . \quad (219)$$

The section is therefore a sphero-quartic counted twice and four circles.

Cor. *The four circles are generators of the sphero-quartic.*

336. We can show geometrically that a generating circle of the cyclide W on each of the eight generating spheres common to W, W′, S′, or to W, W′, T′, is also a generating circle of the cyclide (212), and therefore that these eight circles form the locus which is the intersection of W with $T'^2-4\Delta TW$ (see art. 335). Since S′ and W′ are reciprocals with respect to W, it is evident that, on the eight spheres which are common generators of W, W′, S′, at the points of contact of W, W′ with S′ these spheres are coincident. Hence one of the generating circles of W on each of these generating spheres is also a generating sphere of the cyclide (212); hence W and (212) have eight common generating circles.

337. The cyclide (212) is the same generalization of the developable circumscribed to two quadrics which a cyclide of the second degree in α, β, γ, δ is of a quadric,—thus to the generating lines of one corresponding generating circles of the other. and to the nodal conics corresponding nodal sphero-quartics, and so on. Hence, by the system of substitutions established in Chapter XIII., we can get from the properties proved in Chapter VIII. of the developable Σ, theorems which hold for the cyclide (212). The following are a few illustrations:—

1°. Eight lines of Σ meet any arbitrary line. Hence *eight generating circles of* (212) *meet any arbitrary circle orthogonal to the sphere* U.

2°. The curves of taction of Σ divide homographically the lines of the system. Hence *the curves of taction of* (212) *divide homographically the circles of the system.*

3°. The nodal lines of Σ divide equianharmonically the lines of the system. Hence *the nodal sphero-quartics of* (212) *divide equianharmonically the circles of the system.*

4°. Any line of Σ meet its curves of taction in points the tangents at which to the curves of taction envelope a plane conic.

Hence *any generating circle of* (212) *meet its curves of taction in points, the generating circles of the curves of taction through which are generators of a sphero-quartic.*

338. Since the surface (212) is the envelope of a variable sphere cutting U orthogonally, and whose centre moves along the twisted quartic (VV′), then (VV′) is the déférente. From this generation we can also infer the properties of (212). *Thus the cuspidal edge of* (212) *is the locus of the limiting points composed of the sphere* U *and the osculating planes of* (VV′).

2°. *There are sixteen pairs of stationary points on the cuspidal edge ; these correspond to the stationary planes of* (VV′).

3°. *Any sphere cutting* U *orthogonally meets the cuspidal edge in twelve pairs of inverse points.* This follows from $n=12$ (see art. 224).

4°. *The cuspidal edge is an anallagmatic,* U *being the sphere of inversion.*

339. To find the locus of a pair of inverse points whose polar spheres with respect

to W will be a generating sphere of $W+kW'$. We have then in $\sigma+k\tau+k^2\tau'+k^3\sigma'$ to substitute $\frac{dW}{d\alpha}, \frac{dW}{d\beta}, \frac{dW}{d\gamma}, \frac{dW}{d\delta}$ for $\lambda, \mu, \nu, \varrho$; the result is expressible in terms of the covariants; it is

$$\Delta W+k(\Theta W-\Delta W')+k^2(\Phi W-T')+k^3(\Theta' W-T)=0. \quad . \quad . \quad . \quad . \quad (220)$$

In like manner the locus of inverse pairs of points whose polar spheres with respect to W' are generating spheres of $W+kW'$ is

$$\Theta W-T'+k(\Phi W'-T)+k^2(\Theta W'-\Delta' W)+k^3\Delta' W'=0. \quad . \quad . \quad . \quad . \quad (221)$$

340. To find the locus of a pair of inverse points whose polar spheres with respect to W, W' form a conjugate system with respect to $W+kW'$. Let

$$W\equiv a\alpha^2+b\beta^2+c\gamma^2+d\delta^2, \quad W'\equiv a'\alpha^2+b'\beta^2+c'\gamma^2+d'\delta^2,$$

and the locus will be

$$\begin{vmatrix} a+ka', & 0, & 0, & 0, & a\alpha, \\ 0, & b+kb', & 0, & 0, & b\beta, \\ 0, & 0, & c+kc', & 0, & c\gamma, \\ 0, & 0, & 0, & d+kd', & d\delta, \\ a\alpha, & b\beta, & c\gamma, & d\delta, & 0, \end{vmatrix}=0. \quad . \quad . \quad (222)$$

This can be expressed in terms of the covariants, and is the cyclide

$$\Delta W'+kT'+k^2T'+k^3\Delta' W=0 \text{ (compare 212)}. \quad . \quad . \quad . \quad . \quad . \quad (223)$$

(341). To find the condition that a given circle should have a given pair of inverse points common to the curve of intersection of two given cyclides W, W'. Suppose we have formed the condition (see art. 327) $R=0$, that the given circle should have double contact with W, and that we substitute in it for each coefficient a, $a+ka'$, &c., the condition becomes

$$R+k\pi+k^2R'=0\ ; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (224)$$

and if the given circle has any arbitrary position, we can, by solving this quadratic for k, determine two cyclides through the intersection of W and W', each having double contact with the given circle; but if the given circle has a pair of inverse points in common with the curve (WW'), the cyclides having double contact which can be drawn through (WW') become coincident, and the equation (224) becomes a perfect square. Hence the required condition is the discriminant

$$4RR'-\pi^2=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (225)$$

Cor. $\pi=0$ is the condition that the pair of segments which W intercepts on the given circle should be harmonic conjugates to those which W' intercepts on it.

342. If $W=a\alpha^2+b\beta^2+c\gamma^2+d\delta^2=0$, $W'=a'\alpha^2+b'\beta^2+c'\gamma^2+d'\delta^2=0$, and if the circle be the intersection of the spheres

$$\lambda\alpha+\mu\beta+\nu\gamma+\varrho\delta=0, \quad \lambda'\alpha+\mu'\beta+\nu'\gamma+\varrho'\delta=0,$$

then

$$R = \text{sum of the six terms } \Sigma ab(\nu\varrho'-\nu'\varrho)^2,$$

$$\pi = \text{the sum } \Sigma(ab'+a'b)(\nu\varrho'-\nu'\varrho)^2.$$

Therefore

$$\left.\begin{aligned}\pi^2-4RR' &= \Sigma(ab'-a'b)^2(\nu\varrho'-\nu'\varrho)^4\\ &+2(ab'-a'b)(ac'-a'c)(\mu\varrho'-\mu'\varrho)^2(\nu\varrho'-\nu'\varrho)^2\\ &+2\Sigma\{(ad'-a'd)(cb'-c'b)+(ac'-a'c)(db'-d'b)\}\\ &\times(\lambda\mu'-\lambda'\mu)^2(\nu\varrho'-\nu'\varrho)^2=0.\end{aligned}\right\} \quad . \quad . \quad . \quad (226)$$

343. *The surface generated by the circles which have double contact with the curve of intersection of* W *and* W′ *may be generated as the envelope of a variable sphere which cuts* U *orthogonally and whose centre moves along the cuspidal edge of the developable circumscribed about the focal quadrics of* W, W′ *which correspond to their common sphere of inversion* U. For let us consider any circle having double contact with the curve (WW′). Then, since a circle having double contact with WW′ is orthogonal to the sphere U, it is plain that a line through its centre perpendicular to its plane is a line of the developable circumscribed to the two focal quadrics, and therefore the sphere containing two consecutive circles will have its centre at the point of intersection of two consecutive lines of the developable that is on the cuspidal edge. Hence the theorem is proved.

344. *The curve* (W W′) *is a cuspidal edge on the surface generated by the circles having double contact with* (W W′). This is evident; for any circle having double contact is the characteristic of the surface (see MONGE, 'Application de l'Analyse à la Géométrie,' p. 53), and the points of intersection of each characteristic with the consecutive one form the cuspidal edge. Hence the proposition is proved.

Cor. The cuspidal edge is an anallagmatic.

345. *To find the equation of the surface generated by the circles which have double contact with the curve* (W W′).

Let us consider any pair of inverse points on any circle which has double contact with (W W′). The polar sphere of this pair, with respect either to W or W′, passes evidently through the two points of contact of the circle under consideration with the curve (W W′). The circle of intersection, therefore, of the two polar spheres intersects the curve (W W′) in two points; therefore the equation of the required surface is found by substituting in the equation (226) for λ, μ, ν, ϱ, $\frac{dW}{d\alpha}$, $\frac{dW}{d\beta}$, $\frac{dW}{d\gamma}$, $\frac{dW}{d\delta}$, and for λ', μ', ν', ϱ', $\frac{dW'}{d\alpha}$, $\frac{dW'}{d\beta}$, $\frac{dW'}{d\gamma}$, $\frac{dW'}{d\delta}$.

This surface is of the sixteenth degree, being of the eighth degree in α, β, γ, δ; when we use the canonical forms $a\alpha^2+b\beta^2+c\gamma^2+d\delta^2$, $a'\alpha^2+b'\beta^2+c'\gamma^2+d'\delta^2$ for W, W′, the

equation of the surface becomes

$$\left.\begin{aligned}&\Sigma(ab'-a'b)^4(cd'-c'd)^4\gamma^4\delta^4\\&\quad+2\Sigma(ab'-a'b)(ac'-a'c)(cd'-c'd)^2(bd'-b'd)^2\beta^2\gamma^2\delta^4\\&\quad+2\alpha^2\beta^2\gamma^2\delta^2\{(ab'-a'b)(cd'-c'd)-(ad'-a'd)(bc'-b'c)\}\\&\quad\times\{(ad'-a'd)(bc'-b'c)-(bd'-b'd)(ca'-c'a)\}\\&\quad\times\{(bd'-b'd)(ca'-c'a)-(ab'-a'b)(cd'-c'd)\}.\end{aligned}\right\}\quad . \quad . \quad (227)$$

The imaginary circle at infinity is a multiple curve of the order eight on this surface.

Cor. 1. When we make $\delta=0$ in equation (227) we get a perfect square. Hence each of the four spheres of reference meets the surface in a double line on the surface. These double lines correspond to the double lines of the developable Δ (see Chapter VIII.), and each of them has six double points. Thus the sphero-octavic in which δ intersects the surface is expressed in terms of α, β, γ, and is of the form

$$\frac{(bc'-b'c)^2}{bc\alpha^2}+\frac{(ca'-c'a)^2}{c'a'\beta^2}+\frac{(ab'-a'b)^2}{a'b'\gamma^2}=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (228)$$

Hence the three pairs of points $(\alpha\beta)$, $(\beta\gamma)$, $(\gamma\alpha)$ are double points.

Cor. 2. The equation (227), expressed in terms of the covariant cyclides, is given by the determinant

$$\begin{vmatrix} 2(\Theta WW'-T'W-\Delta W'^2), & \Phi WW'-TW-T'W', \\ \Phi WW'-TW-T'W', & 2(\Theta' WW'-TW'-\Delta' W^2), \end{vmatrix}=0. \quad . \quad . \quad (229)$$

Cor. 3. The surface also meets W in the curve of intersection of W with $T'^2-4\Delta TW'$, which we have shown represents a system of eight circles which are generators of W.

Cor. 4. Any arbitrary circle orthogonal to U meets eight generating circles of the surface, and the spheres determined by the arbitrary circle and the eight meeting circles are generators of a cyclide.

346. If a cyclide W be given by the equation

$$a\alpha^2+b\beta^2+c\gamma^2+d\delta^2=0,$$

and also by this other equation referred to different spheres,

$$a'\alpha'^2+b'\beta'^2+c'\gamma'^2+d'\delta'^2=0,$$

then we can infer, as in SALMON'S 'Geometry of Three Dimensions,' art. 192, the following theorems:—

1°. The eight spheres α, β, γ, δ, α', β', γ', δ' are generators of the same cyclide.

2°. The two quartets of pairs of inverse points

$$(\alpha\ \beta\ \gamma),\quad (\alpha\ \beta\ \delta),\quad (\alpha\ \gamma\ \delta),\quad (\beta\ \gamma\ \delta),$$
$$(\alpha'\ \beta'\ \gamma'),\quad (\alpha'\ \beta'\ \delta'),\quad (\alpha'\ \gamma'\ \delta'),\quad (\beta'\ \gamma'\ \delta'),$$

lie on a cyclide; this theorem (2°) may be inferred by reciprocation from 1°.

347. If W, W′ be two cyclides, $S\equiv\lambda\alpha+\mu\beta+\nu\gamma+\varrho\delta=0$ any sphere, and it be required to find the condition that the binodal cyclides formed by the generating spheres which touch W, W′ along their curves of intersection with S may intersect along a third cyclide W″, the three cyclides being given by the equations

$$W\equiv a\alpha^2+b\beta^2+c\gamma^2+d\delta^2=0,$$
$$W'\equiv a'\alpha^2+b'\beta^2+c'\gamma^2+d'\delta^2=0,$$
$$W''\equiv a''\alpha^2+b''\beta^2+c''\gamma^2+d''\delta^2=0,$$

then we have

$$\sigma\equiv bcd\lambda^2+cda\mu^2+dab\nu^2+abc\varrho^2, \qquad \Delta=abcd,$$
$$\sigma'\equiv b'c'd'\lambda^2+c'da'\mu^2+d'a'b'\nu^2+a'b'c\varrho^2, \qquad \Delta'=a'b'c'd'.$$

Hence (see art. 326) the equations of the binodals which circumscribe W and W′ along the sphero-quartics (WS) and (W′S) are

$$\left(\frac{\lambda^2}{a}+\frac{\mu^2}{b}+\frac{\nu^2}{c}+\frac{\varrho^2}{d}\right)W-S^2=0,$$
$$\left(\frac{\lambda^2}{a'}+\frac{\mu^2}{b'}+\frac{\nu^2}{c'}+\frac{\varrho^2}{d'}\right)W'-S^2=0.$$

Hence we must have, by the given conditions,

$$\left(\frac{\lambda^2}{a}+\frac{\mu^2}{b}+\frac{\nu^2}{c}+\frac{\varrho^2}{d}\right)W-\left(\frac{\lambda^2}{a'}+\frac{\mu^2}{b'}+\frac{\nu^2}{c'}+\frac{\varrho^2}{d'}\right)W'\equiv kW'',$$

where k is some constant; and equating coefficients, we get

$$\mu^2\left(\frac{a}{b}-\frac{a'}{b'}\right)+\nu^2\left(\frac{c}{d}-\frac{c'}{d'}\right)+\varrho^2\left(\frac{a}{d}-\frac{a'}{d'}\right)=ka'',$$
$$\nu^2\left(\frac{b}{c}-\frac{b'}{c'}\right)+\varrho^2\left(\frac{b}{d}-\frac{b'}{d'}\right)+\lambda^2\left(\frac{b}{a}-\frac{b'}{a'}\right)=kb'',$$
$$\varrho^2\left(\frac{c}{d}-\frac{c'}{d'}\right)+\lambda^2\left(\frac{c}{a}-\frac{c'}{a'}\right)+\mu^2\left(\frac{c}{b}-\frac{c'}{b'}\right)=kc'',$$
$$\lambda^2\left(\frac{d}{a}-\frac{d'}{a'}\right)+\mu^2\left(\frac{d}{b}-\frac{d'}{b'}\right)+\nu^2\left(\frac{d}{c}-\frac{d'}{c'}\right)=kd'';$$

multiplying these equations by $\frac{\lambda^2}{aa'}, \frac{\mu^2}{bb'}, \frac{\nu^2}{cc'}, \frac{\varrho^2}{dd'}$, and adding, the left side vanishes identically. Hence the required condition is

$$\frac{\lambda^2a''}{aa'}+\frac{\mu^2b''}{bb'}+\frac{\nu^2c''}{cc'}+\frac{\varrho^2d''}{dd'}=0. \qquad (230)$$

348. The envelope of the sphere $S\equiv\lambda\alpha+\mu\beta+\nu\gamma+\varrho\delta=0$ is the cyclide

$$\frac{aa'}{a''}\alpha^2+\frac{bb'}{b''}\beta^2+\frac{cc'}{c''}\gamma^2+\frac{dd'}{d''}\delta^2=0, \qquad (231)$$

the tangential equation of its focal quadric being (230); I shall denote the cyclide (231) by W''_1.

349. *If the cyclides* W, W', W'' *pass through a common curve, the cyclides* W, W', W''_1 *are inscribed in the same cyclic developable* (see art. 335).

Demonstration. The focal quadrics of W, W', W''_1 are

$$F=\frac{\lambda^2}{a}+\frac{\mu^2}{b}+\frac{\nu^2}{c}+\frac{\varrho^2}{d},$$

$$F'=\frac{\lambda^2}{a'}+\frac{\mu^2}{b'}+\frac{\nu^2}{c'}+\frac{\varrho^2}{d'},$$

$$F''_1=\frac{a''\lambda^2}{aa'}+\frac{b''\mu^2}{bb'}+\frac{c''\nu^2}{cc'}+\frac{d''\varrho^2}{dd'}.$$

Now if $W''=W+kW'$, we must have $a''=a+ka'$ &c. Hence $F''_1=F+kF'$, that is, the focal quadrics pass through a common curve. Hence the proposition is proved.

350. *If the cyclide* W'' *pass through the curve* (W W'), *the cyclide* W''_1 *is inscribed in six cyclides passing through* (W W').

Demonstration. If two cyclides, Ω, Ω', be such that one is inscribed in the other, then the reciprocal of Ω with respect to Ω' is inscribed in Ω', and also the reciprocal of Ω' with respect to Ω.

Let $\Omega=A\alpha^2+B\beta^2+C\gamma^2+D\delta^2$, $\Omega'=A'\alpha^2+B'\beta^2+C'\gamma^2+D'\delta^2$. Hence the required condition will be the determinant

$$\begin{vmatrix} A, & B, & C, & D, \\ A', & B', & C', & D', \\ \frac{A^2}{A'}, & \frac{B^2}{B'}, & \frac{C^2}{C'}, & \frac{D^2}{D'}, \\ \frac{A'^2}{A}, & \frac{B'^2}{B}, & \frac{C'^2}{C}, & \frac{D'^2}{D}, \end{vmatrix}=0,$$

or $\frac{1}{ABCD}\cdot\frac{1}{A'B'C'D'}(AB'-A'B)(AC'-A'C)(AD'-A'D)(BC'-B'C)(CD'-C'D)(BD'-B'D)=0.$

Hence $(AB'-A'B)(AC'-A'C)(AD'-A'D)(BC'-B'C)(CD'-C'D)(BD'-B'D)=0$ (232)

is the condition of Ω being inscribed in Ω', that is, the product of the six determinants of the matrix

$$\begin{Vmatrix} A, & B, & C, & D, \\ A', & B', & C', & D', \end{Vmatrix}$$

as is otherwise evident.

To apply the condition (232), we have

$$W''_1=\frac{aa'}{a+ka'}\alpha^2+\frac{bb'}{b+kb'}\beta^2+\frac{cc'}{c+kc'}\gamma^2+\frac{dd'}{d+ka'}\delta^2=0\,;$$

and let the cyclide through the curve (W W') be

$$(a+k'a')\alpha^2+(b+k'b')\beta^2+(c+k'c')\gamma^2+(d+k'd')\delta^2=0,$$

then we get, after some reduction, the condition of W''_1 being inscribed in $W+k'W'$

given by the equation

$$\left(kk'-\frac{a'b'}{ab}\right)\left(kk'-\frac{a'c'}{ac}\right)\left(kk'-\frac{a'd'}{ad}\right)\left(kk'-\frac{b'c'}{bc}\right)\left(kk'-\frac{b'd'}{bd}\right)\left(kk'-\frac{c'd'}{cd}\right)=0, \quad . \quad (233)$$

a sextic in k'. Hence the proposition is proved.

351. *Given four cyclides* W, W′, W″, W‴, *required to find the locus of a pair of inverse points such that their polar spheres, with respect to the four cyclides, may pass through the same pair of inverse points.*

The required locus is the Jacobian of the four cyclides

$$\begin{vmatrix} \frac{dW}{d\alpha}, & \frac{dW}{d\beta}, & \frac{dW}{d\gamma}, & \frac{dW}{d\delta}, \\ \frac{dW'}{d\alpha}, & \frac{dW'}{d\beta}, & \frac{dW'}{d\gamma}, & \frac{dW'}{d\delta}, \\ \frac{dW''}{d\alpha}, & \frac{dW''}{d\beta}, & \frac{dW''}{d\gamma}, & \frac{dW''}{d\delta}, \\ \frac{dW'''}{d\alpha}, & \frac{dW'''}{d\beta}, & \frac{dW'''}{d\gamma}, & \frac{dW'''}{d\delta}, \end{vmatrix}=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (234)$$

Cor. *The envelope of a sphere whose pole points with respect to four cyclides are homospheric is the Jacobian of the four cyclides.*

352. The Jacobian is the locus of the nodes of all binodals which can be represented by $kW+k'W'+k''W''+k'''W'''$. Thus, there being given six pairs of inverse points, the locus of the nodes of all binodes which can pass through them is an anallagmatic surface of the eighth degree. For if W, W′, W″, W‴ be any cyclides through them, every cyclide through them can be represented by $kW+k'W'+k''W''+k'''W'''$, since this last form contains three independent constants, which are necessary to complete the solution.

Cor. 1. If in any case $kW+k'W'+k''W''+k'''W'''$ can represent two spheres, the intersection of these spheres is a circle on the Jacobian.

Cor. 2. If one of the cyclides W be a perfect square L^2, the Jacobian consists of a sphere and an anallagmatic surface of the third order in $\alpha, \beta, \gamma, \delta$, that is, a surface whose déférente is a surface of the third class.

Cor. 3. If the cyclides have in common four pairs of inverse points which are homospheric, the sphere through these points is a part of the Jacobian.

Cor. 4. If the four cyclides have a sphero-quartic curve common to all, the sphere through the sphero-quartic counts doubly in the Jacobian, which therefore reduces to a cyclide and the square of the sphere.

Cor. 5. The Jacobian of four Cartesian cyclides is a Cartesian cyclide.

353. If F, F′, F″, F‴ be the focal quadrics of W, W′, W″, W‴ in tangential coordinates, the déférente of the Jacobian of W . . . W‴ is the Jacobian in tangential coordinates of

$F \ldots F'''$, that is, the determinant

$$\begin{vmatrix} \frac{dF}{d\lambda}, & \frac{dF}{d\mu}, & \frac{dF}{d\nu}, & \frac{dF}{d\varrho}, \\ \frac{dF'}{d\lambda}, & \frac{dF'}{d\mu}, & \frac{dF'}{d\nu}, & \frac{dF'}{d\varrho}, \\ \frac{dF''}{d\lambda}, & \frac{dF''}{d\mu}, & \frac{dF''}{d\nu}, & \frac{dF''}{d\varrho}, \\ \frac{dF'''}{d\lambda}, & \frac{dF'''}{d\mu}, & \frac{dF'''}{d\nu}, & \frac{dF'''}{d\varrho}, \end{vmatrix} = 0. \qquad (235)$$

354. If a cyclide of the systems $kW+k'W'+k''W''$ have double contact with W''', the points of contact are evidently points on the Jacobian, and therefore lie somewhere on the curve of intersection of W''' with the Jacobian. Again, if a cyclide of the system $kW+k'W'$ have double contact with the curve $(W''\ W''')$, that is to say, if at one of the pairs of inverse points where $kW+'kW'$ meets W'' and W''' the generating spheres of $(kW+k'W')$, W'', W''' intersect in the same circle, the pair of points is evidently a pair of points on the Jacobian. It follows then that sixteen surfaces of the system $kW+k'W'$ can be described to have double contact with the curve $(W''\ W''')$, since the Jacobian is of the fourth degree in $(\alpha, \beta, \gamma, \delta)$, and each of the cyclides W'', W''' of the second degree, and each system of common values of α, β, γ, δ gives a pair of inverse points.

355. *Given three cyclides W, W', W'', the locus of a pair of inverse points whose polar spheres with respect to W, W', W'' have a common circle of intersection is the curve of the twelfth degree, which is common to the system of determinants*

$$\begin{Vmatrix} \frac{dW}{d\alpha}, & \frac{dW}{d\beta}, & \frac{dW}{d\gamma}, & \frac{dW}{d\delta}, \\ \frac{dW'}{d\alpha}, & \frac{dW'}{d\beta}, & \frac{dW'}{d\gamma}, & \frac{dW'}{d\delta}, \\ \frac{dW''}{d\alpha}, & \frac{dW''}{d\beta}, & \frac{dW''}{d\gamma}, & \frac{dW}{d\delta}. \end{Vmatrix} \qquad (236)$$

356. To find the condition in the invariants that two cyclides W, W' shall be so related that four generating spheres α, β, γ, δ of W' shall have the circles $(\alpha\beta)$, $(\alpha\gamma)$, $(\delta\beta)$, $(\delta\gamma)$ lying on W.

The equation of W must be of the form $L\beta\gamma+P\alpha\delta=0$, and the coefficients a, b, c, d must be wanting in the general equation of W'. Hence we have

$$\Delta=L^2P^2, \qquad \Theta=2LP(Lp+Pl),$$

$$\Phi=(Lp+Pl)^2+2LP(lp-mq-nr),$$

$$\Theta'=2(lp-mq-nr)(Lp+Pl).$$

Hence the required condition is

$$4\Delta\Theta\Phi=\Theta^3+8\Delta^2\Theta'. \qquad (237)$$

Cor. The condition that W shall have the circles $(\alpha\beta)$, $(\alpha\gamma)$, $(\delta\beta)$, $(\delta\gamma)$ lying on its surface, while the four pairs of inverse points $(\alpha\beta\gamma)$, $(\alpha\beta\delta)$, $(\alpha\gamma\delta)$, $(\beta\gamma\delta)$ shall lie on the surface of W′, is the equation, reciprocal of the former,

$$4\Delta'\Theta'\Phi=\Theta'^3+8\Delta'^2\Theta. \quad \ldots \ldots \ldots \quad (238)$$

357. We have seen if two cyclides W, W′ be reciprocals with respect to

$$U^2\equiv\alpha^2+\beta^2+\gamma^2+\delta^2,$$

that their focal quadrics are reciprocals with respect to the sphere U. Hence it follows that if the focal quadric of W be a plane conic, the focal quadric of W′ will be a cone. Hence we have the following theorem:—*The reciprocal of a binodal cyclide is a sphero-quartic, and* vice versâ.

358. *If a cyclide* W *breaks up into two spheres, its reciprocal* W′ *breaks up into two spheres.* For if W breaks up into two spheres, its focal quadric is a quadric of revolution circumscribed to U, and the reciprocal of the focal quadric with respect to U is another quadric of revolution circumscribed to U.

This and the last article belong to the Chapter on reciprocation, but were accidentally omitted.

359. If we form the discriminant of $kW+k'W'+k''W''$, the coefficients of the several powers of k, k', k'' will be invariants of the system of cyclides. There are two invariants, however, of the system $kW+k'W'+k''W''$ which, as being combinants, deserve attention. These invariants we shall call I and J.

The combinant I *vanishes whenever any four of the eight generating spheres common to* W, W′, W″ *are connected by a linear relation, that is, pass through the same two points.*

It is easy to see that this is equivalent to the statement that I vanishes for the values of k, k', k'' which will make $kW+k'W'+k''W''$ represent two spheres. The equations of W, W′, W″, as having a common sphere of inversion, may plainly be written in the forms

$$W\equiv a\alpha^2+b\beta^2+c\gamma^2+d\delta^2+e\varepsilon^2,$$
$$W'\equiv a'\alpha^2+b'\beta^2+c'\gamma^2+d'\delta^2+e'\varepsilon^2,$$
$$W''\equiv a''\alpha^2+b''\beta^2+c''\gamma^2+d''\delta^2+e''\varepsilon^2,$$

where $\alpha^2+\beta^2+\gamma^2+\delta^2+\varepsilon^2=0$ identically; and it is clear that I is the product of the ten determinants (a, b', c''), &c. For $(a, b', c'')\alpha^2+(d, b', c'')\delta^2+(e, b', c'')\varepsilon^2$ is evidently a cyclide of the system $kW+k'W'+k''W''$; and this reduces to two spheres if one of the determinants $(a\,b'\,c'')$ vanishes. Hence I is the product of the ten determinants.

Cor. *The combinant* I *vanishes also whenever any four of the eight pairs of inverse points common to* W, W′, W″ *are homospheric.*

360. *The combinant* J *vanishes whenever any two of the eight generating spheres common to* W, W′, W″ *are coincident; or again, when any two pairs of the eight pairs of common inverse points are coincident, so that* J *will be the tact-invariant of the three cyclides.*

If the generating sphere at a pair of inverse points common to the three surfaces pass

through a common circle, the consecutive pair of inverse points on this circle will be common to all the surfaces; such a pair of inverse points will be the two conic nodes of all binodal cyclides of the system $kW+k'W'+k''W''$.

For let the generating spheres at the given pair of inverse points be α, β, and $(a\alpha+b\beta)$, and the equations of the cyclides may be written $\alpha\delta+w_2$, $\beta\delta+w'_2$, and $(a\alpha+b\beta)\delta+w''_2$, where w_2, w'_2, w''_2 denote homogeneous functions of the second degree in α, β, γ; and it is evident that $aW+bW'-W''$ is a binodal cyclide having the given pair of points as conic nodes.

Cor. J will be of the sixteenth degree in the coefficients of W, W′, W″. For if in J we substitute for each coefficient a of W, $a+ka_1$, where a_1 is the corresponding coefficient of a fourth cyclide W_1, it is evident that the degree of the result in k is the same as the number of cyclides of the system $W+kW_1$ which can be drawn to have double contact with the curve of intersections of the cyclides W′ and W″, and the degree is therefore sixteen (see art. 354).

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PHILOSOPHICAL TRANSACTIONS

FOR THE YEAR 1871.

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